Very Large Cardinals and Power Function

Vincenzo Dimonte (in collaboration with Sy Friedman)
TU Wien

Rank-into-rank hypotheses

Rank-into-rank hypotheses are at the top of the large cardinal hierarchy. These are the most known:

I3 $\exists\lambda\ \exists j : V_\lambda \prec V_\lambda$
I1 $\exists\lambda\ \exists j : V_{\lambda+1} \prec V_{\lambda+1}$
I0 $\exists\lambda\ \exists j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$.

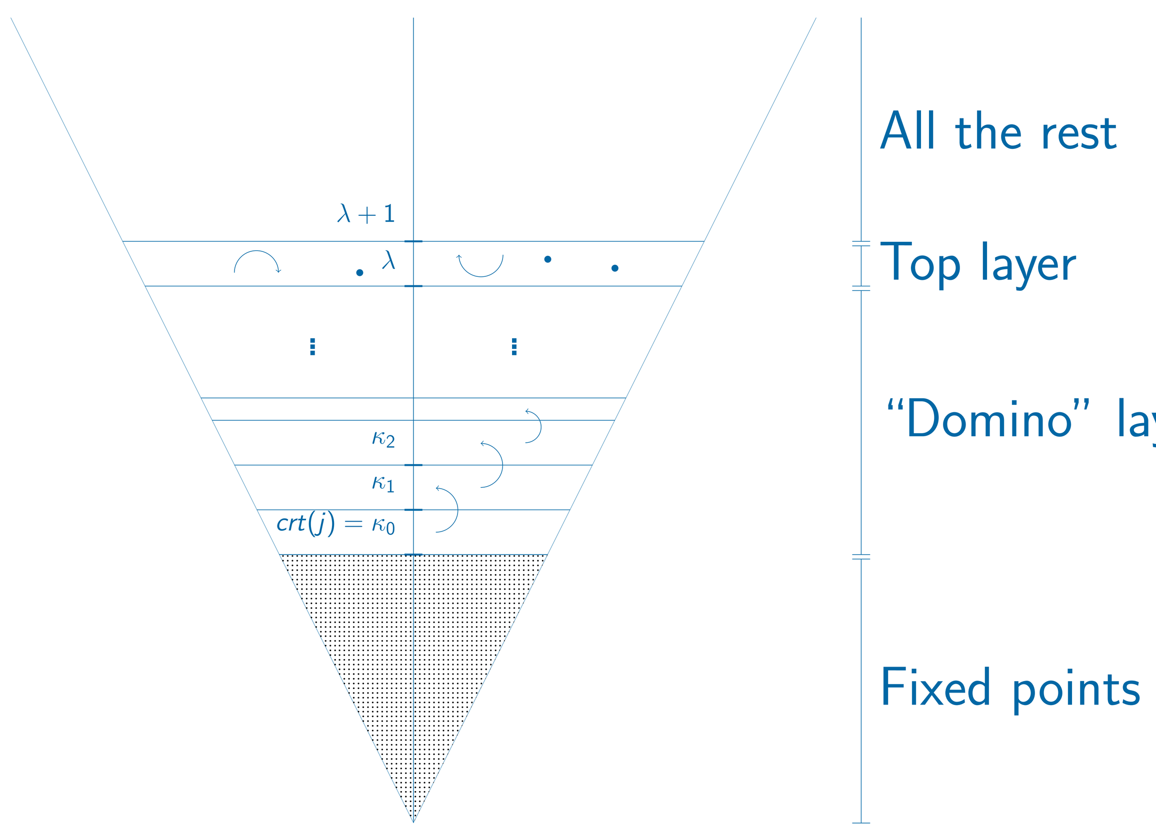
The critical points of such elementary embedding are: measurable, n-huge for every n, supercompact (and strongly compact) in V_λ, etc...
On the other hand, λ is singular, strong limit and Rowbottom.

Question

Are these hypotheses consistent with different behaviours of the power function? E.g., is I0 consistent with GCH?

