Set Theory and C^*-algebras

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Hilbert Spaces

Definition

A (complex) *Hilbert space* is a vector space H together with a complete inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$.

- $H = \ell^2 = \{(x_n) \subseteq \mathbb{C} : \sum |x_n|^2 < \infty\}$; $\langle (x_n), (y_n) \rangle = \sum x_n \overline{y_n}$.
- $\ell^2 \approx$ quantum analog of ω.
- The set of bounded linear operators $\mathcal{B}(H)$ on H is a C*-algebra, i.e. a Banach algebra with involution * satisfying $||xx^*|| = ||x||^2$, (* is the adjoint, i.e. $\langle Tv, w \rangle = \langle v, T^* w \rangle$).

Definition

$P \in \mathcal{B}(H)$ is a *projection* if $P = P^* = P^2$. Denote by $\mathcal{P}(\mathcal{B}(H))$.

- Projections correspond to closed subspaces via $P \mapsto \mathcal{R}(P)$ and

$$PQ = P = QP \quad \Leftrightarrow \quad \mathcal{R}(P) \subseteq \mathcal{R}(Q).$$

- $\mathcal{P}(\mathcal{B}(H)) \approx$ quantum/non-commutative analog of $\mathscr{P}(\omega)$.

Compact Operators

Definition

$T \in \mathcal{B}(H)$ is *compact* if $\overline{T[B_1(H)]}$ is compact. Denote by $\mathcal{K}(H)$.

- $\mathcal{K}(H) = \overline{\{T \in \mathcal{B}(H) : \dim(\mathcal{R}(T)) < \infty\}}$.
- $\mathcal{P}(\mathcal{K}(H)) = \{P \in \mathcal{P}(B)(H) : \dim(\mathcal{R}(T)) < \infty\}$.
- $\mathcal{P}(\mathcal{K}(H)) \approx$ non-commutative analog of $\mathrm{Fin} = [\omega]^{<\omega}$.
- $\mathcal{K}(H)$ is an ideal in $\mathcal{B}(H)$.
- $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is a C*-algebra, the *Calkin Algebra*.
- $\mathcal{P}(\mathcal{C}(H)) \approx$ non-commutative analog of $\mathscr{P}(\omega)/\mathrm{Fin}$.

Order Properties

- $\mathcal{P}(\mathcal{C}(H))$ is separative ($\mathcal{C}(H)$ has real rank zero).
- Unlike $\mathscr{P}(\omega)/\mathrm{Fin}$, $\mathcal{P}(\mathcal{C}(H))$ is *not* a lattice.

$$\exists p \wedge q \quad \Leftrightarrow \quad \sup(\sigma(pq)\setminus\{1\}) < 1 \quad \Leftrightarrow \quad \exists p \vee q.$$

- So maximal centred $\not\Leftrightarrow$ maximal (downwards) directed.

Theorem (B. 2010)

Any $(p_n) \subseteq \mathcal{P}(\mathcal{C}(H))$ has $\leq$-equivalent decreasing $(q_n) \subseteq \mathcal{P}(\mathcal{C}(H))$.

- (ω, ω)-pregaps in $\mathcal{P}(\mathcal{C}(H))$ lift to $\mathcal{P}(\mathcal{B}(H))$ (real rank zero).
- $\mathcal{P}(\mathcal{B}(H))$ is a complete lattice so no (ω, ω)-gaps.

Corollary

No non-trivial finite or countable gaps in $\mathcal{P}(\mathcal{C}(H))$.

Embeddings

- Take a basis $(e_n)_{n\in\omega} \subseteq H$. For $A \subseteq \omega$, define $P_A \in \mathcal{P}(\mathcal{B}(H))$ by $\mathcal{R}(P_A) = \overline{\mathrm{span}(e_n)_{n\in A}}$. Then

$$A \subseteq B \Leftrightarrow P_A \leq P_B \quad \text{and} \quad A \subseteq^* B \Leftrightarrow \pi(P_A) \leq \pi(P_B),$$

  so $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\mathrm{Fin}$ embed in $\mathcal{P}(\mathcal{B}(H))$ and $\mathcal{P}(\mathcal{C}(H))$ resp.
- Also, $A \mapsto P_A$ is continuous w.r.t. weak operator topology.

