NOTES ON BOREL DETERMINACY

SHEHZAD AHMED

Abstract. In this talk, we will provide a high level sketch of Martin's proof of Borel Determinacy.

1. Introduction

Instead of working with games just over the reals, we will actually be working with games over arbitrary game trees. We say T is a *Game Tree* if T is a collection of finite sequences closed under initial segments. We say that x is an infinite branch through T if for all $n \in \omega$, $x|_n \in T$. On the other hand, $p \in T$ is a terminal position if there is no $q \in T$ such that $p \subsetneq q$. Given a game tree T, we define $[T] = \{x : x \text{ is an infinite branch through } T\}$. We take the basic open sets of $[T]$ to be $N_s = \{t \in [T] : s \subset t\}$. For our purposes, it will be beneficial to restrict our attention to trees that have no terminal branches simply because a tree with terminal branches can be turned into a tree without terminal branches without affecting determinacy of games, but there are notational advantages.

Speaking of games, we are still working in the world of two player infinite games with perfect information. However, instead of both players cooperating to make a real, both players are cooperating to create an infinite branch through T. We pick a payoff set $A \in [T]$, and play as follows: We begin by having player I pick x_0 such that $\langle x_0 \rangle \in T$. Player II then responds by picking x_1 such that $\langle x_0, x_1 \rangle \in T$. After n turns, we have a sequence $\langle x_0, x_1, \dots, x_{n-1} \rangle \in T$ that players I and II have cooperated to create. Whichever player is next (I and II alternate) then picks x_n such that $\langle x_0, x_1, \dots, x_{n-1}, x_n \rangle \in T$, and at the end we have an infinite branch $\overline{x} = \langle x_0, x_1, \dots \rangle \in [T]$. I wins if $\overline{x} \in A$, and II wins otherwise. Simply put, we are generalizing the games we have played on ${}^{\omega}\omega$, and we denote these games by $G(A;T)$.

We have seen various determinacy results throughout the semester, all of which have required large cardinal hypotheses. So, it seems natural to see how much determinacy we can get in ZFC. It turns out that we can get quite a bit of determinacy out of the standard axioms, and in fact we see that most sets that the analyst would encounter on a regular basis are determined. We begin by defining the Borel Hierarchy: For ordinals $\alpha < \omega_1$

$$A \in \mathbf{\Sigma^0_1}([T]) (= \mathbf{\Sigma^0_0}([T])) \Leftrightarrow A \text{ is open in } [T]$$

$$A \in \mathbf{\Pi}^{\mathbf{0}}_\alpha([T]) \Leftrightarrow [T] \backslash A \in \mathbf{\Sigma}^{\mathbf{0}}_\alpha([T])$$

$$A \in \mathbf{\Sigma}^{\mathbf{0}}_\alpha([T]) \text{ for } \alpha > 1 \Leftrightarrow A = \bigcup_{n \in \omega} X_n \text{ where each } X_n \text{ is in some } \mathbf{\Pi}^{\mathbf{0}}_{\beta_\mathbf{n}}([T]) \text{ with } \beta_n < \alpha$$

We say that a set is *Borel* if it belongs to some $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ for $\alpha < \omega_1$. Equivalently, a set is Borel if it belongs to the smallest σ-algebra containing the open sets. While the latter is the way Borel sets are usually introduced, the former gives us much more utility when actually dealing with the Borel sets in a set theoretic context.

Now, in 1953, Gale and Stewart published a paper on the determinacy of closed games. One nice thing here is that the determinacy of closed games gives us the determinacy of open games via a dummy move that switches the roles of both players in a sense. At the end of the paper, they asked the natural question of whether or not one could prove full Borel determinacy. This lead to proofs of $\mathbf{\Sigma_2^0}$ determinacy and $\mathbf{\Sigma_3^0}$ determinacy by Wolfe and Davis respectively. Unfortunately, looking at the proofs of these statements, one can see that the combinatorics become rather difficult. In fact, in 1971 Friedman showed it is not possible to prove Borel Determinacy without the replacement axiom, at least for an initial segment of the universe. In other words, if one could prove Borel determinacy, one would have to make use of sets much larger than the reals. This is particularly interesting simply because even Σ_1 replacement will sufice to prove a lot of major results.

