1 Measurable cardinals

1.1 Definition. A filter on a set $X$ is a set $F \subseteq \mathcal{P}(X)$ which is closed under intersections and supersets, i.e., such that

- for all $A, B$ in $F$, $A \cap B \in F$;
- for all $B \subseteq \kappa$ and all $A \in F \cap \mathcal{P}(B)$, $B \in F$.

Given a cardinal $\kappa$, a filter $F$ is $\kappa$-complete if $\bigcap A \in F$ whenever $A$ is a subset of $F$ of cardinality less than $\kappa$.

1.2 Definition. An ultrafilter on a set $X$ is a filter on $X$ such that for all $A \subseteq X$, exactly one of $A$ and $X \setminus A$ is in $U$. The ultrafilter $U$ is nonprincipal if no singleton is a member of $U$.

1.3 Definition. A cardinal $\kappa$ is said to be measurable if there exists a $\kappa$-complete nonprincipal ultrafilter on $\kappa$.

1.4 Exercise. Show that if $\kappa$ is measurable, then $\kappa$ is a regular strong limit cardinal (i.e., $\kappa$ is strongly inaccessible).

Suppose that $U$ is an ultrafilter on a set $X$. Consider the class of functions with domain $X$, and the equivalence relation $\sim$ on this class defined by setting $f \sim g$ if and only if $\{x \in X \mid f(x) = g(x)\} \in U$. For each function $f$ with domain $X$, let $[f]_U$ denote the $\sim$-equivalence class of $f$. Define the relation $E$ on equivalence classes by setting $[f]_U E [g]_U$ if and only if

$$\{x \in X \mid f(x) \in g(x)\} \in U.$$  

Note that this relation does not depend on the representatives chosen from $[f]_U$ and $[g]_U$. Then $\text{Ult}(V, U)$, the ultrapower of $V$ by $U$, consists of the class of all sets of the form $[f]_U$, with the binary relation $E$.

**Theorem 1.5.** Let $U$ be an ultrafilter on a set $X$. For all function $f_1, \ldots, f_n$ with domain $X$, and all $n$-ary formulas $\phi$,

$$(\text{Ult}(V, U), E) \models \phi([f_1]_U, \ldots, [f_n]_U)$$

if and only if

$$\{x \in X \mid \phi(f_1(x), \ldots, f_n(x))\} \in U.$$

**Proof.** By induction on the complexity of formulas. For $=$ and $\in$ this is true by definition. For $\land$ it follows from the fact that $U$ is a filter, and for $\neg$ it follows from the fact that $U$ is an ultrafilter. For $\exists$, note that by Replacement and the Axiom of Choice, if

$$\{x \in X \mid \exists a \phi(a, f_1(x), \ldots, f_n(x))\} \in U$$

then there is a function $g$ with domain $X$ such that

$$\{x \in X \mid \phi(g(x), f_1(x), \ldots, f_n(x))\} \in U.$$

This gives the reverse implication for this step; the other direction is easier. \(\Box\)
1.6 Remark. Eventually, we will want to come back to this proof and think about the fragment of ZFC needed to carry it out.

Corollary 1.7. Let \( U \) be an ultrafilter on a set \( X \), and let \( j : V \to \text{Ult}(V, U) \) be the function which sends each set \( x \) to \([i_x]_U\), where \( i \) is the constant function from \( X \) to \( \{x\} \). Then \( j \) is an elementary embedding.

Lemma 1.8. Suppose that \( U \) is an ultrafilter on a set \( X \), \( j : V \to \text{Ult}(V, U) \) is the corresponding embedding, \( i \) is the identity function on \( X \) and \( f \) is a function with domain \( X \). Then 
\[
[f]_U = j(f)([i]_U),
\]
Proof. It suffices to see that for all functions \( f, g \) with domain \( X \), and any relation \( R \) in \( \{\in, =\} \), \([f]_UR[g]_U \) if and only if \( j(f)([i]_U)Rj(g)([i]_U) \). Since \( j(f) \) and \( j(g) \) are represented by the constant functions from \( X \) to \( \{f\} \) and \( \{g\} \) respectively, both expressions reduce to \( \{x \in X \mid f(x)Rg(x)\} \in U \).

1.9 Remark. By convention, we identify each \([f]_U\) in \( \text{Ult}(V, U) \) for which \((([f]_U, E) \) is wellfounded with its Mostowski collapse, i.e., the unique set \( a \) such that \((([f]_U, E) \) is isomorphic to \((a, \in)\).

1.10 Exercise. Given an ultrafilter \( U \) on a set \( X \), \( \text{Ult}(V, U) \) is wellfounded if and only if \( U \) is countably complete (i.e., closed under countable intersections).