Question

For what other ideals $\mathcal{I}$ does $\mathscr{P}(\omega)/\mathcal{I}$ embed in $\mathcal{P}(\mathcal{C}(H))$? What if we require the embedding to have a continuous lifting?

Gap Spectrum

Theorem (Steprans)

For any $p \in \mathcal{P}(\mathcal{C}(H))$, $\{A \subseteq \omega : \pi(P_A) \leq p\}$ is an analytic p-ideal.

Theorem (Todorcevic/Solecki)

The orthogonal of an analytic p-ideal is countably generated.

Corollary

The map $A \mapsto \pi(P_A)$ is gap preserving.

Theorem (Zamora-Aviles 2009)

$\mathcal{P}(\mathcal{C}(H))$ contains an analytic Hausdorff gap.

- Under (MA) this is a $(\mathfrak{c}, \mathfrak{c})$ gap.
- Under (TA) $\mathscr{P}(\omega)/\mathrm{Fin}$ has only (ω_1, ω_1) and $(\omega, \mathfrak{b})$ gaps.
- (MA)+(TA)+($\mathfrak{c} = \omega_2$), $\mathrm{spec}(\mathscr{P}(\omega)/\mathrm{Fin}) \subsetneq \mathrm{spec}(\mathcal{P}(\mathcal{C}(H)))$.

Automorphisms

- Any (almost) 1-1 onto $f : \omega \to \omega$ yields a *trivial* automorphism on $\mathscr{P}(\omega)/\mathrm{Fin}$ by $A \mapsto f(A)$.
- Any unitary $U \in \mathcal{C}(H)$ ($U^*U = UU^* = 1$) yields an *inner* automorphism on $\mathcal{C}(H)$ by $T \mapsto UTU^*$ ($\mathcal{R}(P) \mapsto U[\mathcal{R}(P)]$).
- $\mathscr{P}(\omega)/\mathrm{Fin}$ has non-trivial automorphisms under CH (Rudin) and none under PFA (Shelah-Steprans).

Theorem (Phillips-Weaver 2007)

Under CH, $\mathcal{C}(H)$ has outer automorphisms.

- Unknown for larger H but same for some higher dimension Calkin-like algebras (Farah-McKenney-Schimmerling).

Theorem (Farah 2010)

Under TA, all automorphisms of $\mathcal{C}(H)$ are inner.

- Same for larger H under PFA (Farah).

Cardinal Invariants

- $\mathscr{P}(\omega)/\mathrm{Fin}$ cardinal invariants have $(\geq)2$ analogs $\because$

$$\begin{array}{ccc} p \leq q & \Rightarrow & p \wedge q^{\perp} = 0 \\ \Updownarrow & \not\Leftarrow & \Updownarrow \\ pq^{\perp} = 0 & & ||pq^{\perp}|| < 1 \end{array}$$

- p strongly splits $q \quad \Leftrightarrow \quad p \wedge q \neq 0 \neq p^{\perp} \wedge q.$

  p weakly splits $q \quad \Leftrightarrow \quad p \wedge q \neq 0$ and $q \not\leq p.$

  $\mathfrak{s}^{\perp} = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(\mathcal{C}(H))$ is a strongly splitting family$\}.$

  $\mathfrak{s}^{*} = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(\mathcal{C}(H))$ is a weakly splitting family$\}.$

Block Subspaces

- V is a block subspace if $\exists$ IP (I_n) and $\exists$ $(v_n) \subseteq H$ s.t. $V = \mathrm{span}(v_n)$ and $\forall n(v_n \in \mathrm{span}\{e_k : k \in I_n\})$.
- Block subspaces are $\leq^*$-dense.

