PCF

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Conventions: When we discuss elementary substructures of H_θ we really mean $(H_\theta, \in, <_\theta)$. For $X \subseteq H_\theta$, $Hull(X)$ is the set of elements definable from parameters in X, that is the least elementary substructure containing X.

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Theorem (Shelah)

There exist an infinite set $A \subseteq \omega$ and a sequence $\langle f_i : i < \aleph_{\omega+1} \rangle$ such that the f_i are increasing and cofinal in $\prod_{n \in A} \aleph_n$ under the eventual domination ordering.

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Scale constructions will be based on a careful study of reduced powers of the ordinals.

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Lemma

There exist a sequence $\langle g_i : i < \aleph_{\omega+1} \rangle$ and a function h with $g_i \in \prod_{n\in\omega} \aleph_n$ and $h \in \prod_{n\in\omega}(\aleph_n + 1)$ such that

1. *The g_i are increasing and cofinal in $\prod_{n\in\omega} h(n)$ under the eventual domination ordering.*
2. *For each m the set $B_m = \{n : cf(h(n)) = \aleph_m\}$ is finite.*

If we now set $A = \{m : B_m \neq \emptyset\}$ and choose cofinal sequences $\langle \alpha^n_\eta : \eta < \aleph_m \rangle$ in $h(n)$ for each $n \in B_m$, it is straightforward to construct the functions $f_i \in \prod_{m \in A} \aleph_m$ as required.

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Associate to each $f \in \prod_{m \in A} \aleph_m$ the function $f^* \in \prod_{n \in \omega} h(n)$ which maps n to $\alpha^m_{f(m)}$ for the unique m such that $B_m \ni n$.

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Now construct f_i inductively by first choosing $\bar{\imath}$ such that $f^*_j <^* g_{\bar{\imath}}$ for all $j < i$ and then choosing f_i such that $g_{\bar{\imath}} < f^*_i$.

The key to the proof is to construct the increasing sequence $\langle g_i : i < \aleph_{\omega+1} \rangle$ in such a way that a suitable h exists. In the next section we describe a very general way of doing this kind of construction.

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The construction will have a "local" character in the sense that we just have to build g_j in a fairly simple way from $\langle g_i : i < j \rangle$.

Fix an index set X and a (proper) ideal I on X. Subsets of X lying in I will often be called *I-small*, while subsets not lying in I will be called *I-positive*. If S is I-positive and the set of $x \in S$ which fail to have a certain property $\phi(x)$ is I-small, we will often say that $\phi(x)$ *holds for I-almost every* $x \in S$.

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Definition

Let f and g in ${}^X ON$ and let R be one of the relations $\{<, =, >, \leqslant, \geqslant\}$ on ON. Then $f \, R_i \, g$ if and only if $f(x) R g(x)$ for I-almost every $x \in X$.

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Remark: The relation $<_I$ is a strict partial ordering, but is not the strict part of $\leqslant_I$ unless the ideal I is prime. $=_I$ is an equivalence relation and we will often write $[f]_I$ for the class of f.

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1. The sequence $\vec{f}$ is *I-increasing* if and only if $f_i <_I f_j$ for all $i < j < \zeta$.
2. The sequence $\vec{f}$ is *strongly I-increasing* if and only if there exist I-small sets $\langle Y_i : i < \zeta \rangle$ such that $f_i(x) < f_j(x)$ for all $i < j$ and all $x \notin Y_i \cup Y_j$.

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3. h is a *exact upper bound (eub) for* $\vec{f}$ if and only if h is an upper bound and for every $f <_I h$ there is $i < \zeta$ such that $f <_I f_i$.

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3. h is a *exact upper bound (eub) for* $\vec{f}$ if and only if h is an upper bound and for every $f <_I h$ there is $i < \zeta$ such that $f <_I f_i$.

Note that these definitions depend only on the classes $[h]_I$ and $[f_i]_I$.

It is easy to see that

- If h_1 and h_2 are both lubs for $\vec{f}$ then $h_1 =_I h_2$.
- If h is an eub for $\vec{f}$ then it is an lub.

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If $\vec{f}$ is increasing and cofinal in $(\prod_{x \in X} h(x), <_I)$ then h is an eub for $\vec{f}$. The converse is false for the trivial reason that for some i the relation $f_i < h$ may fail: however as $f_i <_I h$ we can easily alter the f_i on I-small sets to fix this.

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Example: Let I be the NS ideal on ω_1 and let f_i be the function with constant value i for every $i < \omega_1$. By Fodor the identity function is an lub for $\vec{f}$. However it is not an eub: by Solovay splitting we may construct a function f such that $f <_I id$ and f assumes each value less than ω_1 stationarily often.