1.11 Definition. If \( j : V \to M \) is an elementary embedding, the critical point of \( j \) is the least ordinal \( \alpha \) such that \( j(\alpha) > \alpha \), if one exists.

1.12 Exercise. Suppose that \( U \) is a \( \kappa \)-complete ultrafilter on a regular cardinal \( \kappa \). Show that the critical point of \( j \) is \( \kappa \).

1.13 Exercise (Scott). Show that there are no measurable cardinals in \( L \). (Hint: Let \( \kappa \) be the least measurable cardinal in \( L \), and consider the elementary embedding \( j : L \to M \) given by a \( \kappa \)-complete ultrafilter \( U \) on \( \kappa \) (in \( L \)).)

1.14 Definition. A filter \( F \) on a cardinal \( \kappa \) is said to be normal if for each \( A \in F \) and each regressive function \( f : A \to \kappa \) (i.e., \( f(\alpha) < \alpha \) for all \( \alpha \in A \setminus \{0\} \)) there is an \( \alpha < \kappa \) such that \( f^{-1}[\alpha] \in F \).

1.15 Exercise. Show that if \( U \) is a normal ultrafilter on \( \kappa \), \( U \) is \( \kappa \)-complete.

1.16 Exercise. Show that if \( U \) is a normal ultrafilter on \( \kappa \) and \( i : \kappa \to \kappa \) is the identity function on \( \kappa \), then \([i]_U = \kappa \) (under our identification of elements of \( \text{Ult}(V, U) \) with their Mostowski collapses).

1.17 Exercise. Suppose that \( j : V \to M \) is an elementary embedding with critical point \( \kappa \). Show that \( \{A \subseteq \kappa : k \in j(A)\} \) is a normal ultrafilter on \( \kappa \).

The previous exercise shows that “\( \kappa \) is a measurable cardinal” is equivalent to “there exists an elementary embedding \( j : V \to M \) with critical point \( \kappa \)” and also equivalent to “there exists a normal ultrafilter on \( \kappa \).”
Suppose that \( \exists j: V \to M \) is the corresponding elementary embedding. Show that \( \mathcal{P}(\kappa) \subseteq M \), but \( U \not\subseteq M \). Show that \( M \) is closed under sequences of length \( \kappa \). Show that \( 2^\kappa < j(\kappa) < (2^\kappa)^+ \).

**Theorem 1.19.** Suppose that \( U \) is a normal ultrafilter on \( \kappa \), and that \( \lambda > \kappa \) is a regular cardinal. Let \( X \) be an elementary submodel of \( V_\lambda \) of cardinality less than \( \kappa \), with \( U \subseteq X \). Let \( \gamma \) be any element of \( \bigcap(X \cap U) \), and let \( Y = \{ f(\gamma) \mid f: \kappa \to V_\lambda, f \in X \} \). Then \( X \subseteq Y \), \( Y \prec V_\lambda \), and \( Y \cap \kappa = X \cap \kappa \).

**Proof.** That \( X \subseteq Y \) follows from the fact that there is a constant function in \( X \) for each element of \( X \). The fact that \( Y \cap \kappa = X \cap \kappa \) follows from normality, as follows. If \( f: \kappa \to V_\lambda \) is in \( X \), and \( f(\gamma) < \gamma \), then \( f \) is regressive on a set in \( U \), so there is an \( \alpha < \gamma \) such that \( f^{-1}[\{\alpha\}] \in U \). Then \( \alpha \in X \) and \( f(\gamma) = \alpha \).

For elementarity, by the Vaught-Tarski test we need to see only that if \( a_1, \ldots, a_n \) are in \( Y \) and \( V_\lambda \models \exists x \phi(x, a_1, \ldots, a_n) \), then there is a \( b \in Y \) such that \( V_\lambda \models \phi(b, a_1, \ldots, a_n) \). To see that this holds, fix functions \( f_1, \ldots, f_n \) in \( X \) such that each \( a_i = f_i(\gamma) \). There is in \( V_\lambda \) a function \( g: \kappa \to V_\lambda \) such that, for all \( \alpha < \kappa \), if \( V_\lambda \models \exists y \phi(y, f_1(\alpha), \ldots, f_n(\alpha)) \), then \( V_\lambda \models \phi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha)) \).

By elementarity, there is such a \( g \) in \( X \). Then \( g(\gamma) \in Y \), and

\[
V_\lambda \models \phi(g(\gamma), f_1(\gamma), \ldots, f_n(\gamma)).
\]

\[\Box\]

**1.20 Remark.** Note that in the theorem above, the transitive collapse of \( Y \) is the ultrapower of the transitive collapse of \( X \) by the image of \( U \) under the transitive collapse of \( X \).

