1 Measurable cardinals

1.1 Definition. A *filter* on a set X is a set $F \subseteq \mathcal{P}(X)$ which is closed under intersections and supersets, i.e., such that

- for all A, B in F, $A \cap B \in F$;
- for all $B \subseteq \kappa$ and all $A \in F \cap \mathcal{P}(B)$, $B \in F$.

Given a cardinal κ, a filter F is κ-complete if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a subset of F of cardinality less than κ.

1.2 Definition. An *ultrafilter* on a set X is a filter on X such that for all $A \subseteq X$, exactly one of A and $X \setminus A$ is in U. The ultrafilter U is *nonprincipal* if no singleton is a member of U.

1.3 Definition. A cardinal κ is said to be *measurable* if there exists a κ-complete nonprincipal ultrafilter on κ.

1.4 Exercise. Show that if κ is measurable, then κ is a regular strong limit cardinal (i.e, κ is *strongly inaccessible*).

Suppose that U is an ultrafilter on a set X. Consider the class of functions with domain X, and the equivalence relation $\sim$ on this class defined by setting $f \sim g$ if and only if $\{x \in X \mid f(x) = g(x)\} \in U$. For each function f with domain X, let $[f]_U$ denote the $\sim$-equivalence class of f. Define the relation E on equivalence classes by setting $[f]_U E [g]_U$ if and only if

$$\{x \in X \mid f(x) \in g(x)\} \in U.$$

Note that this relation does not depend on the representatives chosen from $[f]_U$ and $[g]_U$. Then $\mathrm{Ult}(V, U)$, the ultrapower of V by U, consists of the class of all sets of the form $[f]_U$, with the binary relation E.

Theorem 1.5. *Let U be an ultrafilter on a set X. For all function $f_1, \ldots, f_n$ with domain X, and all n-nary formulas ϕ,*

$$(\mathrm{Ult}(V, U), E) \models \phi([f_1]_U, \ldots, [f_n]_U)$$

if and only if

$$\{x \in X \mid \phi(f_1(x), \ldots, f_n(x))\} \in U.$$

Proof. By induction on the complexity of formulas. For $=$ and $\in$ this is true by definition. For $\wedge$ it follows from the fact that U is a filter, and for $\neg$ it follows from the fact that U is an ultrafilter. For $\exists$, note that by Replacement and the Axiom of Choice, if

$$\{x \in X \mid \exists a \phi(a, f_1(x), \ldots, f_n(x))\} \in U$$

then there is a function g with domain X such that

$$\{x \in X \mid \phi(g(x), f_1(x), \ldots, f_n(x))\} \in U.$$

This gives the reverse implication for this step; the other direction is easier. $\square$

1.6 Remark. Eventually, we will want to come back to this proof and think about the fragment of ZFC needed to carry it out.

Corollary 1.7. *Let U be an ultrafilter on a set X, and let $j\colon V \to \mathrm{Ult}(V,U)$ be the function which sends each set x to $[i_x]_U$, where i is the constant function from X to $\{x\}$. Then j is an elementary embedding.*

Lemma 1.8. *Suppose that U is an ultrafilter on a set X, $j\colon V \to \mathrm{Ult}(V,U)$ is the corresponding embedding, i is the identity function on X and f is a function with domain X. Then $[f]_U = j(f)([i]_U)$.*

Proof. It suffices to see that for all functions f, g with domain X, and any relation R in $\{\in, =\}$, $[f]_U R [g]_U$ if and only if $j(f)([i]_U) R j(g)([i]_U)$. Since $j(f)$ and $j(g)$ are represented by the constant functions from X to $\{f\}$ and $\{g\}$ respectively, both expressions reduce to $\{x \in X \mid f(x) R g(x)\} \in U$. $\square$

1.9 Remark. By convention, we identify each $[f]_U$ in $\mathrm{Ult}(V,U)$ for which $([f]_U, E)$ is wellfounded with its Mostowski collapse, i.e., the unique set a such that $([f]_U, E)$ is isomorphic to $(a, \in)$.

