Strong Combinatorial Properties at Small Cardinals
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08/05/13
Large combinatorial properties

0 = 1

→

I0-I3

→

n-huge

\[\exists \]

superhuge

→

huge

\[\exists \]

almost huge

→

Vopenka’s Principle

\[\exists \]

Extendible

→

Supercompact

→

Superstrong

→

Woodin

\[\exists \]

Strong

\[\exists \]

0^+ exists

\[\exists \]

Measurable

\[\exists \]

Ramsey

\[\exists \]

Weakly Compact

Laura Fontanella (KGRC)

Strong Combinatorial Properties at Small Cardinals
Large combinatorial properties

$0=1 \rightarrow I_0-I_3 \rightarrow n$-huge
\[ n \geq \exists \forall \]

\[ \forall \sim \]

\[ \forall \sim \]

\[ \sim \]

\[ \sim \]

almost huge

Vopenka’s Principle

\[ \sim \]

Extendible

Supercompact

Superstrong

Woodin

\[ \forall \sim \]

Strong

\[ \forall \sim \]

$0^+$ exists

\[ \forall \sim \]

Measurable

\[ \forall \sim \]

Ramsey

\[ \forall \sim \]

Weakly Compact

Laura Fontanella (KGRC)
Erdös & Tarski 1961

\( \kappa \) is weakly compact iff it is inaccessible and it satisfies the tree property.


\( \kappa \) is strongly compact iff it is inaccessible and it satisfies the strong tree property.


\( \kappa \) is supercompact iff it is inaccessible and it satisfies the super tree property.
Erdös & Tarski 1961

$\kappa$ is weakly compact iff it is inaccessible and it satisfies the tree property.


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Erdös & Tarski 1961

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$\kappa$ is supercompact iff it is inaccessible and it satisfies the super tree property.
The tree property at small cardinals

The Tree Property

A $\kappa$-tree, for a regular $\kappa$, is a tree of height $\kappa$ and levels of size $< \kappa$. 
The Tree Property

A \( \kappa \)-tree, for a regular \( \kappa \), is a tree of height \( \kappa \) and levels of size \( < \kappa \).

A regular cardinal \( \kappa \) satisfies the tree property if, and only if, every \( \kappa \)-tree has a cofinal branch.
A \( \kappa \)-tree, for a regular \( \kappa \), is a tree of height \( \kappa \) and levels of size \(< \kappa \).

A regular cardinal \( \kappa \) satisfies the tree property if, and only if, every \( \kappa \)-tree has a cofinal branch.
The Tree Property

Definition

A $\kappa$-Aronszajn tree is a $\kappa$-tree with no cofinal branches.

Theorem

- (König’s Lemma 1936) $\aleph_0$ satisfies the tree property;
- (Aronszajn 1934) $\aleph_1$ does not satisfy the tree property;
- (Specker 1949) If $\tau^{<\tau} = \tau$, then the tree property fails at $\tau^+$;
- (Mitchell 1972) If $\text{Cons}(\text{ZFC} + \exists \kappa \text{ weakly compact})$, then for every regular $\tau$ such that $\tau^{<\tau} = \tau$, we have $\text{Cons}(\text{ZFC} + \tau^{++} \text{ has the tree property})$. 
The Tree Property at \textit{Small Cardinals}

Mitchell 1972

Let $n \geq 2$, if $\text{Cons}(ZFC + \exists \kappa$ weakly compact), then $\text{Cons}(ZFC + \aleph_n$ has the Tree Property $)$.

Cummings & Foreman 1998

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $\text{Cons}(ZFC + \forall n \geq 2 (\aleph_n$ has the tree property $))$.

Magidor & Shelah 1996, Sinapova 2012

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $\text{Cons}(ZFC + \aleph_{\omega+1}$ has the tree property $)$. 
The Tree Property at Small Cardinals

Neeman 2012

If Cons(ZFC + ∃⟨κ_n⟩_n<ω supercompact cardinals), then
Cons(ZFC + every regular cardinal ≤ ω_1+1 has the tree property).