  Given inf dim $V \subseteq H$ recursively pick unit vectors $(v_n) \subseteq V$

$$
\begin{aligned}
v_0 &= (0, \frac{1}{5}, \frac{3}{4}, \frac{1}{2}, \frac{1}{10}, \ldots) \quad \text{(arbitrary)} \\
v_1 &= (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{16}}, \ldots) \in V \cap \ell^{2\perp}_{k_0}, \;\; k_0 >> 0 \\
v_2 &= (0, 0, 0, 0 \ldots, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots) \in V \cap \ell^{2\perp}_{k_1}, \;\; k_1 >> k_0 \\
&\vdots
\end{aligned}
$$

$V \supseteq \mathrm{span}(v_n) =^*$ block subspace. $\Box$

Interval Paritions

- So card invs on $\mathcal{P}(\mathcal{C}(H))$ often related to IP card invs.
- Eg. $A \subseteq \omega$ splits IP (I_n) $\Leftrightarrow$ $\exists^\infty n\ I_n \subseteq A$ and $\exists^\infty n\ I_n \subseteq \omega \backslash A$.

  $\mathfrak{s}^{\mathrm{IP}} = \mathsf{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{P}(\omega) \text{ is an IP splitting family}\}.$

  $\mathcal{A} \subseteq \mathscr{P}(\omega)$ IP splitting $\Rightarrow (\pi(P_A))_{A \in \mathcal{A}}$ strongly splitting.

  $\Rightarrow \mathfrak{s}^\perp \leq \mathfrak{s}^{\mathrm{IP}}$.
- $\mathfrak{s}^{\mathrm{IP}} = \mathsf{max}(\mathfrak{s}, \mathfrak{b})$ (A. Kamburelis, B. Weglorz (1995)).

$$\mathfrak{t} \leq \mathfrak{s}^* \leq \mathfrak{s}^\perp \leq \mathsf{max}(\mathfrak{s}, \mathfrak{b})$$

Towers

- $\mathfrak{t} = \mathfrak{b} \Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ generating a non-meagre p-filter. $\Rightarrow (\pi(P_{A_\xi}))$ is a tower in $\mathcal{P}(\mathcal{C}(H))$
- Given IP (I_n) take $v_n = \sum_{k \in I_n} e_k$ and $\mathcal{R}(P) = \overline{\mathrm{span}}(v_n)$.
- If $A \subseteq \omega$ s.t. $|A \cap I_n|/|I_n| \to 1$ then $\pi(P) \leq \pi(P_A)$.
- $\sup_n(\mathfrak{m}(\sigma\text{-}n\text{-linked})) = \mathbf{non}(\mathcal{M}) \;\Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ s.t. $\forall \xi |A_\xi \cap I_n|/|I_n| \to 1 \;\Rightarrow\; (\pi(P_{A_\xi}))$ is not a tower in $\mathcal{P}(\mathcal{C}(H))$.

Theorem (Brendle)

Consistently no towers in $\mathcal{I}^*$ for any analytic p-ideal $\mathcal{I} \subseteq \mathscr{P}(\omega)$.

Corollary

Consistently all towers in $[\omega]^\omega$ remain towers in $\mathcal{P}(\mathcal{C}(H))$.

- $\mathfrak{t}^* = \mathfrak{t}^\perp \geq \mathfrak{t}$ (B., Wofsey, Bell, Malliaris-Shelah).

Other Cardinal Invariants

- $\mathfrak{b}^* = \mathfrak{b}^\perp = \mathfrak{b}$ and $\mathfrak{d}^* = \mathfrak{d}^\perp = \mathfrak{d}$ (Zamora-Aviles).
- $\mathfrak{b} \leq \mathfrak{a}^\perp$ (Brendle). Consistently $\mathfrak{a}^\perp = \mathfrak{a} = \aleph_1 < \mathfrak{c}$ (finite conditions) and $\mathfrak{a}^\perp = \mathfrak{a} = \mathfrak{c} > \aleph_1$ (MA) (Wofsey).
- In the Sacks model $\mathfrak{a}^* = \aleph_1 < \mathfrak{c}$ (B.). Is $\mathfrak{a}^* = \aleph_1$?

Let ϕ be a state on a C*-algebra A. Then

- $\{T \in A : \phi(T^*T) = 0\}$ is a closed left ideal.
- $\{T \in A_+ : \phi(T) = 0\}$ is a closed hereditary cone.
- $\{T \in A^1_+ : \phi(T) = 1\}$ is a norm filter.