Hint: usually they are, if we restrict to regulars. On the singulars, the situation is much more complicated (see below).

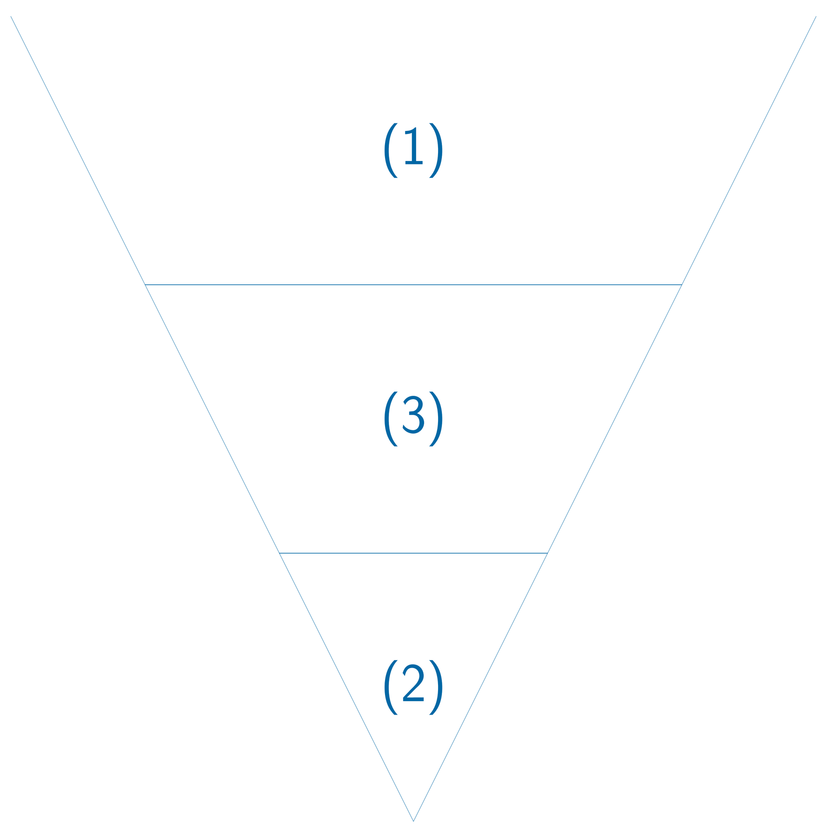
Geography of rank-into-rank hypotheses



- Fixed points: there exists an ordinal κ_0, called critical point, such that $j(\kappa_0) > \kappa_0$ but for any $\beta < \kappa_0$, $j(\beta) = \beta$. Then for any $x \in V_{\kappa_0}$, $j(x) = x$. κ_0 is a large cardinal.
- "Domino" layers: define $j(\kappa_i) = \kappa_{i+1}$. Then for any $x \in V_{\kappa_{n+1}} \setminus V_{\kappa_n}$, $j(x) \in V_{\kappa_{n+2}} \setminus V_{\kappa_{n+1}}$. For Kunen's Theorem, λ must be the supremum of the κ_n's.
- Top layer: $j(\lambda) = \lambda$ and $j(\lambda+1) = \lambda+1$, so for any $x \in V_{\lambda+1} \setminus V_\lambda$, $j(x) \in V_{\lambda+1} \setminus V_\lambda$, but it is no necessarily a fixed point.
- All the rest: very interesting, but not relevant to I3, I1 or this work.

Regular cardinals

Strategy: forcing $2^\kappa = \kappa^{++}$ (or similars) and proving that I^* is preserved in the forcing extension.



