Inverse limit reflection and the structure of $L(V_{\lambda+1})$

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the perfect set property

- $X \subseteq \mathbb{R}$ has the *perfect set property* if either $X$ is countable or $X$ contains a perfect set (and hence $|X| = |\mathbb{R}|$).

- Assuming the Axiom of Choice there is a set reals without the perfect set property, but under ZFC every $\Sigma^1_1$ set of reals has the perfect set property.

- However ZFC does not decide whether $\Sigma^1_2$ sets of reals have the perfect set property. But if enough large cardinals exist, all projective sets of reals have the perfect set property.

- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$. 

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- We can generalize this result by considering sets of reals in the structure $L(\mathbb{R})$. 
the structure $L(\mathbb{R})$

- $L$ is the constructible hierarchy.
  
  $L_0 = \emptyset$, $L_{\alpha+1} = \text{Def}(L_\alpha)$ and $L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$ for $\lambda$ a limit.

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- The above generalizes to: assuming enough large cardinals, every set of reals in $L(\mathbb{R})$ has the perfect set property (Woodin).
In fact, assuming enough large cardinals exist, all classical regularity properties (Lebesgue measurability, property of Baire, etc.) are true for all sets of reals in $L(\mathbb{R})$.

There is in fact a fundamental regularity property called the Axiom of Determinacy (AD) which holds in $L(\mathbb{R})$.

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$$\forall (\text{regularity properties } X)(AD \rightarrow X).$$
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  $$\forall (\text{regularity properties } X)(AD \rightarrow X).$$
Our main goal is to generalize the above situation to the structure $L(V_{\lambda+1})$. That is, we want to find a ‘fundamental regularity property’ for the case of $L(V_{\lambda+1})$. 
Most large cardinals have the following form: there exists an elementary embedding $j : V \rightarrow M$ which is not the identity (non-trivial) such that $M$ is an inner model of $V$, and $M$ has a certain amount of agreement with $V$.

We let $\kappa = \text{crit}(j)$ the critical point of the embedding $j$, which is the least $\kappa$ such that $j(\kappa) \neq \kappa$. In fact $j(\kappa) > \kappa$. 
large cardinals and elementary embeddings

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- We let \( \kappa = \text{crit} (j) \) the critical point of the embedding \( j \), which is the least \( \kappa \) such that \( j(\kappa) \neq \kappa \). In fact \( j(\kappa) > \kappa \).
measurable and strong cardinals

- For instance, \( \kappa \) is *measurable* if there exists a (non-trivial) elementary embedding \( j : V \to M \) such that \( \text{crit} (j) = \kappa \). Automatically

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V_{\kappa+1} \subseteq M.
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- \( \kappa \) is called *2-strong* if there exists a (non-trivial) elementary embedding \( j : V \to M \) such that \( \text{crit} (j) = \kappa \) and

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V_{\kappa+2} \subseteq M.
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- 2-strong cardinals are much stronger than measurable cardinals. For instance if \( \kappa \) is 2-strong then \( \kappa \) is a limit of measurable cardinals.

- In general, the more \( M \) agrees with \( V \), the stronger the large cardinal.
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How much can $M$ agree with $V$?

**Theorem (Kunen)**

(ZFC) There is no (non-trivial) elementary embedding

$$j : V \rightarrow V.$$  

In fact for any $\lambda$ there is no elementary embedding

$$j : V_{\lambda+2} \rightarrow V_{\lambda+2}.$$  

**Definition**

- $I_1$ is the statement: for some $\lambda$, there exists an elementary embedding
  
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- $I_3$ is the statement: for some $\lambda$, there exists an elementary embedding
  
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**Definition (Woodin)**

$I_0$ is the statement: there exists a $\lambda$ such that there is an elementary embedding

$$j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$$

with crit ($j$) $< \lambda$.