In 1975, Martin managed to prove full Borel determinacy, but the proof was a monster. The combinatorial difficulties predicted by previous results were especially apparent in Martin's proof when dealing with the transfinite levels of the Borel Hierarchy. Fortunately for us, this is not the proof we will be working through. In 1982, Martin published another proof of Borel determinacy that was purely inductive and introduced the elegant machinery of Covering Trees and Unravelings.

We are going through this particular proof for a number of reasons, one of which is the obvious insofar that the result itself is interesting. Another reason is simply because this proof is elegant and worth looking at from the standpoint of aesthetics. Finally, the machinery of covering trees and unravelings are of independent interest in the world of determinacy.

2. Preliminaries

Before proving Borel Determinacy we need some closure properties of Borel Sets, and the following theorem due to Gale and Stewart:

Theorem 1 (Gale-Stewart(1953))**.** *Closed games are determined.*

We will sketch the argument.

Proof. Let $A \subset [T]$ be closed. It is enough to show that if II has no winning strategy, then I has a winning strategy. So, suppose that II does not have a winning strategy for $G(A;T)$. Given that II does not have a winning strategy for this game, there should be some x_0 such that II does not have a winning strategy if the game were to start with $\langle x_0 \rangle$ being played. If not, then it would not matter what I's first move was, and II would have a winning strategy. So, we begin by having I play this x_0. Then, II responds by picking some x_1, and we are in a similar position. In other words, we can pick an x_2 so that II has no winning strategy if the game begins at $\langle x_0, x_1, x_2 \rangle$. Continuing on this manner, we are left with a play $\overline{x} = \langle x_0, x_1, x_2, \ldots \rangle$, and we claim that $\overline{x} \in A$.

To see this, note that there are sequences in A that begin with $x_0, x_1, \ldots, x_n$ in A for each $n \in \omega$. We then see that $\overline{x}$ is the limit of these sequences, and since A is closed, $\overline{x} \in A$. ☐

Lemma 1. *Each level of the Borel Hierarchy is closed under continuous preimages.*

Proof. We will prove this through induction on the Borel Hierarchy. We begin by noting that if this holds for a pointclass, it will hold for its dual as well. Clearly this is true of $\mathbf{\Sigma}_1^0$ sets, and therefore of $\mathbf{\Pi}_1^0$ sets. Assume this holds for all $\eta < \gamma$, let f be continuous, and let

$X \in \mathbf{\Sigma}^0_\gamma$. Then, X can be written as $\bigcup_{n\in\omega} X_n$ with each $X_n \in \mathbf{\Pi}^0_{\eta_n}$ with $\eta_n < \gamma$. We see then that $f^{-1}(X) = f^{-1}(\bigcup_{n\in\omega} X_n) = \bigcup_{n\in\omega} f^{-1}(X_n) \in \mathbf{\Sigma}^0_\gamma$. □

With these in hand, we are ready to proceed with the proof of Borel Determinacy. The basic idea is that given $A \subset [T]$, we want to simulate the game $G(A;T)$ with a game $G(A^*;T^*)$ in such a manner that the determinacy of $G(A;T)$ depends on the determinacy of $G(A^*;T^*)$. In order to accomplish this, we make use of the machinery of Covering Trees and Unravelings.

Definition 1. *Given a Tree T, a Covering of T is a triple $\langle T^*, \pi, \psi\rangle$ such that:*

(1) T^* is a game tree;

(2) π maps positions in T^* to positions in T monotonically such that $lh(\pi(s)) = lh(s)$ for $s \in T$;

(3) ψ maps strategies for I in T^* to strategies for I in T (similar for strategies for II) such that $\psi(\sigma^*)$ restricted to positions of length $\leq n$ only depends on σ^* restricted to positions of length $\leq n$ for every n;

(4) Finally, we have a lifting property of sorts for plays insofar as if x is a play in $[T]$ according to a strategy $\psi(\sigma^*)$, then there is an x^* in $[T^*]$ played according to σ^* such that $\pi(x^*) = x$.

Additionally, $\langle T^*, \pi, \psi\rangle$ is a k-covering of T if it is a covering and T restricted to $2k$ is the same as T^* restricted to $2k$, and π is the identity map on T^* restricted to $2k$.

