THE ACADEMY OF SCIENCES OF THE CZECH REPUBLIC

YORK UNIVERSITÉ UNIVERSITY

TOWER GAMES

David Chodounský
david.chodounsky@matfyz.cz

P-filter game

Filter games were originally invented by F. Galvin. Let $\mathcal{F}$ be a filter on ω. In the p-filter game for $\mathcal{F}$, two players play alternatively sets F_i and b_i during ω many moves in the following way:

	move 0	move 1		move i		after ω many moves	
player I	$F_0 \in \mathcal{F}$	$F_1 \in \mathcal{F}$	…	$F_i \in \mathcal{F}$	…	Is $\bigcup_{i\in\omega} b_i \in \mathcal{F}$?	wins if $\bigcup b_i \notin \mathcal{F}$
player II	$b_0 \in [F_0]^{<\omega}$	$b_1 \in [F_1]^{<\omega}$	…	$b_i \in [F_i]^{<\omega}$	…		wins if $\bigcup b_i \in \mathcal{F}$

When the game is over, player II wins if and only if the union of sets he played is in $\mathcal{F}$. In this type of game player II never has a winning strategy.

Theorem (C. Laflamme). *Player I has no winning strategy in the p-filter for $\mathcal{F}$ game iff $\mathcal{F}$ is a non-meager p-filter.*

Tower games

Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be a descending tower in $\mathcal{P}(\omega)$, i.e. $T_\alpha \subset \omega$ and $|T_\beta \setminus T_\alpha| < \omega$ for each $\alpha < \beta \in \kappa$. A game equivalent with the p-filter game for filter $\langle \mathcal{T} \rangle$ (generated by the tower $\mathcal{T}$) is played as follows:

$\alpha_0 \in \kappa, a_0 \in [\omega]^{<\omega}$	$\alpha_1 \in \kappa, a_1 \in [\omega]^{<\omega}$	…	$\alpha_i \in \kappa, a_i \in [\omega]^{<\omega}$	…	$\exists \gamma\colon T_\gamma \subset^* \bigcup_{i\in\omega} b_i$?
$b_0 \in [T_{\alpha_0} \setminus a_0]^{<\omega}$	$b_1 \in [T_{\alpha_1} \setminus a_1]^{<\omega}$	…	$b_i \in [T_{\alpha_i} \setminus a_i]^{<\omega}$	…	

When the game is over, player II wins if and only if there exists $\gamma \in \kappa$ such that T_γ is modulo finite included in $\bigcup_{i\in\omega} b_i$. The theorem for p-filter games implies, that player I does not have winning strategy if and only if $\mathcal{T}$ generates a non-meager filter.

We modify this version of p-filter game by adding a requirement, that player II has to guess the ordinal $\gamma \in \kappa$ witnessing his victory in the p-filter game. We call this a *tower game for $\mathcal{T}$*.

$\alpha_0 \in \kappa, a_0 \in [\omega]^{<\omega}$	…	$\alpha_i \in \kappa, a_i \in [\omega]^{<\omega}$	…	Let $\gamma = \sup_{i\in\omega} \beta_i$.
$\beta_0 \in \kappa, b_0 \in [T_{\alpha_0} \setminus a_0]^{<\omega}$	…	$\beta_i \in \kappa, b_i \in [T_{\alpha_i} \setminus a_i]^{<\omega}$	…	Is $T_\gamma \subset^* \bigcup_{i\in\omega} b_i$?

Player II wins the tower game if and only if $\gamma = \sup_{i\in\omega} \beta_i \in \kappa$ and this γ is an index of a $T_\gamma \in \mathcal{T}$, which is modulo finite included in the union of finite sets player II played, $|T_\gamma \setminus \bigcup_{i\in\omega} b_i| < \omega$.

For player II this game seems to be more difficult than the p-filter game for $\langle \mathcal{T} \rangle$. However, conditions for existence of a winning strategy for player I in the tower game are the same as in the p-filter game.

Grigorieff forcing

A typical application of p-filter game is the proof of properness of Grigorieff forcing $\mathbf{P}$.

Let $\mathcal{F}$ be a filter on ω. Function $p\colon I \to 2$ is a condition in $\mathbf{P}$ iff $\omega \setminus I \in \mathcal{F}$ and $q < p \Leftrightarrow p \subset q$.