Woodin originally introduced $I_0$ in order to show that AD holds in $L(\mathbb{R})$ assuming large cardinals.
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If $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$ is elementary then $\lambda$ is the sup of the critical sequence of $j$. That is, for $\kappa_0 = \text{crit}(j)$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < \omega$, we have

$$\lambda = \sup_{i < \omega} \kappa_i.$$
relationship with $L(\mathbb{R})$

- If $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary and $\text{crit}(j) < \lambda$ then $\lambda$ is the sup of the critical sequence of $j$. So $\text{cof}(\lambda) = \omega$.
- So $L(\mathbb{R}) = L(V_{\omega+1})$ and $L(V_{\lambda+1})$ are both structures of the form $L(V_{\alpha+1})$ for $\alpha$ a strong limit of cofinality $\omega$.
- Furthermore, if AD holds in $L(\mathbb{R})$, then it does not satisfy the axiom of choice. And if $I_0$ holds at $\lambda$ then $L(V_{\lambda+1})$ does not satisfy the axiom of choice.
- Do $L(\mathbb{R})$ and $L(V_{\lambda+1})$ have similar structural properties? For instance does an analogue of the perfect set property hold in $L(V_{\lambda+1})$?
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**Definition**

Let $\Theta = \Theta_\lambda = \sup\{\alpha | \text{(there exists a surjection of } V_{\lambda+1} \text{ onto } \alpha)^{L(V_{\lambda+1})}\}$. 

**Theorem**

Assume AD holds in $L(\mathbb{R})$. Then $L(\mathbb{R})$ satisfies the following:
- $\omega_1$ is measurable. In fact the club filter is an ultrafilter on $\omega_1$ (Solovay).
- $\Theta$ is a limit of measurable cardinals (Kechris and Woodin).

**Theorem (Woodin)**

Assume $I_0$ holds at $\lambda$. Then the following hold in $L(V_{\lambda+1})$.
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Theorem (Davis)

Assume $L(\mathbb{R})$ satisfies AD. Then every set of reals in $L(\mathbb{R})$ has the perfect set property. That is if $X \subseteq \mathbb{R}$ and $X \in L(\mathbb{R})$ then either $X$ is countable or $X$ contains a perfect set and hence $|X| = 2^\omega$.

Theorem (C.)

Assume $I_0$ holds at $\lambda$. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ has the $\lambda$-splitting perfect set property. That is either $|X| \leq \lambda$ or $X$ contains a $\lambda$-splitting perfect set and hence $|X| = 2^\lambda$.

Shi and Woodin originally showed the perfect set property for sets in $L_{\lambda}(V_{\lambda+1})$ using very different techniques, which we will discuss later.
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The club filter

Theorem (Solovay)

Assume that AD holds in $L(\mathbb{R})$. Then in $L(\mathbb{R})$ the club filter is an ultrafilter on $\omega_1$.

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Assume $I_0$ holds at $\lambda$. Let $S_\alpha = \{ \beta < \lambda^+ | \text{cof}(\beta) = \alpha \}$. Then in $L(V_{\lambda+1})$, for all $\alpha < \lambda$ regular, there is a $\delta < \lambda$ and a partition $\langle T_\beta | \beta < \delta \rangle$ of $S_\alpha$ into stationary sets such that for all $\beta < \delta$, the club filter restricted to $T_\beta$ is an ultrafilter.

It is open whether or not the club filter restricted to $S_{\omega}$ is an ultrafilter in $L(V_{\lambda+1})$.

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Assume $I_0$ holds at $\lambda$. Then there are no disjoint stationary subsets $T_1$, $T_2$ of $S_\omega$ (in $V$) such that $T_1, T_2 \in L(V_{\lambda+1})$. 
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The above results point to the possibility that $I_0$ for $L(V_{\lambda+1})$ is analogous to AD for $L(\mathbb{R})$.

There is a problem with this however:

**Definition**

For $X \subseteq V_{\lambda+1}$, let $I_0(X)$ be the statement that there exists an elementary embedding

$$j : L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1})$$

with $\text{crit}(j) < \lambda$.

We have

$$AD \rightarrow \text{the perfect set property}$$

but

$$I_0(X) \nrightarrow \text{the } \lambda\text{-splitting perfect set property}.$$
analog of AD for $L(V_{\lambda+1})$

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We have

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$$I_0(X) \nrightarrow \text{the \lambda-splitting perfect set property}.$$
We will introduce a property called ‘inverse limit reflection’ (ILR) such that if $I_0$ holds at $\lambda$ then $L(V_{\lambda+1})$ satisfies ILR. Furthermore

$$\text{ILR} \rightarrow \text{the } \lambda\text{-splitting perfect set property}.$$ 

So ILR is in this sense a better analog of AD for $L(V_{\lambda+1})$ than $I_0$. 
reflecting \(I_3\) and \(I_1\)

- Recall that if \(\kappa\) is 2-strong then \(\kappa\) is a limit of measurable cardinals. This phenomenon is called reflection.
- Does some large cardinal axiom reflect \(I_3\), \(I_1\), and \(I_0\)? Yes.