Friedman & Halilović 2011

If Cons(ZFC + ∃κ weakly compact hypermeasurable), then
Cons(ZFC + ω has the tree property).
Open question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the tree property?
The tree property at the successor and the double successor of a singular cardinal

Unger 2013

\(\text{Cons}(\text{ZFC} + \exists \kappa \text{ supercompact} + \exists \mu > \kappa \text{ weakly compact})\) implies \(\text{Cons}(\text{ZFC} + \kappa \text{ is singular strong limit of cofinality } \omega, \text{ SCH fails at } \kappa, \text{ there are no special } \kappa^+-\text{Aronszajn trees and } \kappa^{++} \text{ has the tree property})\).

Fontanella & Friedman 2013

\(\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals} + \exists \mu \text{ weakly compact above } \lambda := \sup_{n<\omega} \kappa_n)\) implies \(\text{Cons}(\text{ZFC} + \lambda^+ \text{ and } \lambda^{++} \text{ have the tree property and } 2^{\kappa_0} = 2^\lambda = \lambda^{++})\).

Fontanella & Friedman 2013

\(\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals} + \exists \mu \text{ weakly compact above } \sup_{n<\omega} \kappa_n)\) implies \(\text{Cons}(\text{ZFC} + \aleph_{\omega+1} \text{ has the tree property and } \aleph_{\omega+2} \text{ has the tree property})\).
Magidor & Shelah 1996

Let $\langle \kappa_n \rangle_{n<\omega}$ be a sequence of strongly compact cardinals and let $\lambda := \sup_{n<\omega} \kappa_n$, then $\lambda^+$ has the tree property.
Fontanella & Friedman 2013

Cons(\(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \) supercompact cardinals + \(\exists \mu \) weakly compact above \(\lambda := \sup_{n<\omega} \kappa_n\)) implies Cons(\(\text{ZFC} + \lambda^+ \) and \(\lambda^{++}\) have the tree property and \(2^\kappa_0 = 2^\lambda = \lambda^{++}\)).

\[ \mu\text{ weakly compact} \]
\[ \lambda^+\text{ tree property} \]
\[ \lambda = \sup_{n<\omega} \kappa_n \]
\[ \kappa_n \text{ (indestructible) supercompact} \]
\[ \kappa_1 \text{ (indestructible) supercompact} \]
\[ \kappa_0 \text{ (indestructible) supercompact} \]
Fontanella & Friedman 2013

Cons($ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\lambda := \sup_{n<\omega} \kappa_n$) implies Cons($ZFC + \lambda^+$ and $\lambda^{++}$ have the tree property and $2^{\kappa_0} = 2^\lambda = \lambda^{++}$).

- $\mu$ inaccessible
  + tree property

- $\lambda^+$ tree property

- $\lambda = \sup_{n<\omega} \kappa_n$

- $\kappa_n$ (indestructible) supercompact

- $\kappa_1$ (indestructible) supercompact

- $\kappa_0$ (indestructible) supercompact
Fontanella & Friedman 2013

$$\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n < \omega} \text{ supercompact cardinals} + \exists \mu \text{ weakly compact above } \lambda := \sup_{n < \omega} \kappa_n)$$ implies $$\text{Cons}(ZFC + \lambda^+ \text{ and } \lambda^{++} \text{ have the tree property and } 2^{\kappa_0} = 2^{\lambda} = \lambda^{++})$$.

\[
\begin{align*}
\mu &= \lambda^{++} \\
&\quad + \text{ tree property} \\
\lambda^+ &= \text{ tree property} \\
\lambda &= \sup_{n < \omega} \kappa_n \\
\kappa_n &= \text{ old supercompact} \\
\kappa_1 &= \text{ old supercompact} \\
\kappa_0 &= \text{ old supercompact}
\end{align*}
\]

$\mathcal{M}$ collapses all cardinals between $\lambda^+$ and $\mu$; it makes $2^{\kappa_0} \geq \mu$ (hence $2^{\lambda} \geq \mu$)
Mitchell’s forcing over a singular cardinal

Definition

Conditions of $\mathcal{M}$ are pairs $(p, q)$ such that

1. $p \in Add(\kappa_0, \mu)$;
2. $q$ is a function of size $\leq \lambda$ such that every $\alpha \in \text{dom}(q)$ is a cardinal between $\lambda^+$ and $\mu$, and $\models_{Add(\kappa_0, \alpha)} q(\alpha) \in Add(\lambda^+, 1)$.

