Proper forcing remastered

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Strong properness

Given a model M and a forcing $\mathbb{P}$ we say that M is adequate for $\mathbb{P}$, if whenever $p, q \in \mathbb{P} \cap M$ are compatible, there is $r \in \mathbb{P} \cap M$ that extends both.

$(M, \mathbb{P})$-strong genericity

Suppose $\mathbb{P}$ is a forcing notion and M is adequate for $\mathbb{P}$. We say that a condition p is $(M, \mathbb{P})$-*strongly generic* if p forces that $\dot{G} \cap M$ is a V-generic subset of $\mathbb{P} \cap M$, where $\dot{G}$ is the canonical name for the V-generic filter over $\mathbb{P}$.

Strongly proper forcings

Suppose $\mathbb{P}$ is a forcing notion and $\mathcal{S}$ is a collection of models adequate for $\mathbb{P}$. We say that $\mathbb{P}$ is $\mathcal{S}$-*strongly proper*, if for every $M \in \mathcal{S}$, every condition $p \in \mathbb{P} \cap M$ can be extended to an $(M, \mathbb{P})$-strongly generic condition q.

Approachability

Definition

Let P an elementary submodel of $H(\theta)$ of size $\aleph_1$. We say that P is *internally approachable* if it can be written as the union of an increasing continuous $\in$-chain $\{P_\xi : \xi < \omega_1\}$ of countable elementary submodels of $H(\theta)$ such that $\{P_\xi : \xi < \eta\} \in P_{\eta+1}$, for every ordinal $\eta < \omega_1$.

If let $\trianglelefteq$ be a well ordering of $H(\theta)$ we let $\vec{P}$ be the $\trianglelefteq$-minimal sequence witnessing the approachability of P.

Fact

The set $\mathcal{A}_1^\theta$ is stationary in $[H(\theta)]^{\aleph_1}$.

Remark: If $M \in \mathcal{S}_0^\theta$ and $P \in \mathcal{A}_1^\theta \cap M$, then $M \cap P = P_{\delta_M}$, with $\delta_M = M \cap \omega_1$.

How to prove strong properness?

Lemma

Suppose $\mathbb{P}$ a notion of forcing and M is adequate for $\mathbb{P}$. A condition p is $(M, \mathbb{P})$-strongly generic if and only if for every $r \leq p$ in $\mathbb{P}$ there is a condition $r|M \in \mathbb{P} \cap M$ such that any condition $q \leq r|M$ in M is compatible with r.

Key idea (Todorcevic '80)

Let models be part of our forcing conditions.

Side conditions

Given a regular cardinal θ, we let $\mathbb{M}_0^\theta$ be the poset consisting of conditions $p = \mathcal{M}_p$ such that

- $\mathcal{M}_p$ is a finite $\in$-chain
- every $M \in \mathcal{M}_p$ is an element of $\mathcal{S}_0^\theta$.

We say that $p \leq q$ if $\mathcal{M}_q \subseteq \mathcal{M}_p$.

Theorem

$\mathbb{M}_0^\theta$ is $\mathcal{S}_0^\theta$-strongly proper.

Effect of $\mathbb{M}_0^\theta$

Theorem

The forcing $\mathbb{M}_0^\theta$ preserves ω_1 and collapses θ to ω_1.

Proof

$\mathcal{S}_0^\theta$-strong properness of $\mathbb{M}_0^\theta$ guarantees the preservation of ω_1. Moreover, for every $x \in H(\theta)$, let

$$D_x = \{p \in \mathbb{M}_0^\theta : \exists M \in \mathcal{M}_p \wedge x \in M\},$$

and notice that is a dense subset of $\mathbb{M}_0^\theta$. Then if we let G be a V-generic filter in $\mathbb{M}_0^\theta$ and $\mathcal{M}_G$ be the generic $\in$-chain we have that, in $V[G]$, the structure $H(\theta)$ is covered by $\mathcal{M}_G$. Moreover, since $\mathcal{M}_G$ is also an $\subseteq$-chain (indeed $\mathcal{M}_G$ is a transitive chain) we have that its length is ω_1. Hence the set

$$\{\alpha_M : M \in \mathcal{M}_G \wedge \alpha_M = sup(M \cap \theta)\}$$

has cardinality $\aleph_1$ and it is cofinal in θ.

Motivating example

Force a club in ω_1

Let $\mathbb{C}_1$ be the forcing consisting of triples $p = (F_p, \mathcal{A}_p, \mathcal{M}_p)$, where

1. F_p is a finite set of countable ordinals,
2. $\mathcal{A}_p$ is a finite set of intervals $(\alpha, \beta]$, with $\alpha, \beta \in \omega_1$,
3. $\mathcal{M}_p \in \mathbb{M}_0^\theta$,
4. $F_p \cap \bigcup \mathcal{A}_p = \emptyset$,
5. if $M \in \mathcal{M}_p$ and $I \in \mathcal{A}_p$, then either $I \subseteq M$ or $I \cap M = \varnothing$.

Theorem

The forcing $\mathbb{C}_1$ is $\mathcal{S}_0^\theta$-strongly proper. And $\bigcup_{p \in G} F_p$, for $G \subseteq \mathbb{C}_1$ a V-generic filter, is a club in ω_1 that does not contain any infinite subsets that belong to V.