Before moving on, we should note a few things. The first is that the function $\pi : T^* \to T$ gives rise to a continuous function from positions in $[T^*]$ to positions in $[T]$ in the obvious manner. Next, we may find it useful to think of strategies as trees in the sense that a strategy $\sigma \subset T$ for I (this is similarly defined for II) for a game $G(A;T)$ is a nonempty tree in which:

(i) If $\langle x_0, x_1 \ldots, x_{2n}\rangle \in \sigma$, then for each x_{2n+1} such that $\langle x_0, x_1 \ldots, x_{2n}, x_{2n+1}\rangle \in T$, $\langle x_0, x_1 \ldots, x_{2n}, x_{2n+1}\rangle \in \sigma$;
(ii) For each position of odd length $\overline{x} \in \sigma$, there is a unique a such that $\overline{x} \frown a \in \sigma$.

Finally, we see that for a game $G(A;T)$, and a covering $\langle T^*, \pi, \psi\rangle$ of T, the game $G(\pi^{-1}(A), T^*)$ simulates the game $G(A;T)$. In other words, if we have a winning strategy σ^* for I (respectively II) in $G(\pi^{-1}(A), T^*)$, then $\psi(\sigma^*)$ is a winning strategy for I (respectively II) in $G(A;T)$. This brings us to Unravelings:

Definition 2. *A covering $\langle T^*, \pi, \psi\rangle$ of T Unravels A if $\pi^{-1}(A)$ is clopen.*

Given our above discussion it is clear then that if we have a game $G(A;T)$, and a covering that unravels A, then $G(A;T)$ is determined. Thus, we see that Borel determinacy reduces to the following:

Theorem 2 (Martin(1982))**.** *If $A \subset [T]$ is Borel, then for every $k \in \omega$, there is a k-covering of T that unravels A.*

While the use of a k-covering seems a bit superflous, it turns out that we need it to carry through with the induction.

3. Borel Determinacy

We will follow Martin in proving Borel Determinacy by induction. It turns out that the machinery we've developed in the previous section makes the inductive step relatively painless. The main difficulty with the proof comes in proving that closed sets can be unraveled. Even then, the proof is fairly accessible. Now, assuming that closed sets can be unraveled, we immediately get that open sets can be unraveled. At this point, what we want to do is take a $\mathbf{\Sigma}^0_\alpha$ set, write it as a union of lower complexity sets, and use the coverings that unravel those sets to unravel our $\mathbf{\Sigma}^0_\alpha$ set somehow. It turns out that the way we will do this is by taking a limit of these unravelings. It is here that working with k-coverings (rather than just coverings) proves advantageous.

Lemma 2. *Fix a $k \in \omega$. Let $\langle T_{i+1}, \pi_{i+1}, \psi_{i+1}\rangle$ be a $(k+i)$-covering of T_i $i = 0, 1, 2, \ldots$. Then there is a pruned tree T_∞ and $\pi_{\infty,i}$, $\psi_{\infty,i}$ such that the triple $\langle T_\infty, \pi_{\infty,i}, \psi_{\infty,i}\rangle$ is a $(k+1)$-covering of T_i, $\pi_{i+1} \circ \pi_{\infty,i+1} = \pi_{\infty,i}$, and $\psi_{i+1} \circ \psi_{\infty,i+1} = \psi_{\infty,i}$.*

We see here that this lemma allows us to construct a covering that will unravel our $\mathbf{\Sigma}^0_\alpha$ set from already existing coverings. We sketch the argument for this lemma:

Proof. In a very real sense, we will construct this unraveling in the obvious way. We begin by noting that for any finite sequence s, if $2(k+i) \geq lh(s)$, then whether or not $s \in T_i$ is independent of i. So we set

$$s \in T_\infty \Leftrightarrow s \in T_i \text{ for any } i \text{ with } 2(k+i) \geq lh(s).$$

In other words because we are working with a sequence of $(k+1)$-coverings, we know that past some point the first $2(k+i)$ levels will look the same. We use this idea to construct our k-covering, and we see here why we want to have k-coverings in our theorem. With that in mind, we define $\pi_{\infty,i}$ as follows: If $lh(s) < 2(k+i)$, we simply set $\pi_{\infty,i}(s) = s$. If $lh(s) > 2(k+i)$, and $2(k+j) \geq lh(s)$, we set $\pi_{\infty,i}(s) = \pi_{i+1} \circ \pi_{i+1} \circ \cdots \circ \pi_j(s)$. Note that this is all independent of j given our above discussion.

