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Finite Solutions for Infinite Games

Abstract

Our work focuses on the development of tools, such as the hierarchization and localization, in order to find explicit winning strategies for some particular classes of well-founded combinatorial games.
Specifically, this work relates to two-player well-founded games which satisfy the symmetry property and also recursively split into independent subgames.
A classical example, which can be kept in mind as a reference, is the game of Sprouts.

Symmetric games

A combinatorial game is a two-player turn-based game with perfect information and no random elements, a classical example being the game of Chess. While the rules of Chess would allow loops (or draws at best) we restrict our attention to well-founded games: this ensures that these games are determined.
Among these games we restrict further to the ones which satisfy the symmetry property:
if A,B are two positions and a player can move from A to B, then also the other player can move from A to B.
A symmetric game can be viewed as a general well-founded relation on a set, and in case it satisfies extensionality it can also be viewed (by Mostowski's collapse) as a set.

Independent games

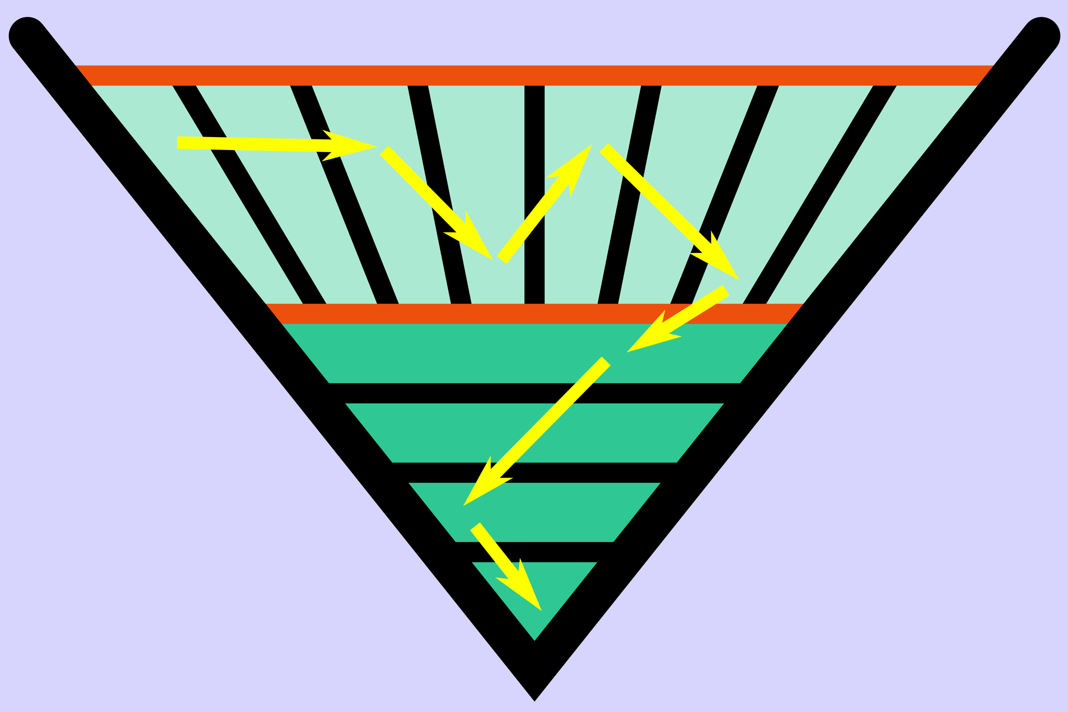
Given two positions A, B the compound position A∗B is a new position where the moving player chooses one of the two positions and moves from that position without altering the other one. In the case of sets, it is defined by well-founded recursion as follows: A∗B={ a∗B, A∗b | A→a, B→b }.
If C=A∗B then we say that C allows a decomposition in two independent subpositions.
To check whether such a compound is a winning or losing positions we can define a suitable sum ⊕ and product ⊗ on the ordinals which turn them into an algebraically closed (class-)field of characteristic 2.
It turns out that there is a surjective monoid homomorphism G from (V,∗) onto (On,⊕) with the class of losing games being the class [0].

Periodic games

In the case of hereditarily finite sets and natural numbers, both G and ⊕ are computable.
Some games are naturally indexed by natural numbers, and many of them show a peculiar behavior: computing the G function on their positions yields (ultimately) periodic results.
An example is this game: given a line of stones, a move consists in removing two adjacent stones. If a player can't move, he loses the game. This game is ultimately periodic with period length 34.
Along with the game just shown, many games are actually periodic and some combinatorial arguments assure that being periodic is semidecidable for some classes of games. There are many more games that seem periodic, but there is no known algorithm to check it: one of them is the game of Sprouts.

