Inverse limit reflection and the structure of $L(V_{\lambda+1})$

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the perfect set property

- $X \subseteq \mathbb{R}$ has the *perfect set property* if either X is countable or X contains a perfect set (and hence $|X| = |\mathbb{R}|$).
- Assuming the Axiom of Choice there is a set reals without the perfect set property, but under ZFC every Σ^1_1 set of reals has the perfect set property.
- However ZFC does not decide whether Σ^1_2 sets of reals have the perfect set property. But if enough large cardinals exist, all projective sets of reals have the perfect set property.
- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$.

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- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$.

the structure $L(\mathbb{R})$

- L is the constructible hierarchy.

$$L_0 = \emptyset,\ L_{\alpha+1} = \mathrm{Def}(L_\alpha) \text{ and } L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \text{ for } \lambda \text{ a limit.}$$

- $L(\mathbb{R})$ is the structure created by building L 'on top of the reals $\mathbb{R}$'.

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the structure $L(\mathbb{R})$

- In fact, assuming enough large cardinals exist, all classical regularity properties (Lebesgue measurability, property of Baire, etc.) are true for all sets of reals in $L(\mathbb{R})$.
- There is in fact a fundamental regularity property called *the Axiom of Determinacy* (AD) which holds in $L(\mathbb{R})$.
- AD is a fundamental regularity property in the sense that

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main goal

Our main goal is to generalize the above situation to the structure $L(V_{\lambda+1})$. That is, we want to find a 'fundamental regularity property' for the case of $L(V_{\lambda+1})$.

large cardinals and elementary embeddings

- Most large cardinals have the following form: there exists an elementary embedding $j : V \to M$ which is not the identity (non-trivial) such that M is an inner model of V, and M has a certain amount of agreement with V.
- We let $\kappa = \mathrm{crit}\,(j)$ the critical point of the embedding j, which is the least κ such that $j(\kappa) \neq \kappa$. In fact $j(\kappa) > \kappa$.

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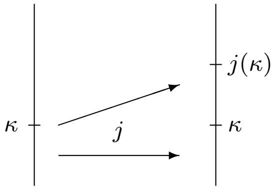

measurable and strong cardinals

- For instance κ is *measurable* if there exists a (non-trivial) elementary embedding $j : V \to M$ such that $\operatorname{crit}(j) = \kappa$. Automatically

$$V_{\kappa+1} \subseteq M.$$

- κ is called 2-*strong* if there exists a (non-trivial) elementary embedding $j : V \to M$ such that $\operatorname{crit}(j) = \kappa$ and

$$V_{\kappa+2} \subseteq M.$$

- 2-strong cardinals are much stronger than measurable cardinals. For instance if κ is 2-strong then κ is a limit of measurable cardinals.
- In general, the more M agrees with V, the stronger the large cardinal.

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the strongest large cardinals

How much can M agree with V?

Theorem (Kunen)

(ZFC) There is no (non-trivial) elementary embedding

$$j : V \to V.$$

In fact for any λ there is no elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Definition

1. I_1 is the statement: for some λ, there exists an elementary embedding

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the axiom I_0

Definition (Woodin)

I_0 is the statement: there exists a λ such that there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\text{crit}\,(j) < \lambda$.

Woodin originally introduced I_0 in order to show that AD holds in $L(\mathbb{R})$ assuming large cardinals.

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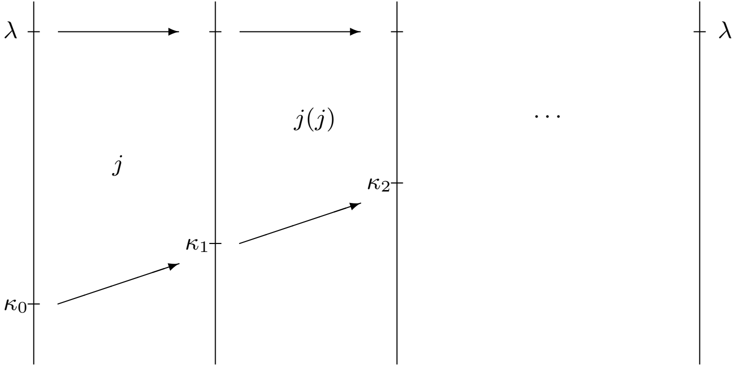
rank into rank embeddings

If $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary then λ is the sup of the *critical sequence* of j. That is, for $\kappa_0 = \operatorname{crit}(j)$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < \omega$, we have

$$\lambda = \sup_{i<\omega} \kappa_i.$$



relationship with $L(\mathbb{R})$

- If $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary and $\mathrm{crit}\,(j) < \lambda$ then λ is the sup of the critical sequence of j. So $\mathrm{cof}(\lambda) = \omega$.
- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for α a strong limit of cofinality ω.
- Furthermore, if AD holds in $L(\mathbb{R})$, then it does not satisfy the axiom of choice. And if I_0 holds at λ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice.
- Do $L(\mathbb{R})$ and $L(V_{\lambda+1})$ have similar structural properties? For instance does an analogue of the perfect set property hold in $L(V_{\lambda+1})$?