**Theorem**

- \((I_1 \text{ reflects } I_3)\) Suppose there is \(V_{\lambda+1} \rightarrow V_{\lambda+1}\) an elementary embedding. Then there is a \(\tilde{\lambda} < \lambda\) and an elementary embedding \(V_\lambda \rightarrow V_\lambda\) (Martin).
- \((I_0 \text{ reflects } I_1)\) Suppose there is \(j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})\) an elementary embedding with \(\text{crit}(j) < \lambda\). Then there is a \(\tilde{\lambda} < \lambda\) and an elementary embedding \(V_{\lambda+1} \rightarrow V_{\lambda+1}\) (Woodin).
- Assume there exists \(j : L_{\lambda+++++\omega+1}(V_{\lambda+1}) \rightarrow L_{\lambda+++++\omega+1}(V_{\lambda+1})\) elementary. Then there exists a \(\tilde{\lambda} < \lambda\) such that there is an elementary embedding \(k : L_{\tilde{\lambda}+}(V_{\tilde{\lambda}+1}) \rightarrow L_{\tilde{\lambda}+}(V_{\tilde{\lambda}+1})\) with \(\text{crit}(k) < \tilde{\lambda}\) (Laver).

Laver used a technique called ‘inverse limits’ to get his reflection result.
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**Theorem**

1. $(I_1 \text{ reflects } I_3)$ Suppose there is $V_{\lambda+1} \rightarrow V_{\lambda+1}$ an elementary embedding. Then there is a $\check{\lambda} < \lambda$ and an elementary embedding $V_\lambda \rightarrow V_\lambda$ (Martin).

2. $(I_0 \text{ reflects } I_1)$ Suppose there is $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ an elementary embedding with $\text{crit}(j) < \lambda$. Then there is a $\check{\lambda} < \lambda$ and an elementary embedding $V_{\lambda+1} \rightarrow V_{\lambda+1}$ (Woodin).

3. Assume there exists $j : L^{\lambda+\omega+1}(V_{\lambda+1}) \rightarrow L^{\lambda+\omega+1}(V_{\lambda+1})$ elementary. Then there exists a $\check{\lambda} < \lambda$ such that there is an elementary embedding $k : L^{\lambda+}(V_{\lambda+1}) \rightarrow L^{\lambda+}(V_{\lambda+1})$ with $\text{crit}(k) < \check{\lambda}$ (Laver).

Laver used a technique called ‘inverse limits’ to get his reflection result.
Reflecting $I_3$ and $I_1$

- Recall that if $\kappa$ is 2-strong then $\kappa$ is a limit of measurable cardinals. This phenomenon is called reflection.
- Does some large cardinal axiom reflect $I_3$, $I_1$, and $I_0$? Yes.

**Theorem**

1. **($I_1$ reflects $I_3$)** Suppose there is $V_{\lambda+1} \rightarrow V_{\lambda+1}$ an elementary embedding. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}} \rightarrow V_{\bar{\lambda}}$ (Martin).

2. **($I_0$ reflects $I_1$)** Suppose there is $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ an elementary embedding with $\text{crit}(j) < \lambda$. Then there is a $\bar{\lambda} < \lambda$ and an elementary embedding $V_{\bar{\lambda}+1} \rightarrow V_{\bar{\lambda}+1}$ (Woodin).

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Laver used a technique called ‘inverse limits’ to get his reflection result.
reflecting $I_0$

**Theorem (C.)**

$(I_0^\# \text{ reflects } I_0)$ Assume there exists an elementary embedding

$$j : L(V_{\lambda+1}^\#) \to L(V_{\lambda+1}^\#)$$

with $\text{crit}(j) < \lambda$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding

$$k : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

with $\text{crit}(k) < \bar{\lambda}$.

