

Universal and ultrahomogeneous Polish metric structures

Michal Doucha

Institute of Mathematics, Czech Academy of Sciences, Prague

Urysohn universal space

The Urysohn universal metric space \mathbb{U} is a Polish metric space which is both universal and homogeneous for the class of all finite metric spaces; i.e.

- It contains an isometric copy of any finite metric space (universality)
- Any finite partial isometry $\phi : \{x_1, \dots, x_n\} \subseteq \mathbb{U} \rightarrow \{y_1, \dots, y_n\} \subseteq \mathbb{U}$ can be extended to an isometry $\tilde{\phi} \supseteq \phi : \mathbb{U} \rightarrow \mathbb{U}$ on the whole space (ultrahomogeneity or ω -homogeneity)

These two properties imply that \mathbb{U} is the only such a space up to isometry and it also contains an isometric copy of every separable metric space.

Our motivation

is to enrich \mathbb{U} with some additional structure so that this metric structure is still universal and ultrahomogeneous (the precise definitions are below). This gives us a general way of coding such structures (via the Effros-Borel structure of closed substructures of the universal one).

For clarification, we include the two definitions below. They are safe to skip unless the statements of our theorems are not clear.

Polish metric structures

Let Z_1, \dots, Z_k be a list of Polish metric spaces. A finite or countably infinite set \mathcal{O} is called a signature if it consists of symbols for closed sets. Moreover, there is a function $a : \mathcal{O} \rightarrow ([0, \dots, k] \times \mathbb{N})^{<\omega}$; i.e. to each symbol from \mathcal{O} it assigns a finite sequence of elements (a, b) where $0 \leq a \leq k$ and $b \in \mathbb{N}$. By $a_F(n, i)$, for $i \in \{1, 2\}$, we denote the i -th coordinate of the n -th element of $a(F)$.

A Polish metric structure of signature \mathcal{O} is a Polish metric space (X, d) such that for every $F \in \mathcal{O}$ there is a closed set $F_X \subseteq Z_{a_F(1,1)}^{a_F(1,2)} \times \dots \times Z_{a_F(|a(F)|,1)}^{a_F(|a(F)|,2)}$, where by Z_0 we denote X .

Universality and ultrahomogeneity

Let (X, d, \mathcal{O}_X) be a Polish metric space of some signature \mathcal{O} . We say that (X, d, \mathcal{O}_X) is universal if for any Polish metric space (Y, d, \mathcal{O}_Y) of the same signature \mathcal{O} there is an isometric embedding $\phi : Y \hookrightarrow X$ that moreover reduces F_Y into F_X (for every $F \in \mathcal{O}$): i.e. for any (y_1, \dots, y_n) , if $I \subset \{1, \dots, n\}$ is the set of coordinates such that $y_i \in Y$ iff $i \in I$, we have $(y_1, \dots, y_n) \in F_Y \Leftrightarrow (x_1, \dots, x_n) \in F_X$, where $x_i = \phi(y_i)$ if $i \in I$ and $x_i = y_i$ otherwise.

We say that (X, d, \mathcal{O}_X) is ultrahomogeneous if any isomorphism between two finite (metric) substructures $(F_1, d, \mathcal{O}_{F_1})$ and $(F_2, d, \mathcal{O}_{F_2})$ of (X, d, \mathcal{O}_X) extends to an automorphism of the whole (X, d, \mathcal{O}_X) .

Theorem 1

Let $n_1 \leq \dots \leq n_m$ be an arbitrary non-decreasing sequence of natural numbers. Then there exist closed relations (subsets) $F_{n_i} \subseteq \mathbb{U}^{n_i}$, for $i \leq m$, such that the structure $(\mathbb{U}, F_{n_1}, \dots, F_{n_m})$ is universal and ultrahomogeneous for the class of Polish metric spaces equipped with closed relations of arities n_1, \dots, n_m .

Moreover, it is unique (up to isometry preserving the relations) with this property.