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It easy to see that if λ is regular, $\vec{f}$ is I-increasing of length λ and h is an eub then $cf(h(x)) \leqslant \lambda$ for I-almost all x.

It is consistent (for example under MA with large continuum) that there are long sequences in ${}^{\omega}\omega$ which are increasing and cofinal mod finite. Here an eub is the function with constant value ω.

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Lemma

If $\vec{f}$ is I-increasing and there is g such that the sequence of sets $\{x : f_j(x) < g(x)\}$ is not eventually constant mod I for large j, then $\vec{f}$ has no eub.

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Proof.

If h is an eub then let $S = \{x : g(x) < h(x)\}$ and use the fact that h is an eub to find i such that $g(x) < f_i(x)$ for I-almost every $x \in S$. Then for $j \geqslant i$ we have that $g(x) < f_j(x)$ for I-almost every $x \in S$, and since $f_j <_I h$ also that $f_j(x) < h(x) \leqslant g(x)$ for I-almost every $x \in S^c$. So $\{x : f_j(x) < g(x)\} = S^c \mod I$ for $j \geqslant i$, contradiction. □

Lemma

Let $\vec{f}$ be I-increasing and let μ be a regular cardinal. Suppose that there exist sets of ordinals $\langle S_x : x \in X \rangle$ of size less than μ and an ultrafilter U on X disjoint from I, such that for every α there exist $g \in \prod_x S_x$ and β with $f_\alpha <_U g <_U f_\beta$. Then there does not exist an eub h such that $cf(h(x)) \geqslant \mu$ for all x.

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Proof.

Suppose for a contradiction that such an eub h exists. Without loss of generality $S_x \subseteq h(x)$ for all x. Define a function g_0 by $g_0(x) = \sup(S_x)$, so that $g_0 < h$. Find α such that $g_0 <_I f_\alpha$, and then $g \in \prod_x S_x$ and $\beta > \alpha$ such that $f_\alpha <_U g <_U f_\beta$. Since U is disjoint from I we actually have $g_0 <_U f_\alpha <_U g <_U f_\beta$, so we may find a coordinate x such that $g_0(x) < f_\alpha(x) < g(x) < f_\beta(x)$; this is a contradiction as $g_0(x) = \sup(S_x)$ and $g(x) \in S_x$. □

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Theorem (Trichotomy theorem)

Let $|X| = \kappa$, let $\mu > \kappa$ be regular and let $\vec{f}$ be I-increasing of regular length $\lambda > \mu$. Then at least one of the following holds:

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(Bad case) There exist nonempty sets $S_x \subseteq ON$ for $x \in X$ such that $|S_x| < \mu$, and an ultrafilter U on X with $U \cap I = \emptyset$, such that for every $\alpha < \lambda$ there exist $g \in \prod_x S_x$ and $\beta < \lambda$ with $f_\alpha <_U g <_U f_\beta$.

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(Ugly case) There exists $g \in {}^X ON$ such that the sequence of sets $\{x : f_\alpha(x) < g(x)\}$ is not eventually constant modulo I.

Sketchy proof: Assume we are not in Bad or Ugly case. It follows from not-Ugly that an lub is an eub and from not-Bad that any eub has cofinality at least μ almost everywhere.

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$h^j_\alpha(x) = \min\{S^j_x \setminus (f_\alpha(x) + 1)\}$.
Claim that h^j_α stabilises mod I for large α.

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Claim that h^j_α stabilises mod I for large α. If not then for all α we have that f_β beats h^j_α on a positive set for all large α, and by not-Ugly this set is stable mod I for large β.

Sketchy proof: Assume we are not in Bad or Ugly case. It follows from not-Ugly that an lub is an eub and from not-Bad that any eub has cofinality at least μ almost everywhere. Assume for contradiction there is no lub. We build upper bounds g_j for $j < \mu$ such that $g_j \leqslant_I g_i$ and $g_j \neq_I g_i$ for $i < j$, starting with $g_0 > f_\alpha$ for all α. Successor step is easy as g_i is not a lub. For limit j we define $S^j_x = \{g_i(x) : i < j\}$ and then $h^j_\alpha(x) = \min\{S^j_x \setminus (f_\alpha(x) + 1)\}$.
Claim that h^j_α stabilises mod I for large α. If not then for all α we have that f_β beats h^j_α on a positive set for all large α, and by not-Ugly this set is stable mod I for large β. Appealing to ZL we get U disjoint from I such that for all α there is β and $f_\alpha < h^j_\alpha <_U f_\beta$, and we are Bad. Choose g_j as the stable value.

Now let $S_x^\infty = \bigcup_{j<\mu} S_x^j = \{g_i(x) : i < \mu\}$ and $h_\alpha^\infty(x) = \min\{S_x^\infty \setminus (f_\alpha(x)+1)\}$.

