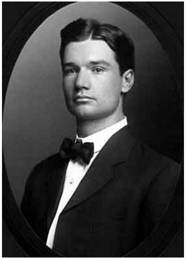
Set Theory Seminar: Fall 2012

The Normal Moore Space Conjecture



R.L. Moore

Outline

A space with a "development"

Let (X, τ) be a topological space.

A sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of *open* covers is a development if:

For each $x \in X$,
For each selection $(U_n : n \in \mathbb{N})$ such that:

1. For each n, $x \in U_n$ and
2. For each n, $U_n \in \mathcal{U}_n$,

$\{U_n : n \in \mathbb{N}\}$ is a neighborhood basis for x.

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Regular Hausdorff spaces

(X, τ) is a **Hausdorff space** if:
For all $\{x, y\} \in [X]^2$ there are disjoint open sets

$$U_x \text{ and } U_y$$

such that $x \in U_x$ and $y \in U_y$.

(X, τ) is a **regular space** if:
For each closed $C \subset X$ and each $x \in X \setminus C$ there are disjoint open sets

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Definition of a Moore space

A space (X, τ) is a **Moore** space if it is regular, Hausdorff, and has a development.

For the rest of the presentation spaces are assumed to be Hausdorff and regular.

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Easy facts about Moore spaces.

1. Every metrizable space is a Moore space.
2. Each Moore space is first countable (each element has a countable neighborhood base).
3. Every subspace of a Moore space is a Moore space.
4. If a Moore space is Lindelöf then it is second countable.
5. A second countable regular space is metrizable (Urysohn).

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Metrizability and Moore spaces.

There are examples of Moore spaces that are not metrizable.

All early examples of non-metrizable Moore spaces are also not normal spaces.

(X, τ) is a **normal space** if:

For each pair of disjoint closed sets $C \subset X$ and $D \subset X$ there are disjoint open sets

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The Normal Moore Space Conjecture (NMSC)

Conjecture(F.B. Jones, 1933) Any normal Moore space is metrizable.

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Separable Moore spaces.

Theorem (R.L. Moore)

Hereditarily separable normal Moore spaces are metrizable.

Theorem (F.B. Jones)

If $2^{\aleph_0} < 2^{\aleph_1}$ then each separable normal Moore space is metrizable.

Thus, if there is a separable non-metrizable Moore space then $2^{\aleph_0} = 2^{\aleph_1}$.

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Q-sets

A set Y of real numbers is a Q-*set* if:

1. Y is uncountable and
2. each subset of Y is a G_δ set in the topology of Y.

Theorem (Rothberger)

If $\mathfrak{p} > \aleph_1$, then each set of reals of cardinality $\aleph_1$ is a Q-set.

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An equivalence.

Theorem (Heath)

There is a separable normal non-metrizable Moore space if, and only if, there is a Q-set.

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Consistency Results.

Theorem (MA+$\neg$CH)

There is a non-metrizable normal Moore space.

Theorem (Fleissner)

CH *implies that there is a non-metrizable normal Moore space.*

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Metacompactness: A problem of Alexandroff

A space (X, τ) is *metacompact* if:

there is for every open cover $\mathcal{U}$ an open cover $\mathcal{V}$ such that:

1. For each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subseteq U$, and
2. For each $x \in X$, $\{V \in \mathcal{V} : x \in V\}$ is finite.

Problem(**Alexandroff**) Is every normal metacompact Moore space metrizable?

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Positive partial results

Theorem (Traylor)

If a metacompact normal Moore space is separable, then it is metrizable.

Heath showed separability is needed in Traylor's Theorem.

Theorem (Fleissner)

V=L *implies that every locally compact normal Moore space is metrizable.*

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Collectionwise normal spaces.

A family $\mathcal{D}$ of subsets of a space (X, τ) is *discrete* if:

For each $x \in X$ there is an open set U such that:

1. $x \in U$ and
2. $|\{C \in \mathcal{C} : C \cap U \neq \emptyset\}| \leq 1$.

The space (X, τ) is *collectionwise normal* if:

For each discrete family $\mathcal{C}$ of closed sets

there is a disjoint family $\{U_C : C \in \mathcal{C}\}$ of open sets such that

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The space (X, τ) is *collectionwise normal* if:

For each discrete family $\mathcal{C}$ of closed sets

there is a disjoint family $\{U_C : C \in \mathcal{C}\}$ of open sets such that

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Collectionwise normality and Moore spaces

Theorem (R.H. Bing)

Every collectionwise normal Moore space is metrizable.

Problem Is every first countable normal space collectionwise normal?
Yes $\Rightarrow$ NMSC.

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Outline

The Product Measure Extension Axiom

PMEA For each infinite set X, the product measure on $\{0, 1\}^X$ can be extended to a measure of all subsets of $\{0, 1\}^X$ such that:

For each family of fewer than $2^{\aleph_0}$ measure zero sets, the measure of their union is zero.

Theorem (Kunen)

If it is consistent that there is a strongly compact cardinal, then it is consistent that the Product Measure Extension Axiom holds.

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Lower bounds on the strength of NMSC

A cardinal κ is a *strong limit* cardinal if for each cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.

Theorem (Fleissner)

If each normal Moore space is metrizable, then

for each strong limit cardinal κ of countable cofinality (including $\aleph_0$),

1. $2^\kappa > \kappa^+$, or
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