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Definition

Let $\Theta = \Theta_\lambda = \sup\{\alpha | \text{ (there exists a surjection of } V_{\lambda+1} \text{ onto } \alpha)^{L(V_{\lambda+1})}\}$.

Theorem

Assume AD holds in $L(\mathbb{R})$. Then $L(\mathbb{R})$ satisfies the following:

1. *ω_1 is measurable. In fact the club filter is an ultrafilter on ω_1 (Solovay).*
2. *Θ is a limit of measurable cardinals (Kechris and Woodin).*

Theorem (Woodin)

Assume I_0 holds at λ. Then the following hold in $L(V_{\lambda+1})$.

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Theorem (Davis)

Assume $L(\mathbb{R})$ satisfies AD. Then every set of reals in $L(\mathbb{R})$ has the perfect set property. That is if $X \subseteq \mathbb{R}$ and $X \in L(\mathbb{R})$ then either X is countable or X contains a perfect set and hence $|X| = 2^\omega$.

Theorem (C.)

Assume I_0 holds at λ. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ has the λ-splitting perfect set property. That is either $|X| \leq \lambda$ or X contains a λ-splitting perfect set and hence $|X| = 2^\lambda$.

Shi and Woodin originally showed the perfect set property for sets in $L_\lambda(V_{\lambda+1})$ using very different techniques, which we will discuss later.

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the club filter

Theorem (Solovay)

Assume that AD holds in $L(\mathbb{R})$. Then in $L(\mathbb{R})$ the club filter is an ultrafilter on ω_1.

Theorem (Woodin)

Assume I_0 holds at λ. Let $S_\alpha = \{\beta < \lambda^+ | \, cof(\beta) = \alpha\}$. Then in $L(V_{\lambda+1})$, for all $\alpha < \lambda$ regular, there is a $\delta < \lambda$ and a partition $\langle T_\beta | \, \beta < \delta \rangle$ of S_α into stationary sets such that for all $\beta < \delta$, the club filter restricted to T_β is an ultrafilter.

It is open whether or not the club filter restricted to S_ω is an ultrafilter in $L(V_{\lambda+1})$.

Theorem (C.)

Assume I_0 holds at λ. Then there are no disjoint stationary subsets T_1, T_2 of S_ω (in V) such that $T_1, T_2 \in L(V_{\lambda+1})$.

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analog of AD for $L(V_{\lambda+1})$

- The above results point to the possibility that I_0 for $L(V_{\lambda+1})$ is analogous to AD for $L(\mathbb{R})$.
- There is a problem with this however:

Definition

For $X \subseteq V_{\lambda+1}$, let $I_0(X)$ be the statement that there exists an elementary embedding

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with $\operatorname{crit}(j) < \lambda$.

- We have

$$AD \to \text{the perfect set property}$$

but

$$I_0(X) \nrightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

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inverse limit reflection

We will introduce a property called 'inverse limit reflection' (ILR) such that if I_0 holds at λ then $L(V_{\lambda+1})$ satisfies ILR. Furthermore

$$\text{ILR} \rightarrow \text{the } \lambda\text{-splitting perfect set property.}$$

So ILR is in this sense a better analog of AD for $L(V_{\lambda+1})$ than I_0.

reflecting I_3 and I_1

- Recall that if κ is 2-strong then κ is a limit of measurable cardinals. This phenomenon is called reflection.
- Does some large cardinal axiom reflect I_3, I_1, and I_0? Yes.

Theorem

1. *(I_1 reflects I_3) Suppose there is $V_{\lambda+1} \to V_{\lambda+1}$ an elementary embedding. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}} \to V_{\bar{\lambda}}$ (Martin).*
2. *(I_0 reflects I_1) Suppose there is $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an elementary embedding with $crit(j) < \lambda$. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$ (Woodin).*
3. *Assume there exists $j : L_{\lambda^+ + \omega+1}(V_{\lambda+1}) \to L_{\lambda^+ + \omega+1}(V_{\lambda+1})$ elementary. Then there exists a $\bar{\lambda} < \lambda$ such that there is an elementary embedding $k : L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$ with $crit(k) < \bar{\lambda}$ (Laver).*

Laver used a technique called 'inverse limits' to get his reflection result.