Finally, we need to define $\psi_{\infty,i}$. We do this by fixing a strategy σ_∞ in T_∞, and letting $\psi_{\infty,i}(\sigma_\infty)$ be σ_∞ for the first $2(k+i)$ levels. For the $j > i$, we define $\psi_{\infty,i}(\sigma_\infty)$ in the same way we did with $\pi_{\infty,i}$, by pulling back through successive levels of our coverings. At this point, we actually need to check that this is a covering, but most of the work is done simply by how we defined everything. In fact we only need to check that we have the lifting property, but that is done basically through unpacking defnitions. ☐

At this point, we are ready to proceed with the inductive argument. We begin with the case where our sets are closed.

Lemma 3. *Let T be a game tree, and let $A \subset [T]$ be closed. For each $k \in \omega$, there is a k-covering of T that unravels A.*

It turns out that there is a lot of book-keeping in this part of the argument, so we present only a sketch.

Proof. We begin by fixing a $k \in \omega$, a game tree T, and a closed $A \subset [T]$. We will informally describe our k-covering of T by descirbing the legal moves in T^*. Since we are aiming for T^* to be a k-covering, the first legal $2k$ moves in T^* will consist of the first $2k$ legal moves in T. More formally, the games in T^* start off with the moves $x_0, x_1, \ldots, x_{2k-2}, x_{2k-1}$ where $\langle x_0, x_1, \ldots, x_i\rangle \in T$ for every $i \leq 2k-1$. At this point, I plays $\langle x_{2k}, \Sigma_I\rangle$ where

$\langle x_0, x_1, \ldots, x_{2k-1}, x_{2k}\rangle \in T$, and Σ_I is an object that allows II to predict I's plays. More precisely, Σ_I is a subtree of T that starts with the position $\langle x_0, x_1, \ldots, x_{2k-1}, x_{2k}\rangle$ fixed, and only restricts I's moves.

At this point, II plays x_{2k+1} such that $\langle x_0, x_1, \ldots, x_{2k}, x_{2k+1}\rangle \in T$, and some additional object. One option is that she plays u, a sequence of even length such that u is consistent with $\langle x_0, x_1, \ldots, x_{2k}, x_{2k+1}\rangle$, and u lands us in Σ_I, but not in the tree of A. The other option is that II plays $\Sigma_{II} \subset \Sigma_I$ which is similar to Σ_I, except it restricts II's moves and has $\langle x_0, x_1, \ldots, x_{2k}, x_{2k+1}\rangle$ fixed. If II plays something of the form $\langle x_{2k+1}, u\rangle$, then I and II may only pick elements that are compatible with u. On the other hand, if II plays Σ_{II}, then both players may only pick from legal moves in Σ_{II}. In the end, we have two sorts of sequences that can appear in T^*:

$$\langle x_0, x_1, \ldots, \langle x_{2k}, \Sigma_I\rangle, \langle x_{2k+1}, \langle 1, u\rangle\rangle, \ldots x_m\rangle \tag{1}$$

$$\langle x_0, x_1, \ldots, \langle x_{2k}, \Sigma_I\rangle, \langle x_{2k+1}, \langle 2, \Sigma_{II}\rangle\rangle, \ldots x_m\rangle \tag{2}$$

Now that we've defined our tree, it turns out that our map π is relatively simple as we just set $\pi(\langle x_0, x_1, \ldots, \langle x_{2k}, *\rangle, \langle x_{2k+1}, *\rangle, \ldots x_m\rangle) = \langle x_0, x_1, \ldots, x_{2k}, x_{2k+1}, \ldots x_m\rangle$. In addition, we see that $\overline{x} \in \pi^{-1}(A) \Leftrightarrow \overline{x}$ is a sequence of type (2), and thus that $\pi^{-1}(A)$ is clopen. All we have left to do is define the map ψ, and here is where we will omit some details.