1. If $(V_{\lambda+1})^{V[G]} = V_{\lambda+1}$, then there is nothing to prove, I^* is untouched by the forcing.
2. If $\mathbb{P} \in V_{\kappa_0}$, then there is a standard way to deal with the problem: as the forcing is "small", every element of $(V_{\lambda+1})^{V[G]}$ has a name in $V_{\lambda+1}$. Define $j(\tau_G) = j(\tau)_G$. This extension witnesses I^*.
3. Beware of the "domino" effect: if $2^{\kappa_0} = (\kappa_0)^{++}$, at the same time it must be $2^{\kappa_1} = (\kappa_1)^{++}$, $2^{\kappa_2} = (\kappa_2)^{++}$, and so on, by elementarity. It is natural to employ definable iterations.

Theorem

Theorem Suppose $I^*(\lambda)$ and let E be an Easton function such that $E \upharpoonright \lambda$ is definable over V_λ. Then there exists a forcing extension $V[G]$ that preserves $I^*(\lambda)$ such that $V[G] \vDash \mathrm{GCH}$, or for any κ regular, $V[G] \vDash 2^\kappa = E(\kappa)$.

Strong limit singular cardinals

Strategy: as before. Failing that, creating $I^*(\kappa)$ in a forcing extension, where κ satisfies what we want.
Fundamentally limiting Theorem:

Theorem (Solovay)

Let κ be a strongly compact cardinal. Let λ be a singular strong limit cardinal greater than κ. Then $2^\lambda = \lambda^+$.

In (1) and (2) the restrictions are independent from I^*.
For (3), as κ_0 is strongly compact in V_λ, all the singular strong limit cardinals between κ_0 and λ satisfy GCH. The only remaining case is λ.
Gitik defined a forcing that forces $2^\lambda = \lambda^{++}$, and does not add sets in V_λ. For I3, this suffices.

Remark

This means that we can force λ to be smaller than any strongly compact cardinal!

It does not work for I1: to force this, many elements are added to $V_{\lambda+1}$, so that it is impossible to have names for all of them in the domain of j. We use a workaround.

Suppose $I0(\lambda)$. Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$.
Let $j_{0,\omega} : L(V_{\lambda+1}) \prec M_\omega$ be the ω-th iterate of j. Then $j_{0,\omega}(\kappa_0) = \lambda$, therefore λ in M_ω is a large cardinal with all the properties of κ_0. In particular it is measurable.
Note that $\langle \kappa_i : i \in \omega\rangle$ is generic for the Prikry forcing on λ in M_ω, so

Theorem (Woodin)

Let α be "big enough" (see Scott Cramer's seminar), e.g., less than λ^+. Then there exists $\pi : (L_\alpha(V_{\lambda+1}))^{M_\omega[\langle[\kappa_i : i\in\omega\rangle]} \prec L_\alpha(V_{\lambda+1})$.

Therefore $M_\omega[\langle \kappa_i : i \in \omega] \vDash I1(\lambda)$. By elementarity of $j_{0,\omega}$, there exists a forcing extension of V that satisfies $I1(\kappa_0)$. If $V \vDash 2^{\kappa_0} = E(\kappa_0)$ we have the theorem:

Theorem

Suppose $I0(\lambda)$ and let E be an Easton function such that $E \upharpoonright \lambda$ is definable over V_λ. Then there exists a forcing extension $V[G]$ in which $I1(\kappa_0)$ holds and $2^{\kappa_0} = E(\kappa_0)$.

Finer results

The pedantic people can consider "I0mini" hypotheses, i.e,
$\exists\lambda\ \exists j : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$.

The theorem works in the same way, as long as β is less than the α above.

Corollary (Woodin)

Suppose the $U(j)$-conjecture is true. Than $I0(\lambda)$ is consistent with $2^\lambda = \lambda^{++}$.

Open Problems

- What is the *real* consistency strength of I1 and the failure of GCH at λ?
- Can λ be the first ordinal in which GCH fails?
- Can we derive I0 and the failure of GCH at λ from something else, maybe $\exists\lambda\ \exists j : L(V_{\lambda+1}, (V_{\lambda+1})^\sharp) \prec L(V_{\lambda+1}, (V_{\lambda+1})^\sharp)$, maybe I0 itself?