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3. *Assume there exists $j : L_{\lambda^+ + \omega+1}(V_{\lambda+1}) \to L_{\lambda^+ + \omega+1}(V_{\lambda+1})$ elementary. Then there exists a $\bar{\lambda} < \lambda$ such that there is an elementary embedding $k : L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$ with $crit(k) < \bar{\lambda}$ (Laver).*

Laver used a technique called 'inverse limits' to get his reflection result.

reflecting I_3 and I_1

- Recall that if κ is 2-strong then κ is a limit of measurable cardinals. This phenomenon is called reflection.
- Does some large cardinal axiom reflect I_3, I_1, and I_0? Yes.

Theorem

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Laver used a technique called ‘inverse limits’ to get his reflection result.

reflecting I_0

Theorem (C.)

($I_0^\#$ reflects I_0) Assume there exists an elementary embedding

$$j : L(V^\#_{\lambda+1}) \to L(V^\#_{\lambda+1})$$

with crit$(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding

$$k : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

with crit$(k) < \bar{\lambda}$.

The proof uses inverse limits as well.

definition of inverse limits

Definition (Laver)

An inverse limit $(J, \langle j_i | \, i < \omega \rangle)$ is a tuple such that the following hold:

1. For all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. $\text{crit}\,(j_0) < \text{crit}\,(j_1) < \text{crit}\,(j_2) < \cdots < \lambda$.
3. $\sup_{i<\omega} \text{crit}\,(j_i) = \bar{\lambda} < \lambda$.
4. $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is defined by: for all $a \in V_{\bar{\lambda}}$,

$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a) = (j_0 \circ j_1 \circ \cdots)(a).$$

- If $(J, \langle j_i | \, i < \omega \rangle)$ is an inverse limit then we write

$$J = j_0 \circ j_1 \circ \cdots .$$

- We can rewrite an inverse limit as a direct limit as follows:

$$J = \cdots \circ j_0(j_1(j_2)) \circ j_0(j_1) \circ j_0.$$

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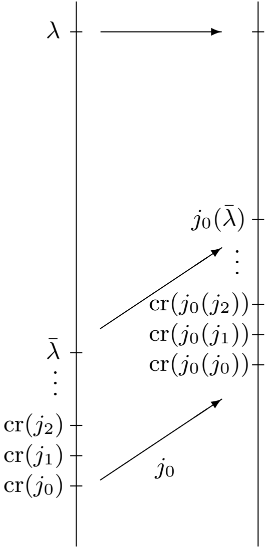

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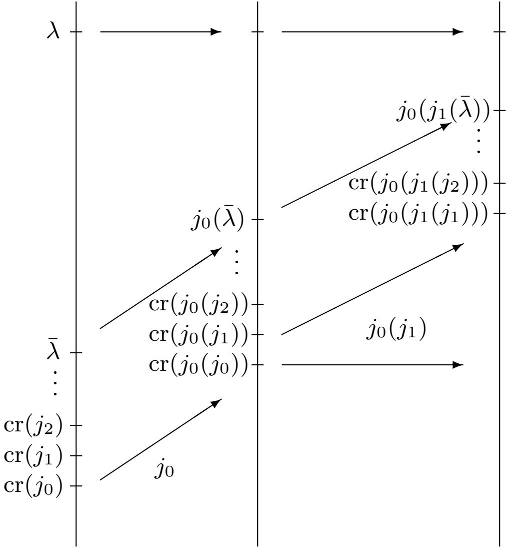

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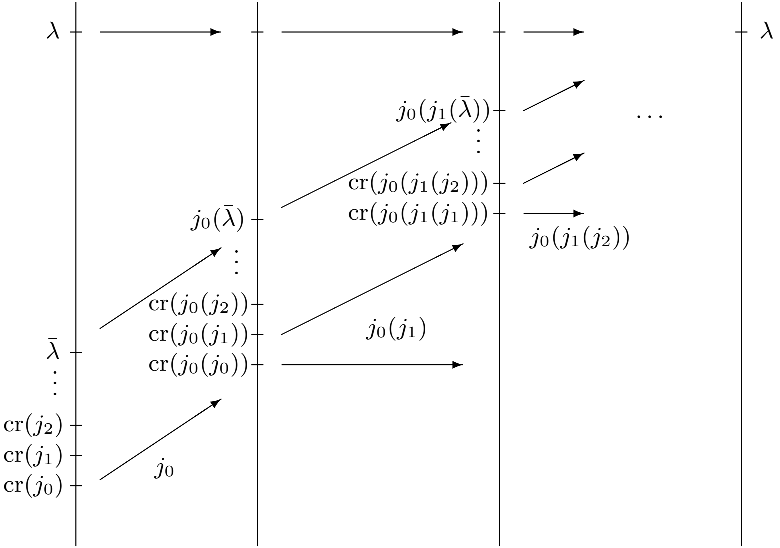