Say that we are playing the game $G(A;T)$ from I's perspective. In this case, we imagine that I is playing the game $G(\pi^{-1}(A);T^*)$ off to the side in order to inform her moves in $G(A;T)$. Since the two games look the same for the first $2k$ moves, I simply plays by a given strategy σ^* for $G(\pi^{-1}(A);T^*)$ in our original game. In the next step, she plays $\langle x_{2k}, \Sigma_I\rangle$ in $G(\pi^{-1}(A);T^*)$ according to σ^*, and x_{2k} in $G(A;T)$. At this point, II plays x_{2k+1} in $G(A;T)$, but I must provide the extra move in the game $G(\pi^{-1}(A);T^*)$ that she is playing on the side. The idea here is that she assumes II is playing optimally, and makes II's choice accordingly as both u and Σ_{II} predict II's moves and the direction of the game to varrying extents. From then on, σ resumes following σ^*, and since I assumed II was playing optimally, she may continue to follow σ^* without issue (even if II decides to play suboptimally). Playing from II's perspective has a similar flavor.

With that, we see that we end up with a covering of T that unravels A. □

Finally, we move on to the induction.

Proof. (Of Theorem 2)

By Lemma 3, we know that our hypothesis holds for $\mathbf{\Pi}^0_1$ sets. Note that if a k-covering unravels A, then it will unravel its relative complement. So, this tells us that our hypothesis holds for $\mathbf{\Sigma}^0_1$ sets as well. Moving on to the induction fix $\gamma < \omega_1$, and assume that our hypothesis holds for every $\eta < \gamma$. In other words, for every $A \in \mathbf{\Pi}^0_\eta$, there is a k-covering that unravels A and consequently its relative complement. Let $A \in \mathbf{\Sigma}^0_\gamma$ and fix a $k \in \omega$. Then, we can write $A = \bigcup_{n\in\omega} A_n$ with each $A_n \in \mathbf{\Pi}^0_{\eta_n}$, and we can unravel each A_n. Let $\langle T_1, \pi_1, \psi_1\rangle$ be a k-covering of $T_0 = T$ that unravels A_0. Then, $\pi_1^{-1}(A_n) \in \mathbf{\Pi}^0_{\eta_n}$ for each $n \geq 1$ by Lemma 1. We now recursively define $\langle T_{n+1}, \pi_{n+1}, \psi_{n+1}\rangle$ to be a $(k+n)$-covering of T_n that unravels $\pi_n^{-1} \circ \pi_{n-1}^{-1} \circ \ldots \pi_1^{-1}(A_n)$. Let $\langle T_\infty, \pi_{\infty,n}, \psi_{\infty,n}\rangle$ be as in Lemma 2, then

$\langle T_\infty, \pi_{\infty,0}, \psi_{\infty,0}\rangle$ unravels each A_n, and so $\pi^{-1}_{\infty,0}(A)$ is open as it is the countable union of open sets. By Lemma 3 again, let $\langle T^*, \pi, \psi\rangle$ by a k-covering of T_∞ that unravels $\pi^{-1}_{\infty,0}(A)$. Then we see that $\langle T^*, \pi_{\infty,0} \circ \pi, \psi_{\infty,0} \circ \psi\rangle$ is a k-covering of T that unravels A.

□

4. Closing Remarks

While Borel Determinacy is a very nice result, it gives us some insight into the structure of the reals as well. For example, it gives us a measure on the set of Turing Degrees, and it can be used to show that Wadge Reducibility induces a well-ordering on the Borel Sets. So, we see that this is a rather powerful result insofar as the structure of the reals is concerned. Not only that, but we actually had to make use of ω_1-many powerset operations through the course of the proof. In other words, sets that are much larger than the reals affect the structure of the reals in an entirely nontrivial manner.

Finally during the course of this proof, we developed the machinery of covering trees and unravelings. As we noted earlier, provided that we have a covering that unravels A, we know that the Gale-Stewart game over A is determined. This tells us that its worthwhile to investigate unravelings of sets of higher complexity. Neeman has shown that it is possible to unravel $\mathbf{\Pi}^1_1$ sets under appropriate large cardinal hypotheses. Interestingly enough, we require more than just a measurable cardinal to accomplish this, and we get a bit more determinacy out of this than just $\mathbf{\Pi}^1_1$ determinacy.

References

[1] A. S. Kechris. *Classical Descriptive Set Theory.* Springer-Verlag, New York, 1995.