1.10 Exercise. Given an ultrafilter U on a set X, $\mathrm{Ult}(V,U)$ is wellfounded if and only if U is countably complete (i.e., closed under countable intersections).

1.11 Definition. If $j\colon V \to M$ is an elementary embedding, the *critical point* of j is the least ordinal α such that $j(\alpha) > \alpha$, if one exists.

1.12 Exercise. Suppose that U is a κ-complete ultrafilter on a regular cardinal κ. Show that the critical point of j is κ.

1.13 Exercise (Scott). Show that there are no measurable cardinals in L. (Hint : Let κ be the least measurable cardinal in L, and consider the elementary embedding $j\colon L \to M$ given by a κ-complete ultrafilter U on κ (in L).)

1.14 Definition. A filter F on a cardinal κ is said to be *normal* if for each $A \in F$ and each regressive function $f\colon A \to \kappa$ (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A \setminus \{0\}$) there is an $\alpha < \kappa$ such that $f^{-1}[\alpha] \in F$.

1.15 Exercise. Show that if U is a normal ultrafilter on κ, U is κ-complete.

1.16 Exercise. Show that if U is a normal ultrafilter on κ and $i\colon \kappa \to \kappa$ is the identity function on κ, then $[i]_U = \kappa$ (under our identification of elements of $\mathrm{Ult}(V,U)$ with their Mostowski collapses).

1.17 Exercise. Suppose that $j\colon V \to M$ is an elementary embedding with critical point κ. Show that $\{A \subseteq \kappa : k \in j(A)\}$ is a normal ultrafilter on κ.

The previous exercise shows that "κ is a measurable cardinal" is equivalent to "there exists an elementary embedding $j\colon V \to M$ with critical point κ" and also equivalent to "there exists a normal ultrafilter on κ."

1.18 Exercise. Suppose that U is a κ-complete ultrafilter on κ, and let $j\colon V \to M$ be the corresponding elementary embedding. Show that $\mathcal{P}(\kappa) \subseteq M$, but $U \notin M$. Show that M is closed under sequences of length κ. Show that $2^\kappa < j(\kappa) < (2^\kappa)^+$.

Theorem 1.19. *Suppose that U is a normal ultrafilter on κ, and that $\lambda > \kappa$ is a regular cardinal. Let X be an elementary submodel of V_λ of cardinality less than κ, with $U \in X$. Let γ be any element of $\bigcap(X \cap U)$, and let $Y = \{f(\gamma) \mid f\colon \kappa \to V_\lambda,\ f \in X\}$. Then $X \subseteq Y$, $Y \prec V_\lambda$, and $Y \cap \kappa = X \cap \kappa$.*

Proof. That $X \subseteq Y$ follows from the fact that there is a constant function in X for each element of X. The fact that $Y \cap \kappa = X \cap \kappa$ follows from normality, as follows. If $f\colon \kappa \to V_\lambda$ is in X, and $f(\gamma) < \gamma$, then f is regressive on a set in U, so there is an $\alpha < \gamma$ such that $f^{-1}[\{\alpha\}] \in U$. Then $\alpha \in X$ and $f(\gamma) = \alpha$.

For elementarity, by the Vaught-Tarski test we need to see only that if $a_1, \ldots, a_n$ are in Y and $V_\lambda \models \exists x \phi(x, a_1, \ldots, a_n)$, then there is a $b \in Y$ such that $V_\lambda \models \phi(b, a_1, \ldots, a_n)$. To see that this holds, fix functions $f_1, \ldots, f_n$ in X such that each $a_i = f_i(\gamma)$. There is in V_λ a function $g\colon \kappa \to V_\lambda$ such that, for all $\alpha < \kappa$, if $V_\lambda \models \exists y \phi(y, f_1(\alpha), \ldots, f_n(\alpha))$, then $V_\lambda \models \phi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha))$. By elementarity, there is such a g in X. Then $g(\gamma) \in Y$, and

$$V_\lambda \models \phi(g(\gamma), f_1(\gamma), \ldots, f_n(\gamma)).$$

□

1.20 Remark. Note that in the theorem above, the transitive collapse of Y is the ultrapower of the transitive collapse of X by the image of U under the transitive collapse of X.

