Well-behaved measures and weak covering for derived models

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Define a closure operation on $\wp(\mathbb{R})$:

Definition (Martin)

Let S be a collection of sets of reals. Then $\overline{S}$ is the set of $A \subset \mathbb{R}$ such that for every countable $\mathcal{I} \subset \mathbb{R}$ there is a set $A' \in S$ with $A \cap \mathcal{I} = A' \cap \mathcal{I}$.

We say A is "countably approximated by sets in S."

- $S \subseteq \overline{S}$
- $\overline{S} = \overline{\overline{S}}$
- $S_1 \subset S_2 \implies \overline{S_1} \subset \overline{S_2}$

Main example of a "bar-closed" pointclass:

Fact (AD)

$\overline{\mathrm{OD}} = \mathrm{OD}$, *where* OD *is the collection of ordinal-definable sets of reals.*

Meaning: "countably approximated by OD sets" implies OD.

Proof.

Well-order $\overline{\mathrm{OD}}$ by $A < B$ if $A_d <_{\mathrm{OD}} B_d$ for a cone of Turing degrees d where

- $\mathcal{I}_d = \{x : x \leq_{\mathrm{T}} d\}$
- $A_d \in \mathrm{OD}$ is $<_{\mathrm{OD}}$-least with $A \cap \mathcal{I}_d = A_d \cap \mathcal{I}_d$
- $B_d \in \mathrm{OD}$ is $<_{\mathrm{OD}}$-least with $B \cap \mathcal{I}_d = B_d \cap \mathcal{I}_d$ □

Other examples of "bar-closed" pointclasses are given by local ordinal-definability or relativized ordinal-definability.

Example (AD)

$\bar{S} = S$ if

- S consists of $A \subset \mathbb{R}$ of the form

$$\{x \in \mathbb{R} : L[x] \models \varphi[\alpha, x]\}$$

  for ordinals α, or
- S consists of $A \subset \mathbb{R}$ that are OD from a parameter π.

Assume AD and let $\kappa < \Theta$.
Take a surjection $\pi : \mathbb{R} \to \wp(\kappa)$ (by coding lemma.)

Definition (Code set of a measure)

For a countably complete measure μ on κ

$$\mathrm{Code}_\pi(\mu) = \{x \in \mathbb{R} : \pi(x) \in \mu\}.$$

Given countable $\mathcal{I} \subset \mathbb{R}$, approximate by principal measures:

$$(\exists \alpha < \kappa)\,(\forall x \in \mathcal{I})\,\big(\pi(x) \in \mu \iff \alpha \in \pi(x)\big)$$

So $\mathrm{Code}_\pi(\mu) \in \overline{\mathrm{OD}_\pi}$

Given surjection $\pi : \mathbb{R} \to \wp(\kappa)$ and c.c. measure μ on κ, recall $\mathrm{Code}_\pi(\mu)$ is in $\overline{\mathrm{OD}_\pi}$.

Fact (AD)

- $\overline{\mathrm{OD}_\pi} = \mathrm{OD}_\pi$.
- $\mathrm{Code}_\pi(\mu) \in \mathrm{OD}_\pi$.

So far we are just approaching Kunen's theorem in a roundabout way:

Theorem (Kunen, AD)

$\mu \in \mathrm{OD}$.

Now don't assume AD, but let M be a model of AD with $\mathbb{R} \subset M$. Following Kechris–Woodin, we prove:

Key Lemma (W.)

Every $A \in \overline{\mathrm{OD}^M}$ is determined.

Sketch of proof

Suppose $A \in \overline{\mathrm{OD}^M}$ is not determined.
Given $t \in \mathbb{R}$ take Skolem hull to get countable Turing ideal $\mathcal{I} \ni t$ such that the restricted game $G_A \restriction \mathcal{I}$ is not determined.
(Every strategy with a code in $\mathcal{I}$ loses to some play in $\mathcal{I}$.)
Let A_t be $<_{\mathrm{OD}^M}$-least A' such that $G_{A'} \restriction \mathcal{I}$ is not determined.
Working in M, piece together the restricted games $G_{A_t} \restriction \mathcal{I}$ into an undetermined game, contradicting AD in M.

Corollary

If M is a model of AD with $\mathbb{R} \subset M$, then

$$\overline{\mathrm{OD}^M} \subseteq \mathrm{OD}_M$$

(Superscript means OD *in*, and subscript means OD *from*.)

Proof.