**1.21 Remark.** By the previous theorem, applied repeatedly, if \( \kappa \) is a measurable cardinal, and \( \lambda > \kappa \) is a regular cardinal, then for every countable \( X \prec V_\lambda \) there is a \( Y \prec V_\lambda \) such that \( X \cap \omega_1 = Y \cap \omega_1 \) and \( Y \cap \kappa \) has ordertype \( \omega_1 \). Taking the transitive collapse of \( Y \), we get another contradiction to \( V = L \), since for each \( \alpha < \omega_1^+ \) there is a \( \beta < \omega_1^+ \) such that \( \alpha \) is countable in \( L_\beta \).

## 2 Precipitous ideals

**2.1 Definition.** Given a set \( X \), a set \( I \subseteq \mathcal{P}(X) \) is an ideal on \( X \) if \( I \) is closed under supersets and unions (i.e., if \( \{X \setminus A : A \in I \} \) is a filter. We call \( \{X \setminus A : A \in I \} \) the filter dual to \( I \). The ideal \( I \) is said to be \( \kappa \)-complete (for some cardinal \( \kappa \)) or normal if and only if its dual filter is.

**2.2 Exercise.** If \( \gamma \) is an ordinal of cofinality \( \kappa \), every normal ideal on \( \gamma \) is \( \kappa \)-complete.

**2.3 Definition.** Given a model \( M \) of a sufficient fragment of ZFC, and a set \( X \in M \), an \( M \)-ultrafilter is a filter \( U \) on \( X \) such that
• $U \subseteq \mathcal{P}(X) \cap M$;
• for all $A \in \mathcal{P}(X) \cap M$, $|U \cap \{A, X \setminus A\}| = 1$.

If $X$ is an ordinal in $M$, $U$ is said to be $M$-normal if for all $A \in U$ and all regressive $f: A \rightarrow X$, $f$ is constant on a set in $U$.

2.4 Remark. If $U$ is an $M$-ultrafilter, we can form $\text{Ult}(M, U)$ as before, and get an elementary embedding from $M$ into $\text{Ult}(M, U)$, as in Theorem 1.5. If $U$ is an normal $M$-ultrafilter on an ordinal $\kappa$ of $M$, then the critical point of the corresponding embedding is $\kappa$. The corresponding versions of Lemma 1.8 and Exercise 1.16 also go through in this context.

2.4 Exercise. If $I$ is an ideal on a set $X$, and consider the Boolean algebra $\mathcal{P}(X)/I$, consisting of the powerset of $X$ modulo the equivalence relation $A \sim B \iff A \triangle B \in I$. Removing the class corresponding to $I$, we consider this a forcing notion, under the order $[A] \leq [B] \iff A \setminus B \in I$. Often for the sake of convenience we identify this with the non-separative partial order $(\mathcal{P}(X) \setminus I, \subseteq)$, since as forcing notions they are equivalent. If $I$ contains a cofinite set, then this partial order is trivial as a forcing construction, so in general we will assume otherwise.

2.5 Exercise. If $I$ is an ideal on a set $X$, and $I$ contains no cofinite sets, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter, then $\{A \mid [A] \in G\}$ is a $V$-ultrafilter on $X$ which is disjoint from $I$. If $X$ is an ordinal and $I$ is normal, then the generic ultrafilter added by $\mathcal{P}(X)/I$ is also normal.

2.6 Definition. We say that an ideal $I$ on a set $X$ is precipitous if whenever $U$ is a $V$-ultrafilter added by forcing with $\mathcal{P}(X)/I$, $\text{Ult}(V, U)$ is wellfounded.

Usually we identify $U$ and $G$, and write $\text{Ult}(V, G)$.

2.7 Remark. Suppose that $I$ is an ideal on a set $X$, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter for which $\text{Ult}(V, G)$ is wellfounded. Then some condition $A \in G$ forces the ultrapower to be wellfounded, which implies that the ideal generated by $I \cup \{X \setminus A\}$ is precipitous.

We will not prove (or use) the following theorem, as its proof would take us too far afield.

Theorem 2.8 ([2]). If there is a $\kappa$-complete precipitous ideal on an ordinal $\kappa$, then there is an inner model in which $\kappa$ is a measurable cardinal.

Theorem 2.9 ([2]). If there is a measurable cardinal, then there is a forcing extension in which there is a normal precipitous ideal on $\omega_1$.

Before we begin the proof of Theorem 2.9, we introduce some terminology.