1.21 Remark. By the previous theorem, applied repeatedly, if κ is a measurable cardinal, and $\lambda > \kappa$ is a regular cardinal, then for every countable $X \prec V_\lambda$ there is a $Y \prec V_\lambda$ such that $X \cap \omega_1 = Y \cap \omega_1$ and $Y \cap \kappa$ has ordertype ω_1. Taking the transitive collapse of Y, we get another contradiction to $V = L$, since for each $\alpha < \omega_1^L$ there is a $\beta < \omega_1^L$ such that α is countable in L_β.

2 Precipitous ideals

2.1 Definition. Given a set X, a set $I \subseteq \mathcal{P}(X)$ is an *ideal* on X if I is closed under supersets and unions (i.e., if $\{X \setminus A : A \in I\}$ is a filter. We call $\{X \setminus A : A \in I\}$ the filter *dual* to I. The ideal I is said to be *κ-complete* (for some cardinal κ) or *normal* if and only if its dual filter is.

2.2 Exercise. If γ is an ordinal of cofinality κ, every normal ideal on γ is κ-complete.

2.3 Definition. Given a model M of a sufficient fragment of ZFC, and a set $X \in M$, an *M-ultrafilter* is a filter U on X such that

- $U \subseteq \mathcal{P}(X) \cap M$;
- for all $A \in \mathcal{P}(X) \cap M$, $|U \cap \{A, X \setminus A\}| = 1$.

If X is an ordinal in M, U is said to be M-*normal* if for all $A \in U$ and all regressive $f\colon A \to X$, f is constant on a set in U.

2.4 Remark. If U is an M-ultrafilter, we can form $\mathrm{Ult}(M, U)$ as before, and get an elementary embedding from M into $\mathrm{Ult}(M, U)$, as in Theorem 1.5. If U is an normal M-ultrafilter on an ordinal κ of M, then the critical point of the corresponding embedding is κ. The corresponding versions of Lemma 1.8 and Exercise 1.16 also go through in this context.

Suppose that I is an ideal on a set X, and consider the Boolean algebra $\mathcal{P}(X)/I$, consisting of the powerset of X modulo the equivalence relation $A \sim B \Leftrightarrow A \triangle B \in I$. Removing the class corresponding to I, we consider this a forcing notion, under the order $[A] \leq [B] \Leftrightarrow A \setminus B \in I$. Often for the sake of convenience we identify this with the non-separative partial order $(\mathcal{P}(X) \setminus I, \subseteq)$, since as forcing notions they are equivalent. If I contains a cofinite set, then this partial order is trivial as a forcing construction, so in general we will assume otherwise.

2.5 Exercise. If I is an ideal on a set X, and I contains no cofinite sets, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter, then $\{A \mid [A] \in G\}$ is a V-ultrafilter on X which is disjoint from I. If X is an ordinal and I is normal, then the generic ultrafilter added by $\mathcal{P}(X)/I$ is also normal.

2.6 Definition. We say that an ideal I on a set X is *precipitous* if whenever U is a V-ultrafilter added by forcing with $\mathcal{P}(X)/I$, $\mathrm{Ult}(V, U)$ is wellfounded.

Usually we identify U and G, and write $\mathrm{Ult}(V, G)$.

2.7 Remark. Suppose that I is an ideal on a set X, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter for which $\mathrm{Ult}(V, G)$ is wellfounded. Then some condition $A \in G$ forces the ultrapower to be wellfounded, which implies that the ideal generated by $I \cup \{X \setminus A\}$ is precipitous.

We will not prove (or use) the following theorem, as its proof would take us too far afield.

Theorem 2.8 ([2])**.** *If there is a κ-complete precipitous ideal on an ordinal κ, then there is an inner model in which κ is a measurable cardinal.*

Theorem 2.9 ([2])**.** *If there is a measurable cardinal, then there is a forcing extension in which there is a normal precipitous ideal on ω_1.*

Before we begin the proof of Theorem 2.9, we introduce some terminology.

