On the complexity of the embeddability relation between uncountable models

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Joint work with H. Mildenberger

The framework

Framework: Establish connections between (basic) Model Theory (MT) for infinitary languages and Descriptive Set Theory (DST).

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Given an $\mathcal{L}_{\kappa\lambda}$-sentence φ, we set

$$\mathrm{Mod}^{\mu}_{\varphi} = \{x \in \mathrm{Mod}^{\mu}_{\mathcal{L}} \mid x \models \varphi\}.$$

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Let's focus on the second item!

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Borel reducibility: $R \leq_B S$ iff $\exists$ Borel $f \colon \mathrm{dom}(R) \to \mathrm{dom}(S)$ such that

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Intuitively: if $R \leq_B S$ then R is not more complicated than S.

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Every two uncountable standard Borel spaces are Borel isomorphic: hence w.l.o.g. one may assume $\mathrm{dom}(R) = 2^{\omega}$.

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This is abbreviated with: $\sqsubseteq\restriction \mathrm{Mod}^{\omega}_{\mathcal{L}}$ is (strongly) invariantly universal.

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Fix an uncountable cardinal κ. The generalized Cantor space 2^κ consists of all binary κ-sequences endowed with the (bounded) topology τ_b generated by the (cl)open sets $\mathbf{N}_s = \{x \in 2^\kappa \mid s \subseteq x\}$ for $s \in {}^{<\kappa}2$.

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Examples: κ^κ (endowed with the analogous of τ_b), any $A \in \mathrm{Bor}_\kappa(2^\kappa)$, ...

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- given a (κ-)analytic quasi-order R, E_R and $\hat{R}$ are defined as before.

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Definition (κ^+-Borel (bi-)reducibility and isomorphism)

Let R, S be two (κ-)analytic quasi orders.

κ^+-Borel reducibility: $R \leq_B^\kappa S \iff \exists\ \kappa^+$-Borel $f \colon \mathrm{dom}(R) \to \mathrm{dom}(S)$ reducing R to S;

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So one may again assume w.l.o.g. that $\mathrm{dom}(R) = 2^\kappa$.

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Proof.

$(2^\kappa, \tau_b)$ is κ-compact: since $h \circ f$ is continuous, $(h \circ f)(2^\kappa)$ is κ-compact and hence closed in 2^κ (because 2^κ is Hausdorff and κ is regular). Let $U = 2^\kappa \setminus (h \circ f)(2^\kappa)$: then $h^{-1}(U) = \mathrm{Mod}^\kappa_{\varphi_U}$, and hence it is enough to let φ be $\Psi \wedge \neg\varphi_U$. □

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Same argument as before to get that $g^{-1}(\mathbf{N}_s) = \mathrm{Mod}^{\kappa}_{\varphi_s}$ for some $\mathcal{L}_{\kappa^+\kappa}$-sentence φ_s, and then use the generalized Lopez-Escobar theorem (in the other direction w.r.t. the countable case!). □

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Therefore we have shown:

Theorem (M.)

Let κ be a weakly compact cardinal. For every κ-analytic quasi-order R on 2^κ there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that $R \simeq^{\kappa}_{B} \sqsubseteq\restriction \mathrm{Mod}^{\kappa}_{\varphi}$ (i.e. $\sqsubseteq$ on $\mathrm{Mod}^{\kappa}_{\mathcal{L}}$ is strongly invariantly universal). In particular, $\sqsubseteq\restriction \mathrm{Mod}^{\kappa}_{\mathcal{L}}$ is also complete.

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Notice that in the previously sketched argument we heavily used (twice) the fact that the topological space $(2^{\kappa}, \tau_b)$ is κ-compact, which is equivalent to κ being weakly compact: therefore we necessarily need to use different ideas...

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5. $\forall s \in {}^{<\kappa} \exists^{<\omega}(u, v) \in ({}^{<\kappa}2)^2$ *s.t.* $(u, v, s) \in T'$.

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5. $\forall s \in {}^{<\kappa} \exists^{<\omega}(u, v) \in ({}^{<\kappa}2)^2$ *s.t.* $(u, v, s) \in T'$.

The last condition reduces the mentioned "technical problem" to that of finding a branch through a subtree of ${}^{<\kappa}2$ of height κ with all levels finite: hence, if $\mathrm{cof}(\kappa) > \omega$ we are done.

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All the above procedure can be formalized by an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ, so that

$$X \models \varphi \iff \exists x \in 2^\kappa (f(x) \cong X).$$

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Therefore $\mathrm{Sat}(f(2^\kappa)) = \mathrm{Mod}^\kappa_\varphi$.

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To show that g is κ^+-Borel we use a similar procedure: given $s \in {}^{<\kappa}2$ add to φ the $\mathcal{L}_{\kappa^+\kappa}$-formalization of: "the $x \in 2^\kappa$ uniquely decoded from the substructure X_0 of X is such that $x \restriction \mathrm{length}(s) = s$".

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Therefore we obtained:

Theorem (Mildenberger-M.)

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then $\sqsubseteq\restriction \mathrm{Mod}^\kappa_{\mathcal{L}}$ is strongly invariantly universal, that is: for every κ-analytic quasi-order R there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that $R \simeq^\kappa_B \sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$.

In particular, $\sqsubseteq\restriction \mathrm{Mod}^\kappa_{\mathcal{L}}$ is complete.

Some corollaries

Under the same assumptions on κ:

Corollary

Let R be a binary relation on a standard Borel κ-space. Then R is an analytic quasi-order iff $R \simeq_B^\kappa \sqsubseteq \restriction \mathrm{Mod}_\varphi^\kappa$ for some $\mathcal{L}_{\kappa^+\kappa}$-sentence φ.

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Remarks:

- According to Sy's classification, these are results of "Type 1": however, the techniques used in the countable/uncountable cases are radically different.
- The situation for the isomorphism relation $\cong$ on $\mathrm{Mod}_\mathcal{L}^\kappa$ is more delicate: its complexity depends on axioms beyond ZFC and/or on the model we are working in (see Sy's tutorial).

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The counterexample remains valid even if we allow arbitrary reductions!

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- the quasi-order S is just an instantiation of a more general phenomenon, which involves all possible analytic quasi-orders ($\sqsubseteq \restriction \mathrm{Mod}^\kappa_{\mathcal{L}}$ is complete).

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- $\kappa = \omega$: combinatorial trees, i.e. connected acyclic graphs;
- $\kappa > \omega_1$: generalized trees, i.e. partial orders in which the set of predecessors of every element is linearly ordered;
- $\kappa = \omega_1$: mixed trees, i.e. a combination of combinatorial trees and generalized trees. This is because we need ω_1-many $\sqsubseteq$-incomparable countable labels, hence we cannot have used generalized trees for them: $\sqsubseteq$ on countable generalized trees may be a wqo!

Some open problems

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Thank you for your attention!