A dichotomy for $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ sets of reals, with applications to generic absoluteness

Trevor Wilson

Young Set Theory Workshop, Oropa, Italy

June 12, 2013

We begin by introducing some basic notions of descriptive set theory.

- Descriptive set theory deals with nice sets of reals, understood in terms of complexity.
- By "nice" we mean in particular less complex than a well-ordering of the reals.
- Above this level of complexity, the subject would turn into combinatorics of the continuum.

Remark

By convention, we identify the set of reals $\mathbb{R}$ with the set of integer sequences ω^ω.

What are "nice" properties?

- regularity properties (*e.g.* property of Baire, Lebesgue measurability, perfect set property)
- determinacy

What are not?

- well-orderings of $\mathbb{R}$
- uncountable well-orderings

Possible definitions of "nice":

- Universally Baire
- ∞-Homogeneous (Hom_∞)

Definition

For a tree $T \subset \omega^{<\omega} \times \mathrm{Ord}^{<\omega}$ we write $[T] \subset \omega^\omega \times \mathrm{Ord}^\omega$ for the set of branches of T, and $p[T] \subset \omega^\omega$ for its projection.

Definition (Feng–Magidor–Woodin)

A set of reals A is κ-universally Baire if $A = p[T]$ for some pair of trees $(T, \tilde{T})$ that is κ-absolutely complementing: $p[\tilde{T}] = \mathbb{R} \setminus p[T]$ in all forcing extensions by posets of size $<\kappa$.

Example

$\utilde{\mathbf{\Sigma}}^1_1$ and $\utilde{\mathbf{\Pi}}^1_1$ sets of reals are universally Baire (κ-uB for all κ.)

Remark

Universal Baire-ness is a natural strengthening of the Property of Baire, and it also implies Lebesgue measurability.

Definition (Kechris, Martin)

Let A be a set of reals and κ an uncountable cardinal.

- A is κ-Homogeneous, or Hom_κ, if there is a continuous function f from $\mathbb{R}$ to sequences of κ-complete measures such that $x \in A \iff f(x)$ is a well-founded tower.
- A is ∞-Homogeneous, or Hom_∞, if it is κ-homogeneous for every κ.

Example (Martin)

If κ is measurable then every $\underset{\sim}{\mathbf{\Pi}}^1_1$ set of reals is Hom_κ.

Theorem (Martin)

Hom_κ sets are determined (for $\kappa > \omega$.)

Theorem (Martin–Solovay)

If the set $A \subset \mathbb{R}^{n+1}$ is Hom_κ, then the projection $pA \subset \mathbb{R}^n$ is κ-universally Baire.

Corollary

If κ is measurable, then every $\utilde{\mathbf{\Sigma}}^1_2$ set is κ-universally Baire.

Their proof shows more:

Theorem (Martin–Solovay)

If κ is measurable, then the Shoenfield tree for Σ^1_2 is κ-absolutely complemented.

Corollary

If κ is a measurable cardinal, then Σ^1_3 statements about reals are absolute between $<\kappa$-generic extensions.

Theorem (Martin–Steel)

If κ is a limit of Woodin cardinals, then the class of $\mathrm{Hom}_{<\kappa}$ sets is closed under real quantifiers $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$.

Corollary

If there are infinitely many Woodin cardinals, then all projective sets are determined.

Remark

The hypothesis of infinitely many Woodin cardinals is more than enough for PD—by a theorem of Woodin it is equiconsistent with $\mathsf{AD}^{L(\mathbb{R})}$.

A Woodin cardinal is a large cardinal property between strongs and superstrongs in consistency strength.

- Every superstrong cardinal is Woodin and is a limit of Woodin cardinals.
- If there is a Woodin cardinal, then "there is a strong cardinal" is consistent, but does not necessarily hold in V.

Theorem (Martin–Steel + Woodin)

If κ is a limit of Woodin cardinals, a set of reals is κ-universally Baire if and only if it is $\mathrm{Hom}_{<\kappa}$.

If there is a proper class of Woodin cardinals, then Hom_∞ (= universally Baire) is a natural class of "nice" sets of reals.

Remark

Hom_∞ is not necessarily closed under quantification over *sets* of reals, as we will discuss next.

