Rank-into-rank hypotheses

Rank-into-rank hypotheses are at the top of the large cardinal hierarchy. These are the most known:

1. Let \( \exists \lambda \exists j : V_\lambda < V_{\lambda+1} \)
2. Let \( \exists \lambda \exists j : V_{\lambda+1} < V_{\lambda+1} \)
3. Let \( \lambda \exists j : L(V_{\lambda+1}) < L(V_{\lambda+1}), \) with \( \operatorname{crit}(j) < \lambda. \)

The critical points of such elementary embedding are: measurable, \( n \)-huge for every \( n \), supercompact (and strongly compact) in \( V_\lambda \), etc...

On the other hand, \( \lambda \) is singular, strong limit and Rowbottom.

**Question**

Are these hypotheses consistent with different behaviours of the power function? E.g., is \( \mathbb{I} \) consistent with \( \mathbb{GCH} \)?

**Hint:** usually they are, if we restrict to regulars. On the singulars, the situation is much more complicated (see below).

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**Strong limit singular cardinals**

**Strategy** as before. Failing that, creating \( \mathbb{I}^*(\kappa) \) in a forcing extension, where \( \kappa \) satisfies what we want.

**Fundamentally limiting Theorem:**

**Theorem (Solovay)**

Let \( \kappa \) be a strongly compact cardinal. Let \( \lambda \) be a singular strong limit cardinal greater than \( \kappa \). Then \( 2^\lambda = \lambda^+ \).

In (1) and (2) the restrictions are independent from \( \mathbb{I}^* \).

For (3), as \( \kappa_0 \) is strongly compact in \( V_\lambda \), all the singular strong limit cardinals between \( \kappa_0 \) and \( \lambda \) satisfy \( \mathbb{GCH} \). The only remaining case is \( \lambda \).

Gitik defined a forcing that forces \( 2^\lambda = \lambda^+ \), and does not add sets in \( V_\lambda \). For \( \mathbb{I}^* \), this suffices.

**Remark**

This means that we can force \( \lambda \) to be smaller than any strongly compact cardinal!

It does not work for \( \mathbb{I} \): to force this, many elements are added to \( V_{\lambda+1} \) so that it is impossible to have names for all of them in the domain of \( j \). We use a workaround.

Suppose \( \mathbb{I}^*(\lambda) \). Let \( j : L(V_{\lambda+1}) < L(V_{\lambda+1}) \).

Let \( j_{\omega_2} : (V_{\lambda+1})^{V_{\lambda+1}} \to M_\omega \) be the \( \omega \)-th iterate of \( j \). Then \( j_{\omega_2}(\kappa_0) = \lambda \), therefore \( \lambda \) in \( M_\omega \) is a large cardinal with all the properties of \( \kappa_0 \). In particular it is measurable.

Note that \( \langle \kappa_i : i \in \omega \rangle \) is generic for the Prikry forcing on \( \lambda \) in \( M_\omega \).

**Theorem (Woodin)**

Let \( \alpha \) be “big enough” (see Scott Cramer’s seminar), e.g., less than \( \lambda^{\beta^+} \). Then there exists \( \exists \pi : (L_\alpha(V_{\lambda+1}))^{M_\omega(V_{\lambda+1})} \to L_\alpha(V_{\lambda+1}) \).

Therefore \( M_\omega(V_{\lambda+1}) \models \mathbb{I}(\lambda) \). By elementarity of \( j_{\omega_2} \), there exists a forcing extension of \( V \) that satisfies \( \mathbb{I}(\kappa_0) \). If \( V \models \mathbb{I}(\kappa_0) \) we have the theorem:

**Theorem**

Suppose \( \mathbb{I}(\lambda) \) and let \( E \) be an Easton function such that \( E \) is definable over \( V_\lambda \). Then there exists a forcing extension \( V[G] \) in which \( \mathbb{I}(\kappa_0) \) holds and \( 2^{\lambda^+} = E(\kappa_0) \).

**Finer results**

The pedantic people can consider “10mini” hypotheses, i.e.,

\[ \exists \lambda \exists j : L(V_{\lambda+1}) < L(V_{\lambda+1}), \] with \( \operatorname{crit}(j) < \lambda. \)

The theorem works in the same way, as long as \( \beta \) is less than the \( \alpha \) above.

**Corollary (Woodin)**

Suppose the \( U(j) \)-conjecture is true. Then \( \mathbb{I}(\lambda) \) is consistent with \( 2^\lambda = \lambda^{\beta^+} \).

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**Open Problems**

- What is the real consistency strength of \( \mathbb{I} \) and the failure of \( \mathbb{GCH} \) at \( \lambda \)?
- Can \( \mathbb{I} \) be the first ordinal in which \( \mathbb{GCH} \) fails?
- Can we derive \( \mathbb{I} \) and the failure of \( \mathbb{GCH} \) at \( \lambda \) from something else, maybe \( \exists \lambda \exists j : L(V_{\lambda+1}, (V_{\lambda+1})^<) < L(V_{\lambda+1}, (V_{\lambda+1})^<_\mathbb{I}) \)? maybe \( \mathbb{I} \) itself?