We let $(p, q) \leq (p', q')$ if and only if $p \leq p'$, $\text{dom}(q') \subseteq \text{dom}(q)$ and for every $\alpha \in \text{dom}(q')$ $p \upharpoonright \alpha \models q(\alpha) \leq q'(\alpha)$.

Fact

$\mathcal{M}$ is a projection of $Add(\kappa_0, \mu) \times Q$ where $Q := \{(0, q); (0, q) \in \mathcal{M}\}$ is $\lambda^+$-directed closed.
Mitchell’s forcing over a singular cardinal

Definition

Conditions of $M$ are pairs $(p, q)$ such that

1. $p \in Add(\kappa_0, \mu)$;
2. $q$ is a function of size $\leq \lambda$ such that every $\alpha \in \text{dom}(q)$ is a cardinal between $\lambda^+$ and $\mu$, and $\Vdash_{Add(\kappa_0, \alpha)} q(\alpha) \in Add(\lambda^+, 1)$.

We let $(p, q) \leq (p', q')$ if and only if $p \leq p'$, $\text{dom}(q') \subseteq \text{dom}(q)$ and for every $\alpha \in \text{dom}(q') p \upharpoonright \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

Fact

$M$ is a projection of $Add(\kappa_0, \mu) \times Q$ where $Q := \{(0, q); (0, q) \in M\}$ is $\lambda^+$-directed closed.
Fontanella & Friedman 2013

Cons($ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\text{sup}_{n<\omega} \kappa_n$) implies Cons($ZFC + \aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property).

- $\mu = \lambda^{++}$ tree property
- $\lambda^+$ tree property
- $\lambda$
- $\kappa_n$ old supercompact
- $\kappa_1$ old supercompact
- $\kappa_0$ old supercompact
Fontanella & Friedman 2013

Cons($ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\sup_{n<\omega} \kappa_n$) implies Cons($ZFC + \aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property).

\[
\begin{align*}
\mu &= \lambda^{++} = \aleph_{\omega+2} \text{ tree property} \\
\lambda^+ &= \aleph_{\omega+1} \text{ tree property} \\
\lambda &= \aleph_\omega \\
\kappa_n &= \aleph_{n+2} \\
\kappa_1 &= \aleph_3 \\
\kappa_0 &= \aleph_2
\end{align*}
\]
Fontanella & Friedman 2013

Cons($\mathsf{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\sup_{n<\omega} \kappa_n$) implies Cons($\mathsf{ZFC} + \mathcal{R}_{\omega+1}$ has the tree property and $\mathcal{R}_{\omega+2}$ has the tree property).

- $\mu = \lambda^{++} = \mathcal{R}_{\omega+2}$ tree property
- $\lambda^+ = \mathcal{R}_{\omega+1}$ tree property
- $\lambda = \mathcal{R}_\omega$
- $\kappa_n = \mathcal{R}_{n+2}$
- $\kappa_1 = \mathcal{R}_3$
- $\kappa_0 = \mathcal{R}_2$

There exists $\nu < \kappa_0$ singular strong limit of cofinality $\omega$, such that

$$\text{Coll}(\omega, \nu) \times \text{Coll}(\nu^+, < \kappa_0)$$
$$\times \prod_{n<\omega} \text{Coll}(\kappa_n, < \kappa_{n+1}) \times \mathbb{M}$$

forces the tree property at $\mathcal{R}_{\omega+1}$ and $\mathcal{R}_{\omega+2}$. 
The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{ f : X \rightarrow 2 ; \ X \in [\lambda]^{<\kappa} \}$ such that:

1. for all $f \in F$, if $X \subseteq \text{dom}(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $\text{Lev}_X(F) := \{ f \in F ; \ \text{dom}(f) = X \} \neq \emptyset$ and has size $< \kappa$. 
The tree property at small cardinals