2.10 Definition. Given a set X, the partial order $\mathrm{Col}(\omega, {<}X)$ consists of finite partial functions p from $(X \setminus \{0\}) \times \omega$ to X, with the requirement that $p(a, n) \in a$ for all (a, n) in the domain of p.

Forcing with $\mathrm{Col}(\omega, {<}X)$ explicitly adds a surjection from ω onto each element of X.

2.11 Exercise. Show that if γ is a regular cardinal, then $\mathrm{Col}(\omega, {<}\gamma)$ does not collapse γ, so $\gamma = \omega_1$ after forcing with $\mathrm{Col}(\omega, {<}\gamma)$.

2.12 Remark. While it is not strictly necessary for what follows, note that if X and Y are disjoint sets, then $\mathrm{Col}(\omega, {<}(X \cup Y))$ is isomorphic to $\mathrm{Col}(\omega, {<}X) \times \mathrm{Col}(\omega, {<}Y)$, so a generic filter for $\mathrm{Col}(\omega, {<}X)$ can be extended to one for $\mathrm{Col}(\omega, {<}(X \cup Y))$.

Proof of Theorem 2.9. Let κ be a measurable cardinal, and let U be a normal ultrafilter on κ. Let $j \colon V \to M$ be the elementary embedding induced by U. Let $H \subseteq \mathrm{Col}(\omega, {<}j(\kappa))$ be a V-generic filter, and let $G = H \cap \mathrm{Col}(\omega, {<}\kappa)$. Then G is V-generic for $\mathrm{Col}(\omega, {<}\kappa)$, H is M-generic for $\mathrm{Col}(\omega, {<}j(\kappa))$, and $H \setminus G$ is $V[G]$-generic for $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$.

The embedding j lifts to an elementary embedding $j^* \colon V[G] \to M[H]$, defined by setting $j^*(\tau_G) = j(\tau)_H$ for each $\mathrm{Col}(\omega, {<}\kappa)$-name τ. To see that this works, note that j is the identity function on $\mathrm{Col}(\omega, {<}\kappa)$, so if $p \in G$, $\tau_1, \ldots, \tau_n$ are $\mathrm{Col}(\omega, {<}\kappa)$-names and ϕ is a formula such that $p \Vdash^V_{\mathrm{Col}(\omega, <\kappa)} \phi(\tau_1, \ldots, \tau_n)$, then $j(p) = p$ is in H and $j(p) \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \phi(j(\tau_1), \ldots, j(\tau_n))$.

Let I be the set of $A \in \mathcal{P}(\kappa)^{V[G]}$ such that A is disjoint from some set in U. Then I is an ideal on ω_1 in $V[G]$. We claim that for each $A \in \mathcal{P}(\kappa)^{V[G]}$, $A \notin I$ if and only if there is an $s \in \mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega, <[\kappa, j(\kappa)))} \check{\kappa} \in j^*(\check{A}).$$

To see this, suppose that τ is a $\mathrm{Col}(\omega, {<}\kappa)$-name in V such that $A = \tau_G$. For the forward direction, since $A \notin I$, for each $p \in G$ the set

$$\{\alpha < \kappa \mid \exists q \leq p \ \ q \Vdash \check{\alpha} \in \tau\}$$

is in U. It follows then that for each $p \in G$, there is a $q \in \mathrm{Col}(\omega, {<}j(\kappa))$ extending p such that

$$q \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \check{\kappa} \in j(\tau).$$

By genericity, then, there is a $q \in \mathrm{Col}(\omega, {<}j(\kappa))$ such that $q \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \check{\kappa} \in j(\tau)$ and $q \cap \mathrm{Col}(\omega, {<}\kappa) \in G$. Then $q \restriction ([\kappa, j(\kappa)) \times \omega)$ is the desired condition s.

To see the reverse direction, since $A \in I$ there exist $B \in U$ and $p \in G$ such that $p \Vdash^V_{\mathrm{Col}(\omega, <\kappa)} \tau \cap \check{B} = \emptyset$. Since every condition in $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ forces that $\check{\kappa} \in j^*(B)$, no condition in $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ can force that $\check{\kappa} \in j^*(A)$.