Theorem. $\mathbf{P}$ *is proper if $\mathcal{F}$ is a non-meager p-filter.*

Proof (idea). To construct a 'fusion like' sequence of conditions $p_0 > p_1 > p_2 > \dots$, play the p-filter game for $\mathcal{F}$. In move n define p_n, such that $\operatorname{Dom} p_n \cap b_i = \emptyset$ for each $i < n$ and let player I play $F_n = \omega \setminus \operatorname{Dom} p_n$. When the game is over, put $p = \bigcup p_n$. $\operatorname{Dom} p \cap \bigcup b_n = \emptyset$ and if player II won, we have $\omega \setminus \operatorname{Dom} p \in \mathcal{F}$ and thus $p \in \mathbf{P}$. □

strong-Q-sequences

Let $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ be an AD system and let $\mathcal{F} = \{f_\alpha : A_\alpha \to 2\}$ be system of functions. Function $f\colon \omega \to 2$ is a *uniformization* of $\mathcal{F}$ iff $f \restriction A_\alpha \equiv^* f_\alpha$ for each $\alpha \in \omega_1$. If a uniformization exists for each system of functions $\mathcal{F}$, the system $\mathcal{A}$ is called a *strong-Q-sequence* [5].

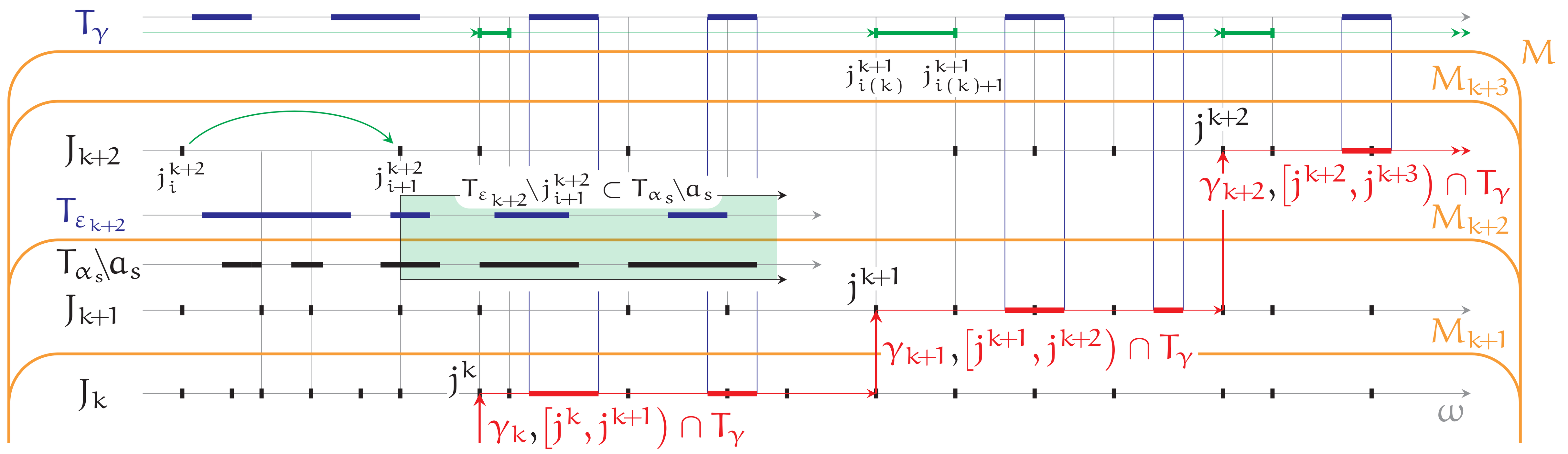
Existence of a strong-Q-sequence is not provable in ZFC (it implies $2^\omega = 2^{\omega_1}$, $\neg$MA) and some AD systems are not strong-Q-sequences in an absolute sense (e.g. Luzin gap). If $\mathcal{P}(\omega)/\mathrm{fin} \cong \mathcal{P}(\omega_1)/\mathrm{fin}$, there is a strong-Q-sequence and $\mathfrak{d} = \omega_1$.