properties of inverse limits

- There are many theorems on inverse limits which take the basic form:

$$\text{property X for the embeddings } k_i \text{ for all } i < \omega$$
$$\Rightarrow \text{property } X \text{ for } K = k_0 \circ k_1 \circ \cdots$$

  We say that property X *transfers* to inverse limits.

- For instance for (certain) inverse limits $K = k_0 \circ k_1 \circ \cdots$ we have for any $a \in V_{\lambda+1}$

$$\forall i < \omega(a \in \operatorname{rng} k_i) \rightarrow a \in \operatorname{rng} K.$$

- Inverse limit reflection (ILR) is basically the statement that 'extension to $L_\alpha(V_{\lambda+1})$' transfers to inverse limits.

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- Inverse limit reflection (ILR) is basically the statement that 'extension to $L_\alpha(V_{\lambda+1})$' transfers to inverse limits.

inverse limit reflection

Definition

Suppose $\alpha < \Theta$. Then *inverse limit reflection at* α is the following statement. There exists $\beta < \Theta$, $\bar{\lambda} < \lambda$, and $\bar{\alpha}$ such that if $(J, \langle j_i | i < \omega\rangle)$ is an inverse limit, $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$, and

$$\forall i < \omega \, \exists \gamma \geq \beta (j_i \text{ extends to } j_i' : L_\gamma(V_{\lambda+1}) \to L_\gamma(V_{\lambda+1}))$$

then J extends to an embedding

$$\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1}).$$

Furthermore there is such an inverse limit.

Theorem

Suppose I_0 holds at λ.

1. *Inverse limit reflection holds at λ^+ (Laver).*
2. *For all $\alpha < \Theta_\lambda$, inverse limit reflection holds at α (C.).*

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applications of inverse limit reflection

Inverse limit reflection seems to be one of the key tools for studying $L(V_{\lambda+1})$, and it was the key tool for the proof of the perfect set property and the result on non-splitting stationary subsets of λ^+. How to extend this property to hierarchies above $L(V_{\lambda+1})$ is still an open problem.

$U(j)$-representations

- Can we connect $L(V_{\lambda+1})$ to actual models of determinacy?
- The perfect set property was originally shown by Woodin and Shi to hold for subsets of $V_{\lambda+1}$ in $L_\lambda(V_{\lambda+1})$. They used a certain representation for subsets of $V_{\lambda+1}$ called a $U(j)$-representation, due to Woodin.
- In fact, Woodin showed that $U(j)$-representations give even stronger properties for $L(V_{\lambda+1})$ than those which have been proven using inverse limits, such as a certain generic absoluteness result. However, it still remains unclear how many subsets of $V_{\lambda+1}$ have $U(j)$-representations.

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Theorem (Woodin)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary with crit$(j) < \lambda$. Then every set $X \subseteq V_{\lambda+1}$, $X \in L_\lambda(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

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The proof uses inverse limit techniques along with theorems of Woodin on $U(j)$-representations.

how far do $U(j)$-representations go?

Because the collection of $U(j)$-representations is closed under complements, along with the above theorems, we would expect the following.

Conjecture ($U(j)$-conjecture)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then every set $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

This would be a stark contrast to scales in $L(\mathbb{R})$, but note that $U(j)$-representations seem not to be related to uniformization.

consequences of the $U(j)$-conjecture

Theorem (Dimonte-Friedman, Woodin independently)

If the $U(j)$-conjecture holds then if I_0 is consistent, it is consistent to have I_0 at some λ and the Singular Cardinal Hypothesis fails at λ.

LST is a strong determinacy axiom whose consistency is (perhaps?) not known to follow from any large cardinal axiom.

Theorem (Woodin)

If the $U(j)$-conjecture holds then under enough large cardinals, there is a model of determinacy which satisfies LST.

The above theorems already show that models of AD do not 'peter out' at the level of I_0.

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open problems and additional topics

There are many (completely) open problems related to $L(V_{\lambda+1})$ including uniformization, Wadge reducibility, regularity of λ^{++}, λ^{+++}, etc., partition properties, ...