An axiomatization of the Mathias model.

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Let Q be the Mathias forcing and Q_α its c.s. iteration of the lenght α.

$$Q = \{\langle s, A\rangle \colon s \in [\omega]^{<\omega}, A \in [\omega]^\omega \;\&\; \max(s) < \min(A)\}$$
$$\langle s, A\rangle \leq_Q \langle t, B\rangle \Leftrightarrow t \subseteq s \subseteq t \cup B \;\&\; A \subseteq B$$

It is well known that $Q \cong Bor(2^\omega) \setminus \mathcal{I}$, where $\mathcal{I}$ is the σ-ideal of Ramsey null sets. We would like to have σ-ideals $\mathcal{I}_\alpha$ for $\alpha < \omega_1$ such that $Q_\alpha \cong Bor(X) \setminus \mathcal{I}_\alpha$, where X is some Polish space.

In [3] S. Shelah and O. Spinas defined posets P_α such that $P_\alpha \cong Q_\alpha$ and P_α provides enough absolutness. For $\alpha < \omega_1$ we put $p \in P_\alpha$ if:

- p is a function and $\mathrm{dom}(p) = \alpha$,
- for $\beta < \alpha$ the value $p(\beta)$ is a Borel function, $p(\beta) \colon (2^\omega)^\beta \to Q$.

To each $p \in P_\alpha$ we assign a Borel set $B_p \subseteq (2^\omega)^\alpha$:

- $\alpha = 0$:

$$P_0 = \{\emptyset\} \text{ and } B_\emptyset = (2^\omega)^0 = \{\emptyset\}$$

- $\alpha = \beta + 1$:

$$\langle r_\gamma \rangle_{\gamma<\alpha} \in B_p \Leftrightarrow \langle r_\gamma \rangle_{\gamma<\beta} \in B_{p\restriction\beta} \text{ and}$$

$$p(\beta)\left(\langle r_\gamma \rangle_{\gamma<\beta}\right) \text{ is in the filter induced by } r_\beta$$

$$\text{(i.e. if } p(\beta)\left(\langle r_\gamma \rangle_{\gamma<\beta}\right) = (s, A) \text{ then } s \sqsubseteq r_\beta \text{ and } r_\beta \subseteq s \cup A)$$

- $\alpha = \bigcup \alpha$:

$$\langle r_\gamma \rangle_{\gamma<\alpha} \in B_p \text{ if for all } \beta < \alpha \text{ we have } \langle r_\gamma \rangle_{\gamma<\beta} \in B_{p\restriction\beta}$$

For $p, q \in P_\alpha$ we define $p \leq_{P_\alpha} q$ if

for each $\langle r_\gamma \rangle_{\gamma<\alpha} \in B_p$ and for each $\beta < \alpha$ we have

$$p(\beta)\left(\langle r_\gamma \rangle_{\gamma<\beta}\right) \leq_Q q(\beta)\left(\langle r_\gamma \rangle_{\gamma<\beta}\right)$$

One can show by induction that $p \leq_{P_\alpha} q \Leftrightarrow B_p \subseteq B_q$.

Definition

For $\alpha < \omega_1$ we define the σ-ideal $\mathcal{I}_\alpha \subseteq Bor((2^\omega)^\alpha)$:

$$B \in \mathcal{I}_\alpha \;\Leftrightarrow\; (\forall p \in P_\alpha)\;(\exists q \leq p)\;(B \cap B_q = \emptyset)$$

Fact

$$Bor((2^\omega)^\alpha) \setminus \mathcal{I}_\alpha \cong Q_\alpha$$

We will use these σ-ideals to state an axiom which captures combinatorial properties of the Mathias model. In fact we have a few of its versions.

Distributivity

A complete Boolean algebra B is κ-distributive if

$$\prod_{\alpha<\kappa} \sum_{i\in I_\alpha} u_{\alpha,i} = \sum_{f\in\prod_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$$

for any $\langle u_{\alpha,i}\colon \alpha<\kappa, i\in I_\alpha\rangle \subseteq B$.

For any partially ordered set $\mathbb{P}$, there is a unique complete Boolean algebra $\mathrm{r.o.}(\mathbb{P})$, such that the separative quotient of $\mathbb{P}$ can be densely embedded into $\mathrm{r.o.}(\mathbb{P})$. The distributivity $\mathfrak{h}(\mathbb{P})$ of $\mathbb{P}$ is defined as the minimal κ such that $\mathrm{r.o.}(\mathbb{P})$ is not κ-distributive.

Notation

$\mathfrak{h}$ denotes the distributivity of $\mathcal{P}(\omega)/\operatorname{fin}$,
$\mathfrak{h}(2)$ denotes the distributivity of $(\mathcal{P}(\omega)/\operatorname{fin}) \times (\mathcal{P}(\omega)/\operatorname{fin})$.