Theorem 2

Let K be an arbitrary compact metric space, Z an arbitrary Polish metric space and $L \in \mathbb{R}^+$. Then there exist a closed subset $C \subseteq \mathbb{U} \times K$ and an L -Lipschitz function $F : \mathbb{U} \rightarrow Z$ such that the structure (\mathbb{U}, C, F) is universal and ultrahomogeneous for the class of Polish metric spaces equipped with a closed subset of the space itself times K and having an L -Lipschitz function to Z . It is again unique.

Comments on the proofs

As in the case of the universal and ultrahomogeneous plain Polish metric space, i.e. the Urysohn universal space, these universal and ultrahomogeneous Polish metric structures can be constructed either via Fraïssé theory or by a generalized Katětov extension technique ([2]).

Let us illustrate the latter with a structure from Theorem 1 as an example: with $m = 1$ and $n_1 = 2$. Let (X, d_X, R) be a Polish metric space equipped with a closed binary relation $R \subseteq X^2$. Consider the set of all quadruples (f, d, d_L, d_R) where

- f is a Katětov function on (X, d_X) , i.e. a 1-Lipschitz real function on (X, d_X) such that $\forall x, y \in X (d_X(x, y) \leq f(x) + f(y))$; it corresponds to a possible one-point metric extension of (X, d_X) where the distance between the new point \dot{f} and any $x \in X$ is $f(x)$
- $d \in \mathbb{R}_0^+$ is a non-negative real number which determines the distance (in the sum metric) of (\dot{f}, \dot{f}) from F_{n_1} (see Theorem 1) satisfying

$$d_X((x, y), R) - f(x) - f(y) \leq d \leq d_X((x, y), R) + f(x) + f(y)$$

where $x, y \in X$

- d_L (resp. d_R) is a function from X to \mathbb{R}_0^+ such that $d_L(x)$ (resp. $d_R(x)$), for $x \in X$, determines the distance (in the sum metric) of (\dot{f}, x) (resp. (x, \dot{f})) from F_{n_1} satisfying

$$d_X((y, z), R) - f(y) - d_X(z, x) \leq d_L(x) \leq d_X((y, z), R) + f(y) + d_X(z, x)$$

where $y, z \in X$, and analogously for d_R

Analogously as in the Katětov construction ([2]) of the Urysohn space, these Katětov quadruples can be used to construct the corresponding universal and ultrahomogeneous Polish metric structure.

Moreover, by a similar technique as Gao and Kechris ([1]) used in their classification of Polish metric spaces we can obtain the following corollary.

Corollary

Let $(\mathbb{U}, \mathcal{O})$ be some universal and ultrahomogeneous Polish metric structure from Theorem 1 or Theorem 2. The relation of isometric isomorphism on $F((\mathbb{U}, \mathcal{O}))$ (closed substructures of $(\mathbb{U}, \mathcal{O})$) is Borel reducible to an orbit equivalence relation induced by an action of the group of all isometric automorphisms on $(\mathbb{U}, \mathcal{O})$.

Some classes of Polish groups admitting universal objects

Let \mathcal{K} be some class of Polish groups (closed under taking closed subgroups). We say that \mathcal{K} has a universal object if there is some $G \in \mathcal{K}$ such that for every $H \in \mathcal{K}$ there is a topological isomorphic embedding of H onto a closed subgroup of G .

Let us consider some natural classes of Polish groups:

- The class of all Polish groups has a universal object - $\text{Iso}(\mathbb{U})$ or $\text{Homeo}([0, 1]^\omega)$ ([5],[4]).
- The class of all abelian Polish groups has a universal object ([3]).
- The class of all locally compact Polish group does not have a universal object.

Consider the class of all abelian periodic (torsion) Polish groups. We can use the previous techniques to show that this class has a universal object, and even something stronger holds.

Theorem 3

There exists a universal and ultrahomogeneous periodic abelian Polish metric group. In particular, the class of all abelian periodic Polish groups has a universal object.

Comments

Let us comment on the statement of Theorem 3. There exists a periodic abelian Polish metric group G such that any other periodic abelian Polish metric group can be embedded onto a closed subgroup of G by an isometric isomorphism.

Moreover, any isometric homomorphism between two finitely generated subgroups of G extends to an isometric automorphism of G .

Since any periodic abelian Polish group admits a (both-sided invariant) compatible complete metric, the assertion about universal object follows.

References

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