Theorem (J. Steprāns). *Existence of a strong-Q-sequence is consistent with ZFC.*

Tower game - defence of player II

Theorem 1. *Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be a descending tower in $\mathcal{P}(\omega)$ generating a non-meager filter. Player I has no winning strategy in the tower game for $\mathcal{T}$.*

Proof (sketch). Let $\mathcal{S} = (\alpha_s, a_s)$ be a strategy of player I, where α_s, a_s is his response to the finite sequence s of moves of player II. Fix a sequence of countable elementary submodels M_k for $k \in \omega$, such that $M_k \prec M_{k+1} \prec H(\theta)$; $\mathcal{T}, \mathcal{S} \in M_k \in M_{k+1}$ and put $M = \bigcup M_k$, $\varepsilon_k = \sup M_k \cap \kappa$ and $\gamma = \sup M \cap \kappa$. Fix a sequence of ordinals $\langle \gamma_k \rangle_{k\in\omega}$ such that $\gamma_k \in M_k$ and $\gamma_k \nearrow \gamma$.
Player II constructs a sequence of increasing sequences of natural numbers $\langle J^k = \langle j^k_i \rangle_{i\in\omega} \rangle_{k\in\omega}$, such that $J^k \in M_{k+1}$ and J^{k+1} is a subsequence of J^k. For each $k \in \omega$ start with j^k_0, such that $T_\gamma \setminus j^k_0 \subset T_{\varepsilon_k}$. Suppose j^k_i is defined and let $O^k_i \in M_k$ be the finite set of all sequences of moves of player II of length j^k_i containing only ordinals γ_m for $m \le k$ and finite subsets of j^k_i. Choose $j^k_{i+1} \in J^{k-1}$, such that $T_{\varepsilon_k} \setminus j^k_{i+1} \subset T_{\alpha_s} \setminus a_s$ for each $s \in O^k_i$. Hence each such s can be legally extended by the move β, b of player II, if $b \in [T_\gamma \setminus j^k_{i+1}]^{<\omega}$.



Player II constructs an increasing sequence $\langle j^k \rangle_{k\in\omega}$ of indexes of moves, in which he will play a nonempty finite set b. If j^k is defined, find $j^{k+1} = j^{k+1}_{i(k)} \in J^{k+1}$, such that $[j^{k+1}_{i(k)}, j^{k+1}_{i(k)+1}) \cap T_\gamma = \emptyset$. This is possible, since $\mathcal{T}$ generates a non-meager filter. In move j^k player II plays γ_k, $b_k = [j^k, j^{k+1}) \cap T_\gamma$ and then keeps playing $\gamma_k, \emptyset$ until move j^{k+1}. In the end $\bigcup_{k\in\omega} b_k =^* T_\gamma$ and $\sup_{k\in\omega} \gamma_k = \gamma$, i.e. player II wins. □

Adding uniformizations

Let $\mathcal{T} = \{T_\alpha : \alpha \in \omega_1\}$ be a descending tower in $\mathcal{P}(\omega)$ generating a non-meager filter. The family $\mathcal{A} = \{A_\alpha = T_\alpha \setminus T_{\alpha+1} : \alpha \in \omega_1\}$ is an AD system. Let $\mathcal{F} = \{f_\alpha : A_\alpha \to 2\}$ be system of functions. The forcing $\mathbf{P}_\mathcal{F}$ consists of functions $p\colon I \to 2$, such that $I =^* \omega \setminus T_{\alpha(p)}$ for some $\alpha(p) \in \omega_1$ and $p \restriction A_\beta \equiv^* f_\beta$ for $\beta < \alpha(p)$ – condition (∗). Define $q < p \Leftrightarrow p \subset q$. The generic real is a uniformization of $\mathcal{F}$. A similar forcing appeared in [3].

Theorem 2. *If $\mathcal{T}$ generates a non-meager filter, then $\mathbf{P}_\mathcal{F}$ is a proper and ${}^\omega\omega$ bounding forcing.*

Proof (idea). Recycle the proof of properness for Grigorieff forcing. To fulfill (∗) for the union p of a 'fusion like' sequence $p_0 > p_1 > p_2 > \dots$, play the tower game for $\mathcal{T}$ instead of p-filter game and ensure that $\alpha(p_i) \nearrow \alpha(p)$. □

Applications

We can iterate adding uniformizations to prove:

Theorem 3. *It is consistent with ZFC that there is a strong-Q-sequence and $\mathfrak{d} = \omega_1$.*

We can also simplify the proof of the following.

Theorem (H. Judah, S. Shelah). *It is consistent with ZFC that there is a Q-set of reals and a set of reals of cardinality ω_1 which is not Lebesgue measurable.*

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