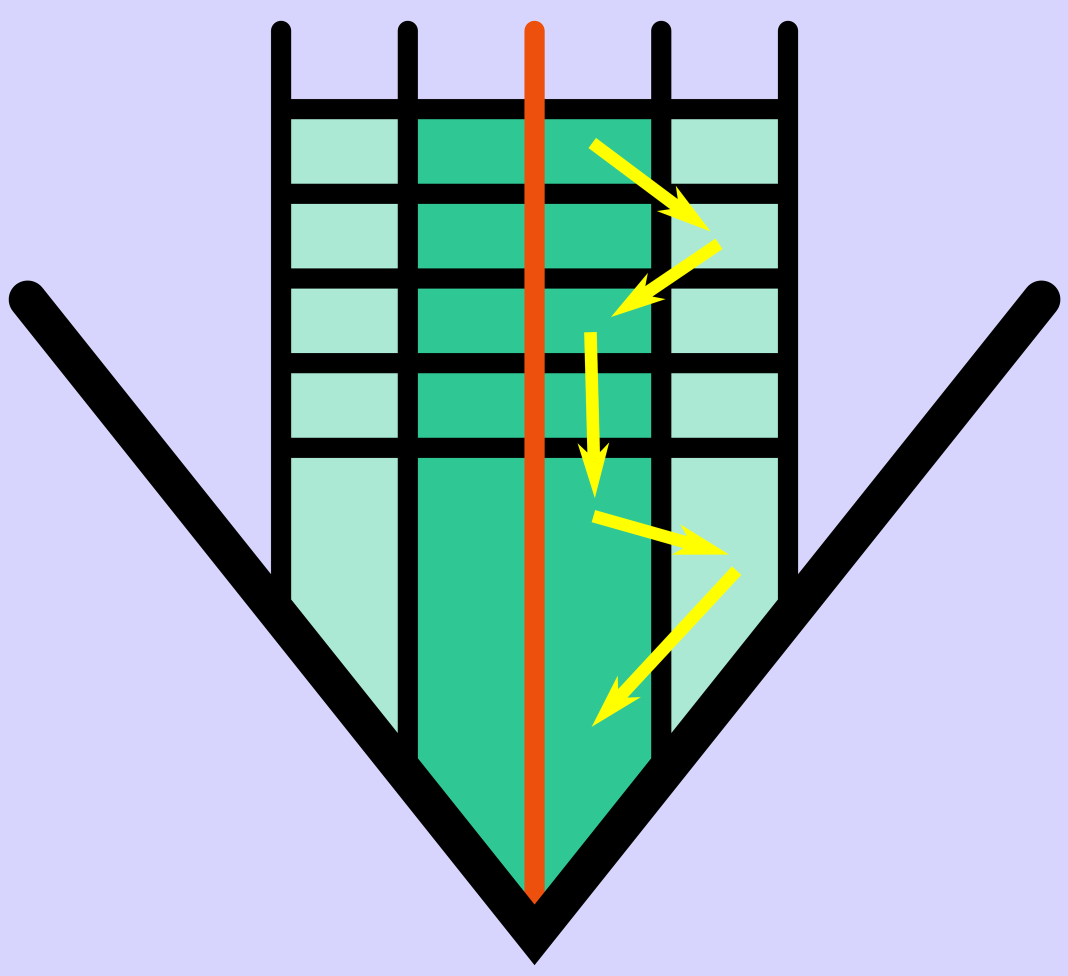
Hierarchization

Branching game such as Sprouts naturally have a hierarchical structure the same way sets have. The set rank, while still very useful, it is often not the best choice in these cases because it does not have a meaningful role in the game (resulting in levels of the hierarchy which are hard to describe).
In fact, it is often convenient to consider other hierarchies where position sequences are weakly decreasing. This approach allows the use of the hierarchy and all its related tools and while providing control on the levels of the hierarchy.
Well-founded recursion and induction allow the use of periodicity theorems on these custom hierarchies.
The picture on the right summarizes a a proof of periodicity, where the sublevels analyzed rest upon the levels where the game is known to be periodic.



Localization

Given a branching game (such as Sprouts), one might want to analyze just a subset A of its positions.
In the case where C is not transitive the hierarchization approach fails. Instead, the idea is to avoid to analyze the full transitive closure of A and instead trying to recursively confine the analysis to a small subset B of its transitive closure.
This is what every chess player automatically does when playing a game: he plays its game with many more rules in his mind with the sole purpose to help him finding the right way for the win.
This approach on one hand allows a faster computation, thus enabling to acquire more information on the game analyzed. On the other hand, in case of periodicity, it allows to provide a solution for all of A in the form of a non-well-founded set.
This is done by choosing a set of central position which should allow a complete analysis of A, then choosing buffer positions to jacket the central positions and isolate them from the rest of the game, while not altering the strategic structure of the game.
In the picture on the right, the central line is A, the dark green is the central part of B and the light green is the buffer part of B.



The game of Sprouts

The game of Sprouts is played by two players, starting with a few spots drawn on a sheet of paper. Players take turns, where each turn consists of drawing a line between two spots (or from a spot to itself) and adding a new spot somewhere along the line. The players are constrained by the following rules:

- The line must not touch or cross any other line or spot;
- No spot may have more than three lines attached to it.

The first player who has no available moves loses.
For a sample game with two spots, see below.

This game has been invented at Cambridge University in the 60s by John H. Conway and Michael S. Paterson.
The game is still unsolved, the standing conjecture is that the starting player has a winning strategy if and only if the number of spots in the starting position modulo 6 is equal to 3, 4 or 5. Computation up to 44 spots (completed in 2011) is still supporting the conjecture.