2.10 Definition. Given a set $X$, the partial order $\text{Col}(\omega, <X)$ consists of finite partial functions $p$ from $(X \setminus \{0\}) \times \omega$ to $X$, with the requirement that $p(a, n) \in a$ for all $(a, n)$ in the domain of $p$. 
Forcing with $\text{Col}(\omega, <X)$ explicitly adds a surjection from $\omega$ onto each element of $X$.

**2.11 Exercise.** Show that if $\gamma$ is a regular cardinal, then $\text{Col}(\omega, <\gamma)$ does not collapse $\gamma$, so $\gamma = \omega_1$ after forcing with $\text{Col}(\omega, <\gamma)$.

**2.12 Remark.** While it is not strictly necessary for what follows, note that if $X$ and $Y$ are disjoint sets, then $\text{Col}(\omega, <(X \cup Y))$ is isomorphic to $\text{Col}(\omega, <X) \times \text{Col}(\omega, <Y)$, so a generic filter for $\text{Col}(\omega, <X)$ can be extended to one for $\text{Col}(\omega, <(X \cup Y))$.

**Proof of Theorem 2.9.** Let $\kappa$ be a measurable cardinal, and let $U$ be a normal ultrafilter on $\kappa$. Let $j: V \to M$ be the elementary embedding induced by $U$. Let $H \subseteq \text{Col}(\omega, <j(\kappa))$ be a $V$-generic filter, and let $G = H \cap \text{Col}(\omega, <\kappa)$. Then $G$ is $V$-generic for $\text{Col}(\omega, <\kappa)$, $H$ is $M$-generic for $\text{Col}(\omega, <j(\kappa))$, and $H \setminus G$ is $V[G]$-generic for $\text{Col}(\omega, <[\kappa, j(\kappa)])$.

The embedding $j$ lifts to an elementary embedding $j^*: V[G] \to M[H]$, defined by setting $j^*(\tau_G) = j(\tau)_H$ for each $\text{Col}(\omega, <\kappa)$-name $\tau$. To see that this works, note that $j$ is the identity function on $\text{Col}(\omega, <\kappa)$, so if $p \in G$, $\tau_1, \ldots, \tau_n$ are $\text{Col}(\omega, <\kappa)$-names and $\phi$ is a formula such that $p^V_{\text{Col}(\omega, <\kappa)} \phi(\tau_1, \ldots, \tau_n)$, then $j(p) = p$ is in $H$ and $j(p)^H_{\text{Col}(\omega, <j(\kappa))} \phi(j(\tau_1), \ldots, j(\tau_n))$.

Let $I$ be the set of $A \in \mathcal{P}(\kappa)^{V[G]}$ such that $A$ is disjoint from some set in $U$. Then $I$ is an ideal on $\omega_1$ in $V[G]$. We claim that for each $A \in \mathcal{P}(\kappa)^{V[G]}$, $A \not\in I$ if and only if there is an $s \in \text{Col}(\omega, <[\kappa, j(\kappa)])$ such that

$$s^V_{\text{Col}(\omega, <[\kappa, j(\kappa)])} \in j^*(A).$$

To see this, suppose that $\tau$ is a $\text{Col}(\omega, <\kappa)$-name in $V$ such that $A = \tau_G$. For the forward direction, since $A \not\in I$, for each $p \in G$ the set

$$\{ \alpha < \kappa \mid \exists q \leq p \ q^V_{\alpha} \in \tau \}$$

is in $U$. It follows then that for each $p \in G$, there is a $q \in \text{Col}(\omega, <j(\kappa))$ extending $p$ such that

$$q^V_{\text{Col}(\omega, <j(\kappa))} \in j(\tau).$$

By genericity, then, there is a $q \in \text{Col}(\omega, <j(\kappa))$ such that $q^V_{\text{Col}(\omega, <j(\kappa))} \in j(\tau)$ and $q \cap \text{Col}(\omega, <\kappa) \subseteq G$. Then $q^V_{\text{Col}(\omega, <[\kappa, j(\kappa)])}$ is the desired condition $s$.

To see the reverse direction, since $A \in I$ there exist $B \subseteq U$ and $p \in G$ such that $p^V_{\text{Col}(\omega, <\kappa)} \tau \cap \dot{B} = \emptyset$. Since every condition in $\text{Col}(\omega, <[\kappa, j(\kappa)])$ forces that $\kappa \in j^*(\dot{B})$, no condition in $\text{Col}(\omega, <[\kappa, j(\kappa)])$ can force that $\kappa \in j^*(A)$.