The Strong Tree Property

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Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{f : X \to 2; \ X \in [\lambda]^{<\kappa}\}$ such that:

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\[
\begin{array}{c}
\text{f : X \to 2} \\
X \quad < \kappa
\end{array}
\]
The Strong Tree Property

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\[ f : X \to 2 \]
\[ X \]
\[ UI \]
\[ Y \]
\[ f \upharpoonright Y \]
The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{f : X \rightarrow 2; \ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq \text{dom}(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $\text{Lev}_X(F) := \{f \in F; \ \text{dom}(f) = X\} \neq \emptyset$ and has size $< \kappa$. 

\[ f : X \rightarrow 2 \]

\[ X \]

\[ \text{Lev}_X(F) \]

\[ \text{Lev}_X(F) \]

\[ f \restriction Y \]

\[ f \restriction Z \]

\[ Y \]

\[ Z \]
The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{ f : X \rightarrow 2; \ X \in [\lambda]^{<\kappa} \}$ such that:

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Definition

A cofinal branch for a $(\kappa, \lambda)$-tree $F$ is a function $b : \lambda \rightarrow 2$ such that $b \upharpoonright X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

$\kappa$ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every $(\kappa, \lambda)$-tree has a cofinal branch.
The Strong Tree Property

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Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{ f : X \rightarrow 2 ; \ X \in [\lambda]^{<\kappa} \}$ such that:

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Definition

$\kappa$ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every $(\kappa, \lambda)$-tree has a cofinal branch.
The Super Tree Property

Definition

Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; \ X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.
The strong and super tree properties

The Super Tree Property

Definition

Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.
The Super Tree Property

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Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; \ X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let $F$ be a $(\kappa, \lambda)$-tree and $D := \langle d_X; \ X \in [\lambda]^{<\kappa} \rangle$ an $F$-level sequence. An ineffable branch for $D$ is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa}; \ b \upharpoonright X = d_X\}$$

is stationary.

Definition

$\kappa$ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all $(\kappa, \lambda)$-tree $F$, every $F$-level sequence has an ineffable branch.
The Super Tree Property

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Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

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Let $F$ be a $(\kappa, \lambda)$-tree and $D := \langle d_X; X \in [\lambda]^{<\kappa} \rangle$ an $F$-level sequence. An ineffable branch for $D$ is a cofinal branch $b : \lambda \to 2$ such that

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The Super Tree Property

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Let $F$ be a $(\kappa, \lambda)$-tree and $D := \langle d_X; \ X \in [\lambda]^{<\kappa} \rangle$ an $F$-level sequence. An ineffable branch for $D$ is a cofinal branch $b : \lambda \to 2$ such that

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Definition

$\kappa$ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all $(\kappa, \lambda)$-tree $F$, every $F$-level sequence has an ineffable branch.
Open Question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the strong or the super tree properties?

Weiss 2010

Let $n \geq 2$, if $\text{Cons}(\text{ZFC} + \exists \kappa \text{ supercompact})$, then $\text{Cons}(\text{ZFC} + \aleph_n \text{ has the Super Tree Property})$.

Fontanella 2012, Unger 2012

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(\text{ZFC} + \forall n \geq 2, \aleph_n \text{ has the Super Tree Property})$.

Fontanella 2012

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(\text{ZFC} + \aleph_{\omega+1} \text{ has the Strong Tree Property})$. 
**Open Question**

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the strong or the super tree properties?

**Weiss 2010**

Let $n \geq 2$, if $\text{Cons}(ZFC + \exists \kappa \text{ supercompact })$, then $\text{Cons}(ZFC + \aleph_n \text{ has the Super Tree Property })$.

**Fontanella 2012, Unger 2012**

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(ZFC + \forall n \geq 2, \aleph_n \text{ has the Super Tree Property })$.