Definition [B. (2011)]

$F \subseteq A^1_+$ is a *norm filter* if, whenever $T \in A^1_+$ and

$$\inf\{||(1 - T)S_1 \dots S_n|| : n \in \omega \text{ and } S_1, \dots, S_n \in F\} = 0,$$

we necessarily have $T \in F$.

Pure States to Subsets

Let ϕ be a pure state on a C*-algebra A. Then

- $\{T \in A : \phi(T^* T) = 0\}$ is a maximal left ideal.
- $\{T \in A_+ : \phi(T) = 0\}$ is a maximal hereditary cone.
- $\{T \in A_+^1 : \phi(T) = 1\}$ is maximal norm centred.

Definition [Farah and Weaver ($\sim$2009)]

$F \subseteq A_+^1$ is a *norm centred* if $||S_1 \ldots S_n|| = 1$, for $S_1, \ldots, S_n \in F$.

- F is a (proper) norm filter $\overset{\Rightarrow}{\not\Leftarrow}$ F is norm centred.
- F is a maximal norm filter $\Leftrightarrow$ F is maximal norm centred.
- The above yields a one-to-one correspondence.

The Real Rank Zero Case

- If A has RR0 then $\phi \mapsto \{P \in \mathcal{P}(A) : \phi(P) = 1\}$ takes pure states to maximal norm centred subsets and vice versa.
- If $\mathcal{P}(A)\backslash\{0\}$ is also σ-closed, i.e. decreasing sequences are bounded, (e.g. if $A = \mathcal{C}(H) =$ the Calkin algebra) then

$$F \text{ is norm centred} \quad \Leftrightarrow \quad F \text{ is centred},$$

i.e. finite subsets of F have lower bounds in $\mathcal{P}(A)\backslash\{0\}$.

Conjecture [Kadison-Singer (1959)]

Pure states on atomic MASAs of $\mathcal{B}(H)$ extend uniquely.

- KS $\Leftrightarrow$ whenever $\mathcal{U}$ is an ultrafilter on ω, $\{\pi(P_U) : U \in \mathcal{U}\}$ has a unique maximal centred extension in $\mathcal{P}(\mathcal{C}(H))$.

The Kadison-Singer Conjecture

- True for Q-points [Reid (1970)].
- True for rapid P-points [B. (2011)].

Definition (special ultrafilters)

An ultrafilter $\mathcal{U}$ on ω is

- a *Q-point* if, whenever (I_n) is a partition of ω into finite intervals, we have $U \in \mathcal{U}$ with $|U \cap I_n| \leq 1$, for all n.
- *rapid* if, whenever $f \in \omega^\omega$, we have $U \in \mathcal{U}$ with $|U \cap [1, f(n)]| \leq n$, for all n.
- a *P-point* if $\mathcal{U}$ is σ-closed w.r.t. $\subseteq^*$. $(X \subseteq^* Y \Leftrightarrow X \setminus Y$ is finite$)$

- Q-points are rapid but not vice versa.
- Under MA(countable)($\Leftarrow$CH), there are rapid P-points that are not Q-points (Flaskova ($\sim$2010)).

Paving Conjectures

$\Leftrightarrow$ KS [Anderson (1980)]

Given $\epsilon > 0$ and $P \in \mathcal{P}(\mathcal{B}(H))$ we have a partition $X_1, \ldots, X_n$ of ω s.t. $||PP_{X_m}||^2 + ||P^\perp P_{X_m}||^2 \leq 1 + \epsilon$, for all $m < n$.

$\Leftrightarrow$ KS [Akemann and Anderson (1991), Weaver (2004), B. (2011)].

There exists $\delta > 0$ such that for $P \in \mathcal{P}(\mathcal{B}(H))$ satisfying $\langle Pe_k, e_k \rangle < \delta$, for all k, we have a partition $X_1, \ldots, X_n$ of ω s.t. for all $m < n$, either $||PP_{X_m}|| < 1$ or $||P^\perp P_{X_m}|| < 1$.

Theorem [B. (2011)]

Assume A is a RR0 C*-algebra, $\mathcal{F} \subseteq \mathcal{P}(A)$ is norm centred and $\mathcal{M} \subseteq \mathcal{P}(A)$ is a maximal extension of $\mathcal{F}$. Then either $\mathcal{M}$ is unique or we have $P \in \mathcal{M}$ such that $\mathcal{F} \cup \{P^\perp\}$ is norm centred.