Let us see that $I$ is normal. Fix a set $A \in \mathcal{P}(\kappa)^{V[G]} \setminus I$, a regressive function $f: A \to \kappa$ in $V[G]$ and an $s \in \text{Col}(\omega, <[\kappa, j(\kappa)])$ such that

$$s^V_{\text{Col}(\omega, <[\kappa, j(\kappa)])} \in j^*(A).$$

We may strengthen $s$ to a condition $s'$ deciding $j^*(f)(\kappa)$ to be some fixed ordinal $\alpha$. It follows then that

$$s'^V_{\text{Col}(\omega, <[\kappa, j(\kappa)])} \in j^*(f^{-1}([\alpha])),$$
so \( f^{-1}([\alpha]) \notin I \).

Let \( D = \{ A \in \mathcal{P}(\kappa)^{V[G]} \mid \kappa \in j^*(A) \} \). We wish to see that \( D \cap I = \emptyset \), that \( D \) is \( V[G] \)-generic for \( \mathcal{P}(\kappa)/I \), and that \( \text{Ult}(V[G], D) \) is wellfounded. The first of these follows from the reverse direction of the claim above. To see that \( \text{Ult}(V[G], D) \) is wellfounded, note that it embeds into \( M[H] \) via the function sending \( [f]_D \) to \( j^*(f)(\kappa) \).

Finally, we check that \( D \) is \( V[G] \)-generic for \( \mathcal{P}(\kappa)/I \). Suppose that \( E \) is a subset of \( \mathcal{P}(\kappa)/I \) in \( V[G] \), and that \( E \cap D = \emptyset \). It suffices to show that \( E \) is not dense in \( \mathcal{P}(\kappa)/I \). Let \( r \) be a condition in \( \text{Col}(\omega, <\kappa, j(\kappa)) \cap H \) such that

\[
\forall A \in E \kappa \notin j^*(A).
\]

Let \( f : \kappa \to \text{Col}(\omega, <\kappa) \) be a function in \( V \) such that \( [f]_V = r \), and let

\[
B = \{ \alpha \in \kappa \mid f(\alpha) \in G \}.
\]

Then \( B \in V[G] \) and \( \kappa \in j^*(B) \), so \( B \in D \) and \( B \notin I \).

It suffices now to see that \( B \cap A \in I \) for all \( A \in E \). Fix such an \( A \). If \( B \cap A \notin I \), then there is an \( s \in \text{Col}(\omega, <[\kappa, j(\kappa)]) \) such that

\[
sV[G]\text{Col}(\omega, <[\kappa, j(\kappa)])\kappa \in j^*(A \cap B).
\]

However, \( sV[G]\text{Col}(\omega, <[\kappa, j(\kappa)])\kappa \notin j^*(B) \) implies that

\[
sV[G]\text{Col}(\omega, <[\kappa, j(\kappa)])j^*(\hat{f})(\hat{\kappa}) \notin G^*.
\]

Since \( j^*(f)(\kappa) = j(f)(\kappa) = r \), this implies that \( s \leq r \), which means that \( sV[G]\text{Col}(\omega, <[\kappa, j(\kappa)])\kappa \notin j^*(A) \), giving a contradiction. \( \square \)

2.13 Definition. Let \( A \) be a subset of an ordinal \( \gamma \). The set \( A \) is closed if, for all \( \alpha < \gamma \), \( \sup(A \cap \alpha) \in A \). The set \( A \) is cofinal if, for all \( \alpha < \gamma \), \( A \setminus \alpha \neq \emptyset \). We say that \( A \) is club if it is both closed and cofinal. The set \( A \) is stationary if it intersects every club subset of \( \gamma \), and nonstationary otherwise.

2.14 Exercise. For any ordinal \( \gamma \) of uncountable cofinality \( \kappa \), the set of club subsets of \( \gamma \) is a \( \kappa \)-complete filter (so the set of nonstationary subsets of \( \gamma \) is a \( \kappa \)-complete ideal).

2.15 Definition. The collection of club subsets of an ordinal \( \gamma \) is called the club filter on \( \gamma \), and the set of nonstationary sets is called the nonstationary ideal.

2.16 Remark. It is proved in [2] that the normal precipitous ideal on \( \omega_1 \) can be the nonstationary ideal, but again we will skip the proof.

2.17 Remark. Exercise 1.17 implies that if there is a \( \kappa \)-complete ultrafilter on a cardinal \( \kappa \) then there is a normal ultrafilter on \( \kappa \). Gitik [1] has shown that the existence of a precipitous ideal on \( \omega_1 \) does not imply the existence of a normal precipitous ideal on \( \omega_1 \).
References