Now let $S_x^\infty = \bigcup_{j<\mu} S_x^j = \{g_i(x) : i < \mu\}$ and $h_\alpha^\infty(x) = \min\{S_x^\infty \setminus (f_\alpha(x) + 1)\}$. As $\mu = cf(\mu) > \kappa$, for each α there is j such that $h_\alpha^\infty = h_\alpha^j$.

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Note to the reader: The "classical" version of Trichotomy which I presented at YSTW 2013 is the one with $\mu = \kappa^+$. The proof of this version is essentially the same.

The caveat: in order to be in the Good case it is essential that $\lambda \geqslant \mu$ and we are not in the Bad or Ugly cases. But the hypothesis says that $\lambda > \mu$ (and this is known to be necessary, at least for the classical case when $\mu = \kappa^+$).

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We still need a practical way of ensuring we end up in the Good Case.

Cheap way: if $\mu = \kappa^+$ and $\lambda > 2^\kappa$ then not hard to see that Bad and Ugly cases are impossible.

Definition

Let $|X| = \kappa$, let I be an ideal on X and let $\vec{f}$ be I-increasing. Then $\alpha \leqslant lh(\vec{f})$ is a *good (or flat)* point for $\vec{f}$ if and only if $cf(\alpha) > \kappa$ and there is an eub h for $\vec{f} \restriction \alpha$ such that $cf(h(x)) = cf(\alpha)$ for all x.

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Remark: It is not hard to see that in case $cf(\alpha) > \kappa$, if there exist an eub h and a cardinal $\mu > \kappa$ such that $cf(h(x)) = \mu$ for all x then necessarily $\mu = cf(\alpha)$.

Two I-increasing sequences $\vec{f}$ and $\vec{g}$ are *cofinally interleaved* iff for all $i < lh(\vec{f})$ there is $j < lh(\vec{g})$ with $f_i <_I g_j$, and conversely for all $j < lh(\vec{g})$ there is $i < lh(\vec{f})$ with $g_j <_I f_i$.

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It is not hard to see that if $\vec{f}$ and $\vec{g}$ are cofinally interleaved then h is an eub for $\vec{f}$ iff h is an eub for $\vec{g}$. Conversely if $\vec{f}$ and $\vec{g}$ have a common eub they are cofinally interleaved.

Theorem

Let $|X| = \kappa$, let I be an ideal on X and let $\vec{f}$ be $<_I$-increasing. Let $\alpha \leqslant lh(\vec{f})$ with $cf(\alpha) > \kappa$. The following are equivalent:

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1. *α is good.*
2. *There exists a sequence $\langle H_j : j < cf(\alpha)\rangle$ of functions in ${}^X ON$ which is pointwise increasing and cofinally interleaved with $\vec{f} = \langle f_i : i < \alpha\rangle$.*

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3. *For every unbounded set $B \subseteq \alpha$, there is an unbounded $A \subseteq B$ such that $\langle f_i : i \in A \rangle$ is strongly I-increasing.*

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For 2 implies 3 we find an unbounded set $A \subseteq B$ such that each $\delta \in A$ can be "bracketed" by H's in the sense that $H_{j_\delta} <_I f_\delta < H_{j'_\delta}$ where $j'_\gamma < j_\delta$ for all $\gamma < \delta$. Now we choose I-small sets Y_δ such that $H_{j_\delta}(x) < f_\delta(x) < H_{j'_\delta}(x)$ for $x \notin Y_\delta$, and use them to witness the strong increasingness.

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3 implies 4 is easy.
For 4 implies 1 we thin out A to have order type $cf(\alpha)$, fix $\langle Y_\delta : \delta \in A \rangle$ to witness the strong increasingness, and let $h(x) = \sup\{f_\delta(x) : \delta \in A, x \notin Y_\delta\}$. To see it's an eub let $g <_I h$, choose $\delta_x \in A$ such that $x \notin Y_{\delta_x}$ and $g(x) < f_{\delta_x}(x)$, find $\delta \in A$ with $\delta > \delta_x$ for all x and observe that for $x \notin Y_\delta$ we have $g(x) < f_{\delta_x}(x) < f_\delta(x)$. A similar argument shows $cf(h(x)) = cf(\alpha)$ for I-almost all x.

□

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Theorem

Let $|X| = \kappa < \delta = cf(\delta) < \lambda = cf(\lambda)$, let I be an ideal on X and $\vec{f}$ an I-increasing sequence of length λ. TFAE:

1. *There are stationarily many good points in $\lambda \cap cof(\delta)$.*
2. *There is an eub h for $\vec{f}$ such that $cf(h(x)) > \delta$ for all x.*

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1. *There are stationarily many good points in $\lambda \cap cof(\delta)$.*
2. *There is an eub h for $\vec{f}$ such that $cf(h(x)) > \delta$ for all x.*

It is easy to verify that 1 above is equivalent to the assertion "For every $C \subseteq \lambda$ club there is $A \subseteq C$ of order type δ such that $\langle f_i : i \in A\rangle$ is strongly increasing", which is the form used in Abraham and Magidor's Handbook article.