Well-order $\overline{\mathrm{OD}^M}$ by $A < B$ if $A_d <_{\mathrm{OD}^M} B_d$ for a cone of Turing degrees d.
Let $A_d \in \mathrm{OD}^M$ be $<_{\mathrm{OD}^M}$-least with $A \cap \mathcal{I}_d = A_d \cap \mathcal{I}_d$ and similarly for B_d. $\square$

Let M be a model of AD with $\mathbb{R} \subset M$.
Let $\kappa < \Theta^M$ and take a surjection $\pi : \mathbb{R} \to \wp(\kappa)^M$ in M.
Let μ be a c.c. measure on the σ-algebra $\wp(\kappa)^M$.

Corollary (Definability of measures)

$\text{Code}_\pi(\mu) \in \text{OD}_{M,\pi}$

Kunen's proof generalizes to give:

Corollary (Definability of measures)

$\mu \in \text{OD}_M$

Corollary (Number of measures)

Let M be a model of AD with $\mathbb{R} \subset M$.
Let $\kappa < \Theta^M$, and take a surjection $\pi : \mathbb{R} \to \wp(\kappa)^M$ in M.

- Wadge determinacy: for countably complete measures μ, ν on $\wp(\kappa)^M$ there is a continuous function f such that $\text{Code}_\pi(\mu) = f^{-1}[\text{Code}_\pi(\nu)]$ or $\text{Code}_\pi(\nu) = \mathbb{R} \setminus f^{-1}[\text{Code}_\pi(\mu)]$.
- So there are at most $\mathfrak{c}^+$ many countably complete measures on $\wp(\kappa)^M$, even if $2^{\mathfrak{c}} > \mathfrak{c}^+$.

Let M be a model of AD with $\mathbb{R} \subset M$.
Let $\kappa < \Theta^M$, and take a surjection $\pi : \mathbb{R} \to \wp(\kappa)^M$ in M.

Corollary

It is consistent to have $\leq \mathfrak{c}$ many countably complete measures on $\wp(\kappa)^M$.

Proof.

We can collapse set of measures without adding new ones: measures are OD_M and $\mathrm{Col}(\mathfrak{c}, \mathfrak{c}^+)$ is homogeneous. □

It is also consistent to have $\mathfrak{c}^+$ many countably complete measures on $\wp(\kappa)^M$: say $M = L(\mathbb{R})$, $\kappa = (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$, and V is a generic extension of $L(\mathbb{R})$ by $\mathrm{Col}(\omega_1, \mathbb{R})$ or by $\mathbb{P}_{\max}$.

Definition

Let κ be an ordinal.

- $A \subset \mathbb{R}$ is κ-Suslin if $A = p[T]$ for some tree T on $\omega \times \kappa$.
    - $[T]$ is the set of branches of T
    - $p[T]$ is its projection $\{x \in \omega^\omega : \exists f \in \kappa^\omega \, (x, f) \in [T]\}$.
- $A \subset \mathbb{R}$ is Suslin if it is κ-Suslin for some κ.
- $A \subset \mathbb{R}$ is co-Suslin if its complement is Suslin.

AC implies that every set of reals is $\mathfrak{c}$-Suslin.

Example

The ω-Suslin sets are the analytic ($\undertilde{\boldsymbol{\Sigma}}^1_1$) sets.

Example

Under AD:

- The ω_1-Suslin sets are the $\undertilde{\boldsymbol{\Sigma}}^1_2$ sets
- In $L(\mathbb{R})$, the Suslin sets are the $\undertilde{\boldsymbol{\Sigma}}^2_1$ sets

The theory "AD + every set of reals is Suslin" has higher consistency strength than AD.

Definition

T a tree on $\omega \times \kappa$,
δ an uncountable cardinal.

- T is δ-weakly homogeneous if there is a countable set σ of δ-complete measures on $\kappa^{<\omega}$ such that for every $x \in p[T]$ there is a countably complete tower of measures $\{\mu_0, \mu_1, \ldots\} \subset \sigma$ concentrating on the tree

$$T_x = \{s \in \kappa^n : (x \restriction n, s) \in T\}$$

- T is $<\delta$-absolutely complemented if there is a tree $\tilde{T}$ such that $V^{\mathrm{Col}(\omega,\alpha)} \models p[T] = \omega^\omega \setminus p[\tilde{T}]$ for all $\alpha < \delta$

Theorem (Martin–Solovay)

If a tree T on $\omega \times \kappa$ is δ-weakly homogeneous then it is $<\delta$-absolutely complemented.

- In particular, if T is ω_1-weakly homogeneous then $p[T]$ is co-Suslin

Fact

If M is an inner model and $\mathbb{R} \cup \mathrm{Ord} \cup \{T\} \subset M$, then countably complete partial measures on $\wp(\kappa^{<\omega})^M$ are enough to show that $p[T]$ is co-Suslin.