Definition

A $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ statement about $x \in \mathbb{R}^n$ says

$$\exists A \in \mathrm{Hom}_\infty \ (H_{\omega_1}; A, \in) \models \phi[x].$$

Example

For a real x, the following statements are $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$.

- "x is $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ in a countable ordinal"
- "x is in a premouse with a Hom_∞ iteration strategy"

Remark

- A *premouse* is a generalization of "model of $V = L$" to accomodate large cardinals.
- For a premouse to be a canonical inner model (a "mouse") it is not enough to be wellfounded as with models of $V = L$—it needs to have an iteration strategy.

Remark

- If x is in a premouse with a Hom_∞ iteration strategy, then it is $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ in a countable ordinal.
- The *Mouse Set Conjecture* says roughly the converse.

Example

There is a $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ well-ordering of reals appearing in canonical inner models:

- Define $x < y$ if x is constructed before y in some/all premice with Hom_∞ iteration strategies.

This extends the Σ^1_2 well-ordering of reals in L.

Remark

If V itself is a canonical inner model of a certain type then there is a $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ well-ordering of $\mathbb{R}$, so $(\Sigma^2_1)^{\mathrm{Hom}_\infty} \not\subset \mathrm{Hom}_\infty$.

Open question

Is *every* large cardinal axiom consistent with the existence of a $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ well-ordering of $\mathbb{R}$?

Theorem (Woodin)

Let κ be a limit of Woodin cardinals.

- There is a tree T such that in every $<\kappa$-generic extension, $p[T]$ is the universal $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ set of reals. (Compare to Shoenfield tree for Σ^1_2.)
- If V_κ has a strong cardinal δ, then there is a $<\kappa$-generic extension in which T is κ-absolutely complemented. (Compare to Martin–Solovay tree for Π^1_2.)

Remark

If the tree T for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ is κ-absolutely complemented, then in every generic extension of V_κ we have $(\Sigma^2_1)^{\mathrm{Hom}_\infty} \subset \mathrm{Hom}_\infty$.

Theorem (W.)

If κ is a measurable limit of Woodin cardinals, then either:

1. In cofinally many $<\kappa$-generic extensions there is an uncountable $(\utilde{\boldsymbol{\Sigma}}^2_1)^{\mathrm{Hom}_{<\kappa}}$ well-ordering, or
2. The tree for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ is κ-absolutely complemented in some $<\kappa$-generic extension.

Remark

- Cases 1 and 2 are mutually exclusive.
- If V_κ has a strong cardinal then Case 2 must hold.
- If V_κ has no strong cardinal *and* is a certain type of canonical inner model then Case 1 must hold.

Remark

Suppose Case 2 holds: the tree for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ is κ-absolutely complemented in some $<\kappa$-generic extension.

- This is equivalent to saying that the *derived model* at κ, which is always a model of AD, satisfies "every Π^2_1 set is Suslin" (*i.e.* the projection of a tree on $\omega \times \mathrm{Ord}$.)
- The theory AD + "every Π^2_1 set is Suslin" has high consistency strength: it implies there is an inner model with a cardinal that is strong past a Woodin cardinal.

So we can recover a "trace" of a collapsed $<\kappa$-strong cardinal.

Proof idea

- Assume Case 1 fails: for some $<\kappa$-generic extension, in every further $<\kappa$-generic extension every $(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{Hom}_{<\kappa}}$ well-ordering is countable.
- We want to establish Case 2 by showing that the tree T for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ has a κ-absolute complement $\tilde{T}$ in some $<\kappa$-generic extension.
- If κ were 2^{2^κ}-supercompact, this would come from a theorem of Martin and Woodin.
- In $\mathrm{Ult}(V,\mu)[x_g]$ where μ is on κ and x_g generically codes V_κ, only countably many reals are $(\Sigma^2_1(x_g))^{\mathrm{Hom}_{<j(\kappa)}}$. So there are only κ many partial measures to consider, and measurability of κ is enough (details omitted.)

Now we derive some consequences of this dichotomy related to generic absoluteness.

$(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ generic absoluteness (Woodin)

Assume there is a proper class of Woodin cardinals.

