Applications of the core model induction

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Introduction

We present some applications of the core model indution in computing lower-bound consistency strength. In particular, we outline the construction of a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" from the theories (T_1) and (T_2), where

(T_1) := ZF + DC + ω_1 is supercompact,

where "ω_1 is supercompact"' means for every uncountable set X, there is a normal fine measure on $\mathcal{P}_{\omega_1}(X)$, and

(T_2) := $\omega_2^\omega = \omega_2 + 2_2^\omega = \omega_3 +$ the set of $X \prec H_{\omega_6}$ such that $|X| = \omega_2$, $X^\omega \subseteq X$, and X is ω_2-guessing is stationary.

In the above, X is ω_2-guessing if whenever $y \subseteq z \in X$ is such that for all $b \in X \cap \mathcal{P}_{\omega_2}(X)$, $b \cap y \in X$, then there is some $a \in X$ such that $b \cap X = a \cap X$.

The core model induction and hod mice

Hod mice being constructed here can have a measurable limit of Woodin cardinals. A (minimal) hod premouse $\mathcal{P}$ with a measurable limit of Woodin cardinals satisfies:

- $\delta^{\mathcal{P}}$ is measurable where $\delta^{\mathcal{P}}$ is a limit of $\mathcal{P}$'s Woodin cardinals.
- $\mathcal{P} = ((\mathrm{Lp}_2^{\oplus_{\alpha<\delta^{\mathcal{P}}}\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta^{\mathcal{P}}))^{\mathcal{P}}, E_\mu)$, where E_μ is the active extender witnessing $o(\delta^{\mathcal{P}})^{\mathcal{P}} = 1$.

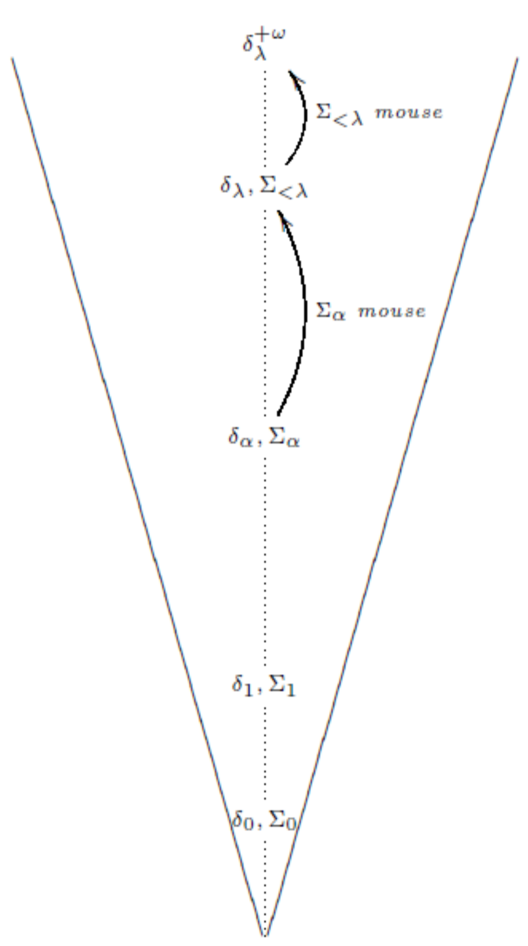

Figure 1. Hod Premouse

Our core model inductions construct hod mice $(\mathcal{P}, \Sigma)$ up to those minimal with a measurable limit of Woodin cardinals and Σ-cmi operators (defined in the last section); these operators in turns generate pointclasses of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measuble".