The proof uses inverse limits as well.
introduction to $L(V_{\lambda+1})$

inverse limit reflection

$U(j)$-representations

definition of inverse limits

**Definition (Laver)**

An inverse limit $(J, \langle j_i \mid i < \omega \rangle)$ is a tuple such that the following hold:

1. For all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. $\text{crit}(j_0) < \text{crit}(j_1) < \text{crit}(j_2) < \cdots < \lambda$.
3. $\sup_{i<\omega} \text{crit}(j_i) = \bar{\lambda} < \lambda$.
4. $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is defined by: for all $a \in V_{\bar{\lambda}}$,
   \[
   J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a) = (j_0 \circ j_1 \circ \cdots)(a).
   \]

- If $(J, \langle j_i \mid i < \omega \rangle)$ is an inverse limit then we write
  \[
  J = j_0 \circ j_1 \circ \cdots.
  \]

- We can rewrite an inverse limit as a direct limit as follows:
  \[
  J = \cdots \circ j_0(j_1(j_2)) \circ j_0(j_1) \circ j_0.
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picture of an inverse limit
picture of an inverse limit
picture of an inverse limit

\[ \lambda \]

\[ \bar{\lambda} \]

\[ \cdots \]

\[ \operatorname{cr}(j_2) \]

\[ \operatorname{cr}(j_1) \]

\[ \operatorname{cr}(j_0) \]

\[ j_0 \]

\[ j_0(j_1) \]

\[ \cdots \]

\[ \operatorname{cr}(j_0(j_2)) \]

\[ \operatorname{cr}(j_0(j_1)) \]

\[ \operatorname{cr}(j_0(j_0)) \]

\[ j_0(j_1(\bar{\lambda})) \]

\[ \cdots \]

\[ \operatorname{cr}(j_0(j_1(j_2))) \]

\[ \operatorname{cr}(j_0(j_1(j_1))) \]
picture of an inverse limit

\[ \begin{array}{c}
\lambda \\
\bar{\lambda} \\
\cr(j_2) \\
\cr(j_1) \\
\cr(j_0) \\
\end{array} \quad \begin{array}{c}
\lambda \\
\bar{\lambda} \\
\cr(j_2) \\
\cr(j_1) \\
\cr(j_0) \\
\end{array} \quad \begin{array}{c}
\lambda \\
\bar{\lambda} \\
\cr(j_2) \\
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\end{array} \quad \begin{array}{c}
\lambda \\
\bar{\lambda} \\
\cr(j_2) \\
\cr(j_1) \\
\cr(j_0) \\
\end{array} \]

\[
\begin{align*}
\cdots & \quad \cdots \\
\cr(j_0(j_1(j_2))) & \quad \cr(j_0(j_1(j_1))) \\
\cr(j_0(j_1)) & \quad \cr(j_0(j_1)) \\
\cr(j_0) & \quad \cr(j_0) \\
\end{align*}
\]
There are many theorems on inverse limits which take the basic form:

property X for the embeddings $k_i$ for all $i < \omega$

$\Rightarrow$ property X for $K = k_0 \circ k_1 \circ \cdots$

We say that property X transfers to inverse limits.

- For instance for (certain) inverse limits $K = k_0 \circ k_1 \circ \cdots$ we have for any $a \in V_{\lambda+1}$

  $\forall i < \omega (a \in \text{rng } k_i) \rightarrow a \in \text{rng } K$.

- Inverse limit reflection (ILR) is basically the statement that ‘extension to $L_\alpha(V_{\lambda+1})$’ transfers to inverse limits.
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Inverse limit reflection (ILR) is basically the statement that ‘extension to $L_\alpha(V_{\lambda+1})$’ transfers to inverse limits.
inverse limit reflection

Definition

Suppose $\alpha < \Theta$. Then inverse limit reflection at $\alpha$ is the following statement. There exists $\beta < \Theta$, $\bar{\lambda} < \lambda$, and $\bar{\alpha}$ such that if $(J, \langle j_i | i < \omega \rangle)$ is an inverse limit, $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$, and

$$\forall i < \omega \exists \gamma \geq \beta (j_i \text{ extends to } j'_i : L_{\gamma}(V_{\lambda+1}) \to L_{\gamma}(V_{\lambda+1}))$$

then $J$ extends to an embedding

$$\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_{\alpha}(V_{\lambda+1}).$$

Furthermore there is such an inverse limit.