Generalized side conditions (after Neeman)

Given a regular cardinal θ, we let $\mathbb{M}_1^\theta$ be the poset consisting of conditions $p = \mathcal{M}_p$ such that

- $\mathcal{M}_p$ is a finite $\in$-chain
- every $M \in \mathcal{M}_p$ is an element of $\mathcal{S}_0^\theta \cup \mathcal{A}_1^\theta$,
- $\mathcal{M}_p$ is closed under intersection; i.e. if $N, M \in \mathcal{M}_p$, then $N \cap M \in \mathcal{M}_p$.

We say that $p \leq q$ if $\mathcal{M}_q \subseteq \mathcal{M}_p$.

Lemma

Suppose $M \in \mathcal{S}_0^\theta \cup \mathcal{A}_1^\theta$ and $p \in \mathbb{M}_1^\theta \cap M$. Then there is a new condition p^M, which is the smallest element of $\mathbb{M}_1^\theta$ extending p and containing M as an element.

Structural properties

Notation:

- For $p \in \mathbb{M}_1^\theta$ let $\pi_0(p) = p \cap \mathcal{S}_0^\theta$ and $\pi_1(p) = p \cap \mathcal{A}_1^\theta$.
- Let $\in^*$ be the transitive closure of the $\in$ relation, i.e. $x \in^* y$ if $x \in \mathrm{tcl}(y)$.
- Given $M, N \in \mathcal{M}_p \cup \{\emptyset, H(\theta)\}$ with $M \in^* N$ let

$$[M, N)_p = \{P \in \mathcal{M}_p : M \in^* P \in^* N\} \cup \{M\}.$$

Facts

- For $p \in \mathbb{M}_1^\theta$, if $M \in \pi_1(p)$, then $p \cap M$ is a $\in^*$-initial segment of p. Therefore, $p \cap M \in \mathbb{M}_1^\theta$.
- Suppose $p \in \mathbb{M}_1^\theta$ and $M \in \pi_0(p)$. Then

$$\mathcal{M}_p \cap M = \mathcal{M}_p \setminus \bigcup \{[M \cap N, N)_p : N \in (\pi_1(p) \cap M) \cup \{H(\theta)\}\}.$$

  Therefore, $p \cap M \in \mathbb{M}_1^\theta$.

$\mathbb{M}_1^\theta$ strong properness

In order to prove the theorem we use the following.

Lemma

Suppose $r \in \mathbb{M}_1^\theta$ and $M \in \mathcal{M}_r$. Let $q \in M$ be such that $q \leq r \cap M$. Then q and r are compatible.

Sketch of the proof

If M is uncountable then one can easily check that $\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain which is closed under intersection. Therefore $s = \mathcal{M}_s$ is a common extension of q and r. Suppose now M is countable. We first check that $\mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain, then we close this chain under intersections and show that the resulting set is still an $\in$-chain.

Theorem

$\mathbb{M}_1^\theta$ is $\mathcal{S}_0^\theta \cup \mathcal{A}_1^\theta$-strongly proper. Hence it preserves ω_1 and ω_2, while collapsing θ to ω_2.

Applications

Thanks to the generalized side conditions we can prove uniformly the following theorems:

After S. D. Friedman and Mitchell

It is possible to force a club in ω_2 with finite conditions such that it does not contain any infinite subsets which are in the ground model.

The forcing for adding the club

Let $\mathbb{M}_2$ be the forcing notion whose elements are triples $p = (F_p, A_p, \mathcal{M}_p)$, where $F_p \in [\omega_2]^{<\omega}$, A_p is a finite collection of intervals of the form $(\alpha, \beta]$, for some $\alpha, \beta < \omega_2$, $\mathcal{M}_p \in \bar{\mathbb{M}}_1^\theta$, and

1. $F_p \cap \bigcup A_p = \emptyset$,
2. if $M \in \mathcal{M}_p$ and $I \in A_p$, then either $I \in M$ or $I \cap M = \varnothing$,

The order on $\mathbb{M}_2$ is coordinatewise reverse inclusion, i.e. $q \leq p$ if $F_p \subseteq F_q$, $A_p \subseteq A_q$ and $\mathcal{M}_p \subseteq \mathcal{M}_q$.

After Koszmider

Assume that Chang conjecture does not hold. Then it is possible to force with finite conditions a chain of length ω_2 in the partial order $(\omega_1^{\omega_1}, <_{\text{fin}})$ of all functions from ω_1 to ω_1 ordered by $f <_{\text{fin}} g$ iff $\{\xi : f(\xi) \geq g(\xi)\}$ is finite.

After Baumgartner, Shelah and Bagaria

It is possible to force with finite conditions the existence of a superatomic Boolean algebra $\mathbb{B}$ (i.e. such that every homeomorphic image of $\mathbb{B}$ is atomic) with $\text{ht}(\mathbb{B}) = \aleph_2$ (where $\text{ht}(\mathbb{B})$ is the smallest ordinal such that the corresponding Cantor-Bendixon's ideal $I_\alpha = \text{ht}(\mathbb{B})$) and $\text{wd}_\alpha(\mathbb{B}) = \aleph_0$ (where $\text{wd}_\alpha(\mathbb{B})$ is the cardinality of $\mathbb{B} \setminus I_\alpha$), for every $\alpha < \text{ht}(\mathbb{B})$.

After folklore

It is possible to force with finite conditions the existence an ω_2-Souslin tree.