**Fontanella 2012**

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property })$. 
The Strong Tree Property at $\aleph_{\omega+1}$

Fontanella 2012

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $\text{Cons}(\text{ZFC} + \aleph_{\omega+1}$ has the Strong Tree Property).
The strong tree property at successors of singular cardinals

The Strong Tree Property at $\aleph_{\omega+1}$

Magidor & Shelah 1996
If $\lambda$ is a singular limit of strongly compact cardinals, then $\lambda^+$ satisfies the Tree Property.

Fontanella 2012 - Key Lemma
If $\lambda$ is a singular limit of strongly compact cardinals, then $\lambda^+$ satisfies the Strong Tree Property.
Fontanella - Key Lemma

If $\lambda$ is a singular limit of strongly compact cardinals, then $\lambda^+$ satisfies the Strong Tree Property.

$\lambda^+$ strong tree property

$\lambda$

$\kappa_n$ strongly compact

$\kappa_1$ strongly compact

$\kappa_0$ strongly compact
Fontanella - Key Lemma

If $\lambda$ is a singular limit of strongly compact cardinals, then $\lambda^+$ satisfies the Strong Tree Property.

\[ \lambda^+ \text{ strong tree property} \]
\[ \lambda \]
\[ \kappa_n \varphi(\kappa_n, \lambda^+) \]
\[ \kappa_1 \varphi(\kappa_1, \lambda^+) \]
\[ \kappa_0 \varphi(\kappa_0, \lambda^+) \]
A Partition Property for Strongly Compact Cardinals

For a cofinal $S \subseteq [\mu]^{<\nu}$, we denote by $[[ S ]]^2$ the set of all pairs $(X, Y) \in S \times S$ such that $X \subseteq Y$.

**Definition**

Let $\lambda \geq \kappa$, the principle $\varphi(\kappa, \lambda)$ establishes that for every $\mu \geq \lambda$ and for every stationary $S \subseteq [\mu]^{<\lambda}$, every $c : [[ S ]]^2 \rightarrow \gamma$ with $\gamma < \kappa$ has a quasi homogenous set $H$ of color $i < \gamma$ which is also stationary, i.e.

for every $X, Y \in H$ there is $Z \supseteq X, Y$ in $H$ such that $c(X, Z) = i = c(Y, Z)$.
A Partition Property for Strongly Compact Cardinals

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A Partition Property for Strongly Compact Cardinals

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Let $\lambda \geq \kappa$, the principle $\varphi(\kappa, \lambda)$ establishes that for every $\mu \geq \lambda$ and for every stationary $S \subseteq [\mu]^{<\lambda}$, every $c : [[ S ]]^2 \rightarrow \gamma$ with $\gamma < \kappa$ has a quasi homogenous set $H$ of color $i < \gamma$ which is also stationary, i.e.

for every $X, Y \in H$ there is $Z \supseteq X, Y$ in $H$ such that $c(X, Z) = i = c(Y, Z)$.
Theorem

Let $\kappa$ be a strongly compact cardinal, then $\varphi(\kappa, \lambda)$ holds for every $\lambda \geq \kappa$.

Fontanella - Key Lemma

If $\lambda = \lim_{n<\omega} \kappa_n$ where every $\kappa_n$ satisfies $\varphi(\kappa_n, \lambda^+)$, then $\lambda^+$ satisfies the Strong Tree Property.
Fontanella 2012

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(ZFC + \kappa_{\omega+1} \text{ has the Strong Tree Property})$.

- $\lambda^+$ strong tree property
- $\lambda$
- $\kappa_n$ supercompact
- $\kappa_1$ supercompact
- $\kappa_0$ supercompact
If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(\text{ZFC} + \aleph_{\omega+1} \text{ has the Strong Tree Property})$.

There exists $\nu < \kappa_0$ singular strong limit of cofinality $\omega$, such that

$$\text{Coll}(\omega, \nu) \times \text{Coll}(\nu^+, < \kappa_0) \times \prod_{n<\omega} \text{Coll}(\kappa_n, < \kappa_{n+1})$$

forces the strong tree property at $\aleph_{\omega+1}$. 
Future work

- Can we find similar characterizations of other large cardinals?
- What cardinals can satisfy those properties?
- How can we use them?
Thank you.