- Let $V[g]$ and $V[g][h]$ be generic extensions and let $x \in \mathbb{R} \cap V[g]$.
- Then any $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ statement about x holds in $V[g]$ if and only if it holds in $V[g][h]$.

Remark

This is analogous to Shoenfield's Σ^1_2 absoluteness.

What is analogous to Martin–Solovay Σ^1_3 absoluteness?

Definition

A $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ statement says that all reals have a $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ property.

Example

"All reals are in a canonical inner model" is $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_\infty}$.

Remark

- Forcing over a canonical inner model to add a Cohen real makes this statement go from true to false.
- Forcing over L to add a Cohen real makes the $\forall^{\mathbb{R}}\Sigma^1_2$ $(= \Pi^1_3)$ statement "all reals are constructible" go from true to false.

Generic absoluteness for $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ holds or fails according to the cases of the dichotomy theorem:

1. An uncountable $(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{Hom}_\infty}$ well-ordering gives a true $\forall^{\mathbb{R}}(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{Hom}_\infty}$ statement that is false after forcing with $\mathrm{Col}(\omega, \mathbb{R})$.[1]
2. An absolute complement $\tilde{T}$ for the tree T for $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ gives $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ generic absoluteness, by the absoluteness of well-foundedness.

[1] Correction added June 16, 2013: "$(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{Hom}_\infty}$ well-ordering" should be replaced here and elsewhere with "$(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{Hom}_\infty}$-*good* well-ordering."

Corollary

Let κ be a measurable limit of Woodins. Then the following statements are equivalent:

(a) In some $<\kappa$-generic extension, $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ generic absoluteness holds between further $<\kappa$-generic extensions

(b) In some $<\kappa$-generic extension, the tree for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ is κ-absolutely complemented.

(c) The derived model at κ satisfies "every Π^2_1 set is Suslin."

Compare:

Theorem (Martin–Solovay + Woodin)

The following statements are equivalent.

(a) Σ^1_3 generic absoluteness.

(b) The Shoenfield tree for Σ^1_2 is absolutely complemented.

(c) Every set has a sharp.

Remark

In this talk "generic absoluteness" always means *two-step* generic absoluteness.

We can get even more consistency strength (in the form of strong determinacy axioms) from the generic absoluteness provided by the following theorem:

Theorem (Woodin)

Assume there is a proper class of Woodin cardinals. If there is a supercompact cardinal δ, then there is a forcing extension in which

- The theory of $L(\mathrm{Hom}_\infty, \mathbb{R})$ is generically absolute for further forcing extensions, and
- $L(\mathrm{Hom}_\infty, \mathbb{R}) \models \mathrm{AD} + \mathrm{DC} +$ "every set of reals is Suslin."

Using the dichotomy, we can get a partial reversal:

Theorem (W.)

Let κ be a measurable limit of Woodin cardinals.
If the theory of $L(\mathrm{Hom}_{<\kappa}, \mathbb{R})$ is $<\kappa$-generically absolute, then $L(\mathrm{Hom}_{<\kappa}, \mathbb{R}) \models \mathrm{AD} + \mathrm{DC} +$ "every set of reals is Suslin."

Remark

The proof uses a relativization of above results for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$, and the equivalence of the following statements for the derived model $L(\mathrm{Hom}^*, \mathbb{R}^*)$.

- Every set of reals is Suslin.
- Every $\Pi^2_1(A)$ set of reals is Suslin for every $A \in \mathrm{Hom}^*$.

Remark

The partial reversal leaves a very big gap.
Ranked in increasing order of consistency strength:

- "there is a proper class of Woodins"
- "there is a measurable limit of Woodins"
- ZF + AD + DC + "every set of reals is Suslin"
- (very big gap)
- "there is a supercompact cardinal."

Questions

- Is the measurability of κ really necessary for anything?
- To get κ-absolute complementation of the tree for $(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ from $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{Hom}_{<\kappa}}$ generic absoluteness, must we go to a forcing extension or does it hold already in V?
- How much more consistency strength can we get from generic absoluteness of the theory of $L(\mathrm{Hom}_{<\kappa}, \mathbb{R})$?