The distributivity game

Let $\mathbb{P}$ be a partial order such that $\mathrm{r.o.}(\mathbb{P})$ is homogenous and let α be an ordinal. Consider the game $G(\mathbb{P}, \alpha)$ between two players PI and PII, which lasts α rounds. At step γ, PI plays (if possible) $p^I_\gamma \in \mathbb{P}$ which is below p^{II}_β for every $\beta < \gamma$, then PII responds with $p^{II}_\gamma \leq_{\mathbb{P}} p^I_\gamma$. PI wins iff he can always pick some legal p^I_γ and there is some $p \in \mathbb{P}$ such that $p \leq p^{II}_\beta$ for all $\beta < \alpha$.

The following conditions are equivalent:

- $\mathbb{P}$ is κ-distributive,
- PII has no winning strategy in the game $G(\mathbb{P}, \alpha)$.

The Covering Property Axiom (CPA)

The CPA game:

Players are Adam and Eve. The game lasts ω_1 rounds.
At each step $\gamma < \omega_1$ Adam plays a triple $(\alpha_\gamma, A_\gamma, f_\gamma)$, where:

- α_γ is a countable ordinal,
- $A_\gamma \in Bor((2^\omega)^{\alpha_\gamma}) \setminus \mathcal{I}_{\alpha_\gamma}$
- $f_\gamma \colon A_\gamma \to 2^\omega$ is a Borel function.

Then Eve responds with some $E_\gamma \subseteq A_\gamma$ from $Bor((2^\omega)^{\alpha_\gamma}) \setminus \mathcal{I}_{\alpha_\gamma}$.
Adam wins iff

$$\bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = 2^\omega$$

CPA

The axiom CPA says that CH fails and Eve has no winning strategy in the CPA game.

Theorem

$$Q_{\omega_2} \Vdash \mathrm{CPA}$$

Sketch of proof:
Fix a CPA-strategy $\sigma \in V[G_{\omega_2}]$ of Eve.
Find $\alpha < \omega_2$ such that $\sigma \restriction V[G_\alpha] \in V[G_\alpha]$.
Wlog $V[G_\alpha] \models \mathrm{CH}$.
Therefore there are $\aleph_1$ possible moves of Adam in $V[G_\alpha]$.
Adam plays all these moves during the game.
This is his winning counterplay against σ.

$\square$

Notation
For $\alpha < \omega_1$ let $\langle \dot{r}_\xi \colon \xi < \alpha \rangle$ be the cannonical name for the sequence of $Bor((2^\omega)^\alpha) \setminus \mathcal{I}_\alpha$-generic reals.

Consequences of CPA

Fact

Let $\mathcal{M}$ [$\mathcal{N}$] denote the σ-ideal of meager [null] sets. Then

$$\mathrm{CPA} \vdash \mathrm{cov}(\mathcal{M}) = \aleph_1 = \mathrm{cov}(\mathcal{N})$$

Proof:

We describe a CPA-strategy σ of Eve. Let $\gamma < \omega_1$ and $(\alpha_\gamma, A_\gamma, f_\gamma)$ be the γ-th move of Adam. Eve plays $E_\gamma \subseteq A_\gamma$ which is an $\mathcal{I}_\alpha$-positive set such that $f_\gamma[E_\gamma]$ is meager [null] in 2^ω.
(Such E_γ exists. Otherwise A_γ would force $f_\gamma(\langle \dot{r}_\xi \colon \xi < \alpha \rangle)$ to be a Cohen [random] real, but Q_{α_γ} adds no Cohen [random] reals.)
By CPA, Adam has a winning counterplay against σ. The play gives

$$\bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = 2^\omega$$

where each $f_\gamma[E_\gamma]$ is meager [null]. $\square$

The distributivity of $\mathbb{R}^*$

For $A, B \subseteq \mathbb{R}$ we say $A \subseteq^* B$ if $A \setminus B$ is bounded. Let $\mathbb{R}^*$ be the set of all unbounded open subsets of $\mathbb{R}$, with the ordering $\subseteq^*$.

A. Dow proved in [2] that $Q_{\omega_2} \Vdash \mathfrak{h}(\mathbb{R}^*) = \omega_1$.

Theorem

$$\mathrm{CPA} \vdash \mathfrak{h}(\mathbb{R}^*) = \omega_1$$

The following lemma gives a strategy σ for Eve; Adam's winning counterplay against σ allows us to define a winning strategy for PII in the distributivity game $G(\mathbb{R}^*, \omega_1)$.

Lemma (A. Dow, [2])
Suppose that $\mathbb{P}$ is a poset that has the Laver property and that G is $\mathbb{P}$-generic. Let $W \in V[G]$ be an unbounded subset of $\mathbb{R}$. Then there is a dense open subset $U \in V$ of $\mathbb{R}$ such that $W \setminus U$ is unbounded.