To get weak homogeneity systems:

Theorem (Woodin)

If T is a tree on $\omega \times \delta$ and δ is 2^{2^δ}-supercompact, then T is δ-weakly homogeneous in $V^{\mathrm{Col}(\omega,\alpha)}$ for some $\alpha < \delta$.

- In particular, $p[T]$ is co-Suslin in the symmetric extension $V(\mathbb{R}^*)$ where $\mathbb{R}^* = \mathbb{R} \cap V^{\mathrm{Col}(\omega,<\delta)}$

If $j : V \to \mathrm{Ult}$ witnesses 2^{2^δ}-supercompactness of δ then

$$\sigma = \{j(\mu) : \mu \text{ is a } \delta\text{-complete measure on } \delta^{<\omega}\}$$

witnesses weak homogeneity of $j(T)$ in Ult when collapsed.

What about AD?

Definition

For a V-generic filter $G \subset \mathrm{Col}(\omega, {<}\delta)$:

- $\mathbb{R}^*_G$ is the set of reals in $V[G \restriction \alpha]$ for some $\alpha < \delta$
- Hom^*_G is the set of $p[T] \cap \mathbb{R}^*_G$ for ${<}\delta$-absolutely complementing trees T in $V[G \restriction \alpha]$ for some $\alpha < \delta$

Hom^*_G consists of all Suslin, co-Suslin sets of reals in $V(\mathbb{R}^*_G)$.

Theorem (Woodin)

If δ is a limit of Woodin cardinals then "the" derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ at δ satisfies AD^+.

What we need to know about AD^+:

Fact

The following theories are equivalent:

- $\mathrm{AD}^+ +$ *"every Suslin set of reals is co-Suslin"*
- $\mathrm{AD} +$ *"every set of reals is Suslin"*

Corollary

If δ is 2^{2^δ}-supercompact, then the derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ at δ satisfies $\mathrm{AD} +$ "every set of reals is Suslin."

Idea

Weaken hypothesis from "δ is 2^{2^δ}-supercompact" to "δ is weakly compact"

- To get AD, also assume δ is a limit of Woodins
- Build weak homogeneity systems with partial measures on $\wp(\kappa^{<\omega})^M$ where M is the derived model at δ
- These measures are "well-behaved"

Theorem (W.)

Suppose δ is a weakly compact limit of Woodin cardinals and $(\delta^+)^{\mathsf{HOD}} < \delta^+$.
Then the derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ at δ satisfies AD + "every set of reals is Suslin."

Remark

- Consistency strength of hypothesis with $(\delta^+)^{\mathsf{HOD}} = \delta^+$ is much weaker than "AD + every set of reals is Suslin"
- Consistency strength lower bound for the hypothesis follows from "Stacking mice" by a different method
- The hypothesis can be forced from a supercompact

Sketch of proof.

δ a weakly compact limit of Woodins with $(\delta^+)^{\mathrm{HOD}} < \delta^+$.
$M = L(\mathbb{R}^*, \mathrm{Hom}^*)$ is DM from generic $G \subset \mathrm{Col}(\omega, {<}\delta)$.
$M \models \mathrm{AD}^+$, so we show in M every Suslin set is co-Suslin:

- Take a tree $T \in M$ on $\omega \times \kappa$ where $\kappa < \Theta^M$
- $\delta = \mathfrak{c}^{V[G]}$ and $(\delta^+)^V = (\mathfrak{c}^+)^{V[G]}$
- Using ordinal definability of measures and $(\delta^+)^{\mathrm{HOD}} < \delta^+$, show that there are $\leq \delta$ many measures on $\wp(\kappa^{<\omega})^M$
- Take $j : N \to N'$ witnessing weak compactness
- Extend to $j^* : N[G] \to N'[H]$ where $H \subset \mathrm{Col}(\omega, {<}j(\delta))$
- $\{j^*(\mu) : \mu$ is a c.c. measure on $\wp(\kappa^{<\omega})^M\} \in N'[H]$
- Then use Woodin's argument □

An analogy with covering for L:

Theorem (Kunen)

If δ is weakly compact and $(\delta^+)^L < \delta^+$, then $0^\sharp$ exists.

Consider also:

Theorem (Jensen)

If δ is a singular cardinal and $(\delta^+)^L < \delta^+$, then $0^\sharp$ exists.

Question

If δ is a singular limit of Woodin cardinals, and $(\delta^+)^{\mathrm{HOD}} < \delta^+$, does the derived model satisfy "every set of reals is Suslin"?