Let us see that I is normal. Fix a set $A \in \mathcal{P}(\kappa)^{V[G]} \setminus I$, a regressive function $f \colon A \to \kappa$ in $V[G]$ and an $s \in \mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega, <[\kappa, j(\kappa)))} \check{\kappa} \in j^*(\check{A}).$$

We may strengthen s to a condition s' deciding $j^*(f)(\kappa)$ to be some fixed ordinal α. It follows then that

$$s' \Vdash^{V[G]}_{\mathrm{Col}(\omega, <[\kappa, j(\kappa)))} \check{\kappa} \in j^*(f^{-1}[\{\alpha\}]),$$

so $f^{-1}[\{\alpha\}] \notin I$.

Let $D = \{A \in \mathcal{P}(\kappa)^{V[G]} \mid \kappa \in j^*(A)\}$. We wish to see that $D \cap I = \emptyset$, that D is $V[G]$-generic for $\mathcal{P}(\kappa)/I$, and that $\mathrm{Ult}(V[G], D)$ is wellfounded. The first of these follows from the reverse direction of the claim above. To see that $\mathrm{Ult}(V[G], D)$ is wellfounded, note that it embeds into $M[H]$ via the function sending $[f]_D$ to $j^*(f)(\kappa)$.

Finally, we check that D is $V[G]$-generic for $\mathcal{P}(\kappa)/I$. Suppose that E is a subset of $\mathcal{P}(\kappa)/I$ in $V[G]$, and that $E \cap D = \emptyset$. It suffices to show that E is not dense in $\mathcal{P}(\kappa)/I$. Let r be a condition in $\mathrm{Col}(\omega, <[\kappa, j(\kappa))) \cap H$ such that

$$r \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \forall A \in \check{E}\ \check{\kappa} \notin j^*(A).$$

Let $f \colon \kappa \to \mathrm{Col}(\omega, <\kappa)$ be a function in V such that $[f]_U = r$, and let

$$B = \{\alpha \in \kappa \mid f(\alpha) \in G\}.$$

Then $B \in V[G]$ and $\kappa \in j^*(B)$, so $B \in D$ and $B \notin I$.

It suffices now to see that $B \cap A \in I$ for all $A \in E$. Fix such an A. If $B \cap A \notin I$, then there is an $s \in \mathrm{Col}(\omega, <[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*((A \cap B)\check{}).$$

However, $s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*(\check{B})$ implies that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} j^*(\check{f})(\check{\kappa}) \in G^*.$$

Since $j^*(f)(\kappa) = j(f)(\kappa) = r$, this implies that $s \leq r$, which means that $s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \notin j^*(\check{A})$, giving a contradiction. $\square$

2.13 Definition. Let A be a subset of an ordinal γ. The set A is *closed* if, for all $\alpha < \gamma$, $\sup(A \cap \alpha) \in A$. The set A is *cofinal* if, for all $\alpha < \gamma$, $A \setminus \alpha \neq \emptyset$. We say that A is *club* if it is both closed and cofinal. The set A is *stationary* if it intersects every club subset of γ, and *nonstationary* otherwise.

2.14 Exercise. For any ordinal γ of uncountable cofinality κ, the set of club subsets of γ is a κ-complete filter (so the set of nonstationary subsets of γ is a κ-complete ideal).

2.15 Definition. The collection of club subsets of an ordinal γ is called the *club filter* on γ, and the set of nonstationary sets is called the *nonstationary ideal.*

2.16 Remark. It is proved in [2] that the normal precipitous ideal on ω_1 can be the nonstationary ideal, but again we will skip the proof.

2.17 Remark. Exercise 1.17 implies that if there is a κ-complete ultrafilter on a cardinal κ then there is a normal ultrafilter on κ. Gitik [1] has shown that the existence of a precipitous ideal on ω_1 does not imply the existence of a normal precipitous ideal on ω_1.

References

[1] M. Gitik, *A model with a precipitous ideal but without normal one*, http://www.math.tau.ac.il/ gitik/nnpi4.pdf

[2] T. Jech, M. Magidor, W. Mitchell, K. Prikry, *Precipitous ideals*, J. Symb. Logic 45 (1980) 1, 1-8