The Covering Property Axiom with Sections (CPAs)

For $X \subseteq (2^\omega)^\alpha$ and $r \in 2^\omega$ let $(X)_r$ denote the section of X at r, i.e

$$(X)_r = \{x \in X \colon x(0) = r\}$$

The CPAs game:

Rules of the CPAs game are the same as in the CPA game, except that Adam wins iff there exists $r \in 2^\omega$ such that

$$\bigcup_{\gamma\in\omega_1} f_\gamma[(E_\gamma)_r] = 2^\omega$$

Obviously CPAs is stronger than CPA. It captures some other properties of the Mathias model:

Theorem

- $Q_{\omega_2} \Vdash \mathrm{CPAs}$
- $\mathrm{CPAs} \vdash \mathfrak{h} > \omega_1$
- $\mathrm{CPAs} \vdash$ Borel Conjecture

The Strong Covering Property Axiom (SCPA)

The SCPA(σ) game:
Fix a CPA-strategy σ of Eve. Consider the game SCPA(σ):
At each step $\gamma < \omega_1$ Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$ as in the CPA game.
Let $E'_\gamma = \sigma\left(\langle \alpha_\xi, A_\xi, f_\xi \rangle_{\xi \leq \gamma}\right)$ be the answer indicated by σ.
Eve plays $E_\gamma \subseteq E'_\gamma$ such that $E'_\gamma \setminus E_\gamma \in \mathcal{I}_\alpha$.
Adam wins if

$$\bigcup_{\gamma < \omega_1} f_\gamma[E_\gamma] = 2^\omega$$

SCPA
The axiom SCPA says that CH fails and for every CPA-strategy σ of Eve, Adam has a winning strategy in the SCPA(σ) game.

Roughly speaking, CPA says that if Adam knows the strategy of Eve, he can win. SCPA says that he can win even if Eve is allowed to modify slightly the answers indicated by her strategy.

Theorem

$$Q_{\omega_2} \Vdash \mathrm{SCPA}$$

The distributivity number $\mathfrak{h}(2)$

S. Shelah and O. Spinas proved in [3] that $Q_{\omega_2} \Vdash \mathfrak{h}(2) = \omega_1$. This result was also considered by A. Dow in [1].

Theorem

$$\mathrm{SCPA} \vdash \mathfrak{h}(2) = \omega_1$$

For the proof we need the following Lemma which was essentially showed in [1]:

Lemma

Suppose $\mathbb{P} = Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property. Let $\dot{r}$ be the cannonical name for the Q-generic real. Assume that

$$\mathbb{P} \Vdash \dot{s} \subseteq \omega \quad \& \quad \forall i \in \omega \; |\dot{s} \cap \dot{r}(i)| \leq i.$$

Then for each $\langle a, A' \rangle * q \in \mathbb{P}$ there exists infinite $A \subseteq A'$ such that for each infinite $C', D' \subseteq \omega$ there are infinite $C \subseteq C', D \subseteq D'$ such that

$$\langle a, A \rangle * q \Vdash |C \cap \dot{s}| < \aleph_0 \text{ or } |D \cap \dot{s}| < \aleph_0.$$

The lemma gives a CPA-strategy σ of Eve. By SCPA, Adam has a winning SCPA(σ)-strategy. We use it to describe a winning strategy for PII in the distributivity game $G((\mathcal{P}(\omega)/\operatorname{fin}) \times (\mathcal{P}(\omega)/\operatorname{fin}), \omega_1)$.

The covering of the Marczewski null σ-ideal s_0

Definition

$X \subseteq 2^\omega$ is Marczewski null ($X \in s_0$) if for each perfect $P \subseteq 2^\omega$ there exists perfect $Q \subseteq 2^\omega$ such that $X \cap Q = \emptyset$.

Lemma

Suppose $\mathbb{P} = Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property and $\mathbb{P} \Vdash \dot{s} \in 2^\omega$. Then for each $\langle a, A' \rangle * q \in \mathbb{P}$ there exists infinite $A \subseteq A'$ and countable $R \subseteq 2^\omega$ such that for every closed $F \subseteq 2^\omega \setminus R$ we have

$$\langle a, A \rangle * q \Vdash \dot{s} \notin F$$

Using the Lemma we obtain the following theorem:

Theorem

$$\mathrm{SCPA} \vdash \mathrm{cov}(s_0) = \omega_1$$

References

[1] A. Dow, *Recent Results in Set-Theoretic Topology*, Recent progress in general topology II (2002), Elsevier Science B.V., pp. 131-152.

[2] A. Dow, *The regular open algebra of* $\beta\mathbb{R} \setminus \mathbb{R}$ *is not equal to the completion of* $\mathcal{P}(\omega)/\operatorname{fin}$, Fund. Math. 157 (1998) 3341.

[3] S. Shelah, O. Spinas *The distributivity numbers of* $\mathcal{P}(\omega)/\operatorname{fin}$ *and its square*, Trans. AMS, 325 (1999), 2023-2047.