Theorem

Suppose $I_0$ holds at $\lambda$.

1. Inverse limit reflection holds at $\lambda^+$ (Laver).
2. For all $\alpha < \Theta_{\lambda}$, inverse limit reflection holds at $\alpha$ (C.).
**Definition**

Suppose $\alpha < \Theta$. Then *inverse limit reflection at $\alpha$* is the following statement. There exists $\beta < \Theta$, $\bar{\lambda} < \lambda$, and $\bar{\alpha}$ such that if $(J, \langle j_i \mid i < \omega \rangle)$ is an inverse limit, $J : V_{\bar{\lambda}+1} \rightarrow V_{\lambda+1}$, and

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**Theorem**

*Suppose $I_0$ holds at $\lambda$.***

1. *Inverse limit reflection holds at $\lambda^+$ (Laver).*
2. *For all $\alpha < \Theta_\lambda$, inverse limit reflection holds at $\alpha$ (C.).*
Inverse limit reflection seems to be one of the key tools for studying $L(V_{\lambda+1})$, and it was the key tool for the proof of the perfect set property and the result on non-splitting stationary subsets of $\lambda^+$. How to extend this property to hierarchies above $L(V_{\lambda+1})$ is still an open problem.
Can we connect $L(V_{\lambda+1})$ to actual models of determinacy?

The perfect set property was originally shown by Woodin and Shi to hold for subsets of $V_{\lambda+1}$ in $L_{\lambda}(V_{\lambda+1})$. They used a certain representation for subsets of $V_{\lambda+1}$ called a $U(j)$-representation, due to Woodin.

In fact, Woodin showed that $U(j)$-representations give even stronger properties for $L(V_{\lambda+1})$ than those which have been proven using inverse limits, such as a certain generic absoluteness result. However, it still remains unclear how many subsets of $V_{\lambda+1}$ have $U(j)$-representations.
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Theorem (Woodin)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary with $\text{crit}(j) < \lambda$. Then every set $X \subseteq V_{\lambda+1}$, $X \in L\lambda(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

Theorem (C. and Woodin)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary with $\text{crit}(j) < \lambda$. Then every set $X \subseteq V_{\lambda+1}$, $X \in L\lambda(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$. In fact for $\kappa$ the least $\Sigma_1$-gap, every $X \in L\kappa(V_{\lambda+1})$ is $U(j)$-representable.

The proof uses inverse limit techniques along with theorems of Woodin on $U(j)$-representations.
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**Theorem (C. and Woodin)**

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary with $\text{crit}(j) < \lambda$. Then every set $X \subseteq V_{\lambda+1}$, $X \in L_{\lambda^+}(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$. In fact for $\kappa$ the least $\Sigma_1$-gap, every $X \in L_{\kappa}(V_{\lambda+1})$ is $U(j)$-representable.

The proof uses inverse limit techniques along with theorems of Woodin on $U(j)$-representations.
how far do $U(j)$-representations go?

Because the collection of $U(j)$-representations is closed under complements, along with the above theorems, we would expect the following.

**Conjecture ($U(j)$-conjecture)**

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then every set $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ is $U(j)$-representable in $L(V_{\lambda+1})$.

This would be a stark contrast to scales in $L(\mathbb{R})$, but note that $U(j)$-representations seem not to be related to uniformization.
consequences of the $U(j)$-conjecture

**Theorem (Dimonte-Friedman, Woodin independently)**

*If the $U(j)$-conjecture holds then if $I_0$ is consistent, it is consistent to have $I_0$ at some $\lambda$ and the Singular Cardinal Hypothesis fails at $\lambda$.*

LST is a strong determinacy axiom whose consistency is (perhaps?) not known to follow from any large cardinal axiom.

**Theorem (Woodin)**

*If the $U(j)$-conjecture holds then under enough large cardinals, there is a model of determinacy which satisfies LST.*

The above theorems already show that models of AD do not ‘peter out’ at the level of $I_0$. 

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The above theorems already show that models of AD do not ‘peter out’ at the level of $I_0$. 

open problems and additional topics

There are many (completely) open problems related to $L(V_{\lambda+1})$ including uniformization, Wadge reducibility, regularity of $\lambda^{++}$, $\lambda^{+++}$, etc., partition properties, ...