P-Points

Theorem [B. (2011)]

If $\mathcal{U}$ is a P-point and $\mathrm{KS}(\mathcal{U})$ holds then the upwards closure $\mathcal{P} \supseteq \pi[P_{\mathcal{U}}]$ is maximal centred.

- Hence $\mathcal{P}$ is directed, i.e. given $p, q \in \mathcal{C}(H)$ with $\phi_{\mathcal{U}}(p) = \phi_{\mathcal{U}}(q) = 1$, there exists $r \leq p, q$ with $\phi_{\mathcal{U}}(r) = 1$.

Theorem [B. (2011)]

If $\mathcal{U}$ is a non-P-point then $\exists p, q \in \mathcal{P}(C(H))$ with $\phi(p) = \phi(q) = 1$ and $\phi(r) = 0$ whenever $r \leq p, q$ and ϕ is a state: $\phi[\pi[P_{\mathcal{U}}]] = \{1\}$.

Theorem [B. (2011)]

Consistently (with ZFC), $\forall$ pure state ϕ on $\mathcal{C}(H)$, $\exists p, q \in \mathcal{P}(C(H))$ with $\phi(p) = \phi(q) = 1$ and $\sup\{\phi(r) : r \leq p, q\} < 1$.

Ultrafilters of Projections

Question

For ultrafilter $\mathcal{U}$ on ω, does $\pi[P_\mathcal{U}]$ have a unique ultrafilter extension?

- Yes for Q-points. In fact, $\pi[P_\mathcal{U}]$ is an ultrafilter base.
- Yes for rapid P-points. Again, $\pi[P_\mathcal{U}]$ is an ultrafilter base.
- Not always an ultrafilter base: Take interval partition (I_n) of ω with $|I_n| \to \infty$ and ultrafilter $\mathcal{U}$ with $\limsup |U \cap I_n|/|I_n| > 0$, for all $U \in \mathcal{U}$. Take P with $\mathcal{R}(P) = \overline{\mathrm{span}}\{\sum_{m \in I_n} e_m : n \in \omega\}^\perp$. Then $\pi(P_U) \nleq \pi(P)$ but $\dim(\mathcal{R}(P_U) \cap \mathcal{R}(P)) = \infty$, for $U \in \mathcal{U}$. So $\{\pi(Q) : Q \in \mathcal{P}(\mathcal{B}(H)) \wedge \exists U \in \mathcal{U}(\mathcal{R}(P_U) \cap \mathcal{R}(P) \subseteq \mathcal{R}(Q))\}$ is a filter properly containing the upwards closure of $\pi[P_\mathcal{U}]$. $\square$

Forcing: $\mathscr{P}(\omega)/\mathrm{Fin}$ vs $\mathcal{P}(\mathcal{C}(H))$

- $\mathscr{P}(\omega)/\mathrm{Fin}$ is σ-closed so forcing with it adds no reals.
- But it does add a canonical subset of reals, namely a selective ultrafilter $\Leftrightarrow$ simultaneously a P-point and a Q-point.
- Likewise $\mathcal{P}(\mathcal{C}(H))$ is σ-closed and adds a 'selective' maximal centred filter $\mathcal{P}$, i.e. $\mathcal{P}$ is σ-closed and, for any basis (e_n) of H and finite interval partition (I_n) of ω, there exists a coarser partition (J_n) and $(v_n) \subseteq H$ with $v_n \in \mathrm{span}_{k \in J_n}(e_k)$ such that $\pi(P) \in \mathcal{P}$, where P is the projection onto $\overline{\mathrm{span}}(v_n)$.

Questions

What other similarities do these forcing extensions have? Could they even be the same? Do (e.g. Ramsey type) theorems about selective ultrafilters in $\mathscr{P}(\omega)/\mathrm{Fin}$ have natural analogs for these 'selective' maximal centred filters in $\mathcal{P}(\mathcal{C}(H))$? What about similar analogous forcing notions (e.g. Mathias forcing)?