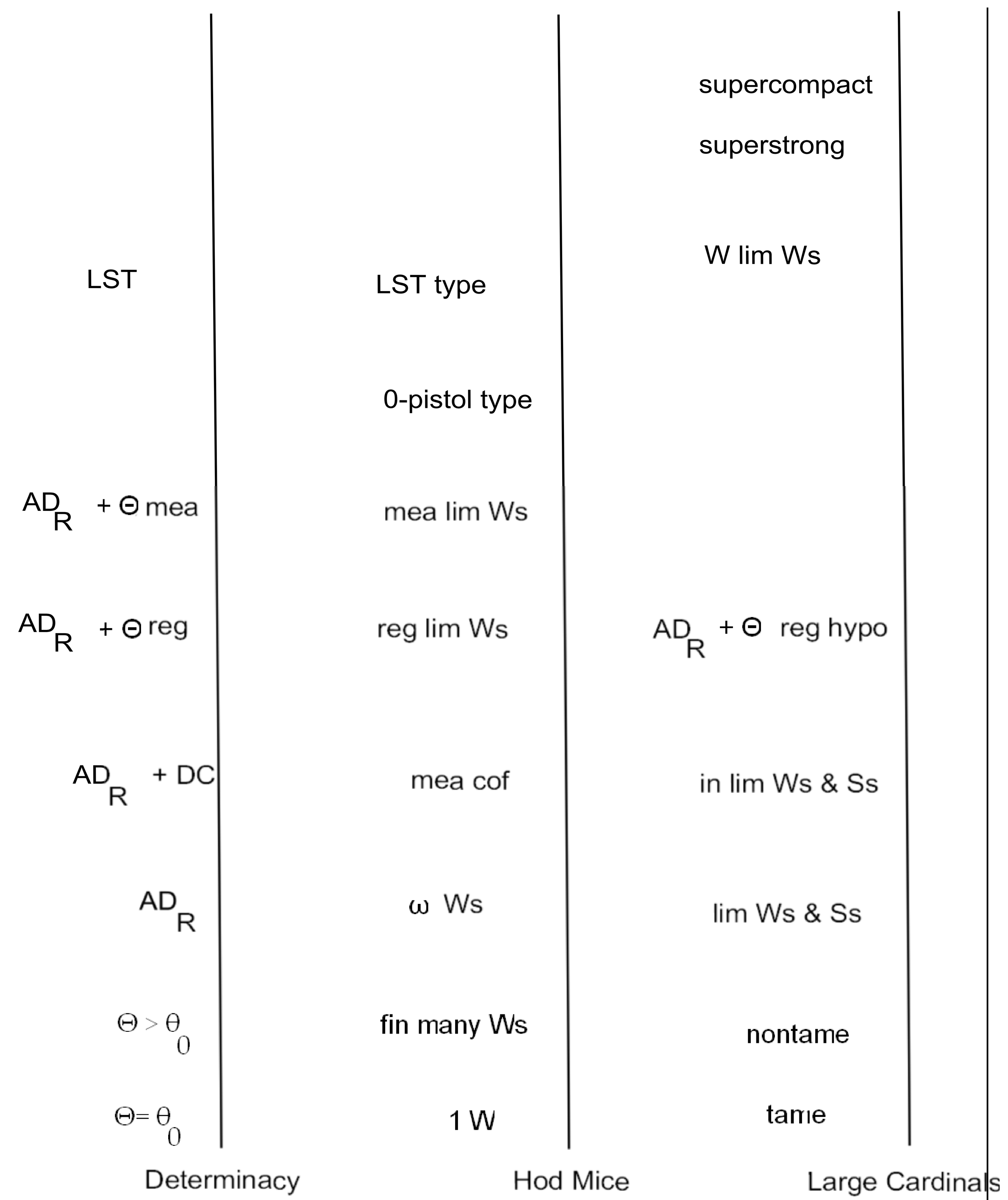

Figure 2. Three hierarchies

Strength from $\mathsf{ZF} + \mathsf{DC} + \omega_1$ is supercompact

Upper-bound (Woodin): A properclass of Woodin limits of Woodin cardinals.

Some technical weakenining of (T_1) and their strength:

- "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC+ there is a measurable cardinal".
- (Woodin) "$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC+ there are ω^2 Woodin cardinals".
- (T., Woodin) "$\Theta > \omega_2 + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC+ there are ω^2 Woodin cardinals".
- (T., Wilson) The minimal model of "$\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$" exists in the presence of a normal fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$.
- (T.) In addition, if one assumes Θ is regular, then the minimal model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" exists. The proof also shows Con(T_1) implies Con($\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable).

Strength from guessing hulls

Upper-bound: a supercompact cardinal. Also, in the natural model where (T_2) holds, there are no ω_1-guessing hulls that are closed under ω sequences.

Previous known lower-bound (JSSS): the existence of a nondomestic mouse, which roughly corresponds to "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$". This is because $\neg\Box(\omega_3) + \neg\Box_{\omega_3}$ is a consequence of (T_2).

The strength of (T_2) is conjectured to be much more than "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable". The next possible milestone is to show one can construct a model of $\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal.

Methods of proofs

Let κ be a cardinal. Working in $V[G]$, where $G \subseteq Col(\omega, \kappa)$ or $G \subseteq Col(\omega, <\kappa)$, we define.

Definition 1:[Hod pair below κ] $(\mathcal{P}, \Sigma)$ is a hod pair below κ if $\mathcal{P} \in V$, $|\mathcal{P}| \leq \kappa$ ($< \kappa$, resp), Σ is a (ω_1, ω_1)-strategy with branch condensation and is fullness preserving. Furthermore, $\Sigma \restriction V \in V$.

Definition 2:[Core model induction operators] Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ. We say F is a Σ core model induction operator or just Σ-cmi operator if in $V[G]$, for some $\alpha \in$ OR, letting $M = \mathrm{Lp}^\Sigma(\mathbb{R})||\alpha$, $M \vDash \mathsf{AD}^+ + \mathsf{MC}(\Sigma)$[1] and one of the following holds:

1. $F = F_J$ (as in the notation of Definition 2.1.8 of [3]), where J is a first order Σ-model operator which condenses and relativizes well. Furthermore, there is some $a \in HC \cap V$ such that J is defined on all $b \in HC$ coding a.
2. For some swo $b \in HC \cap V$ and some Σ-premouse $\mathcal{Q} \in HC \cap V$ over b, F is an (ω_1, ω_1)-iteration strategy for $\mathcal{Q}$ which is $(\mathcal{P}(\mathbb{R}))^M$-fullness preserving, has branch condensation and is guided by some $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD^M_{b,\Sigma,x}$ for some $x \in b$ and α ends either a weak or a strong gap in the sense of [2] and [1] and $\mathcal{A}$ seals a gap that ends at α[2]. Hence F condenses well.

Theorem 1: Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation. Furthermore, assume there is no model M containing $\mathbb{R} \cup OR$ such that $M \vDash \mathsf{AD}^+ + \Theta > \theta_\Sigma$. Then $Lp^\Sigma(\mathbb{R}) \vDash \mathsf{AD}^+ + \theta_\Sigma = \Theta$.

Let Γ be the maximal pointclass (that is Γ consists of sets of reals Wadge reducible to a Σ-cmi operator for some hod pair $(\mathcal{P}, \Sigma)$). Suppose there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

Case 1: Suppose $\Gamma \vDash \Theta = \theta_{\alpha+1}$. Let $(\mathcal{P}, \Sigma)$ be a hod pair such that Σ is Γ-fullness preserving and $\Gamma(\mathcal{P}, \Sigma) = \Gamma|\theta_\alpha$. Let $\Omega = \Sigma_1^2(\Sigma)$. Can then construct a sjs $\mathcal{A}$ that seals $Env(\Omega)$. One can then construct a hod pair $(\mathcal{Q}, \Lambda)$ such that Λ is guided by $\mathcal{A}$ (Λ then is Γ-fullness preserving and has branch condensation). $\Lambda \notin \Gamma$ but $L(\Lambda, \mathbb{R})$ is a model of AD^+. Contradiction.

Case 2: Θ^Γ is a limit in Γ's Solovay sequence. We can construct a hod pair $(\mathcal{Q}, \Lambda)$ such that Λ is Γ-fullness preserving and has branch condenstaion (the strategy Λ in both cases are guided by an uncollapse map), but $\Lambda \notin \Gamma$. One can again show $L(\Lambda, \mathbb{R})$ is a model of AD^+. Contradiction.

So it must be the case that there is a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

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References

[1] F. Schlutzenberg and N. Trang. Scales in $\mathrm{Lp}^J(\mathbb{R})$, in preparation. 2013.

[2] J. R. Steel. Scales in $K(\mathbb{R})$. In *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, volume 31 of *Lect. Notes Log.*, pages 176–208. Assoc. Symbol. Logic, Chicago, IL, 2008.

[3] T. Wilson. *Contributions to descriptive inner model theory*. PhD thesis, UC Berkeley, 2012.

[1] $\mathsf{MC}(\Lambda)$ stands for the Mouse Capturing relative to Λ which says that for $x, y \in \mathbb{R}$, x is $\mathrm{OD}(\Lambda, y)$ iff x is in some Λ-mouse over y.

[2] This means that $\mathcal{A}$ is cofinal in $Env(\Gamma)$, where $\Gamma = \Sigma_1^M$. Note that $Env(\Gamma) = \mathcal{P}(\mathbb{R})^M$ if α ends a weak gap and $Env(\Gamma) = \mathcal{P}(\mathbb{R})^{\mathrm{Lp}^\Sigma(\mathbb{R})|(\alpha+1)}$ if α ends a strong gap.