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For 1 implies 2, first use the hypothesis to show we are not in Bad or Ugly cases of the Trichotomy. Then use it again to argue that the eub assured by the good case has cofinality above δ at I-almost every coordinate.

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We sketch main ideas.
For 1 implies 2, first use the hypothesis to show we are not in Bad or Ugly cases of the Trichotomy. Then use it again to argue that the eub assured by the good case has cofinality above δ at I-almost every coordinate.
For 2 implies 1, fix θ so that all relevant objects are in H_θ and build everything into an internally approachable submodel $M = \bigcup_{i<\delta} M_i$ of cardinality δ. Now let $\alpha = \sup(M \cap \lambda)$ and argue that the functions $h_i : x \mapsto \sup(M_i \cap h(x))$ are pointwise increasing and cofinally interleaved with $\langle f_\eta : \eta < \alpha\rangle$. □

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Theorem (Shelah's Club Guessing Theorem)

Let ρ and σ be regular cardinals with $\rho^+ < \sigma$, and let $S \subseteq \sigma \cap cof(\rho)$ be stationary. Then there exists a club guessing sequence *$\langle C_\eta : \eta \in S \rangle$, that is a sequence such that:*

1. *C_η is a club subset of η with order type ρ.*
2. *For every club subset E of σ there is $\eta \in S$ such that $C_\eta \subseteq E$.*

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Theorem

Let $|X| = \kappa$, let I be an ideal on X. Let ρ, σ and λ be regular cardinals where $\kappa < \rho < \rho^+ < \sigma < \lambda$. Let $\vec{f}$ be a I-increasing sequence of length λ, such that for every $\delta \in \lambda \cap cof(\sigma)$ there is E_δ a club subset of δ such that $\sup_{i \in E_\delta} f_i \leqslant_I f_\delta$. Then there are stationarily many good points in $\lambda \cap cof(\rho)$, so that $\vec{f}$ has an eub h such that $cf(h(x)) > \rho$ for all x.

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Historical note: Shelah's original method for generating I-increasing sequences with such an eub was more indirect, going via a proof that the ideal $I[\lambda]$ contains a stationary set. The Abraham-Magidor argument uses some ideas from this earlier work.

Proof.

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Fix a club guessing sequence $\langle C_\eta : \eta \in \sigma \cap cof(\rho) \rangle$ and let $E \subseteq \lambda$ be club. We build an increasing and continuous sequence $\vec{\gamma}$ of length $\sigma + 1$ of elements of E. To construct γ_{j+1}, for each $\eta \in \sigma \cap cof(\rho)$ we let $h_\eta = \sup_{i \in C_\eta \cap (j+1)} f_{\gamma_i}$. Then choose $\gamma_{j+1} \in E$ such that for all η, if there is $\gamma < \lambda$ with $h_\eta <_I f_\gamma$ then $h_\eta <_I f_{\gamma_{j+1}}$.

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Let $D = \{i < \sigma : \gamma_i \in E_{\gamma_\sigma}\}$, and find $j \in \sigma \cap cof(\rho)$ such that $C_j \subseteq D$.

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Let $D = \{i < \sigma : \gamma_i \in E_{\gamma_\sigma}\}$, and find $j \in \sigma \cap cof(\rho)$ such that $C_j \subseteq D$. For each $j_0 \in C_j$ we considered at stage $j_0 + 1$ a function $h = \sup_{i \in C_j \cap (j_0+1)} f_{\gamma_i}$, and since $C_j \subseteq D$ we have $h \leqslant \sup_{i \in E_{\gamma_\sigma}} f_i \leqslant_I f_{\gamma_\sigma}$.

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If $\bar{\imath} < \bar{\jmath}$ are successor points of C_j and $x \notin Y_{\bar{\jmath}}$ then $f_{\gamma_{\bar{\imath}}}(x) < f_{\gamma_{\bar{\jmath}}}(x)$, so we have a witness that γ_j is good.

□

We can now prove Shelah's result that there exists an infinite set $A \subseteq \omega$ and a sequence $\langle f_i : i < \aleph_{\omega+1} \rangle$ such that the f_i are increasing and cofinal in $\prod_{i \in A} \aleph_i$ under the eventual domination ordering. As we saw, it is enough to prove that there is a *FIN*-increasing sequence in $\prod_n \aleph_n$ with an eub h such that $cf(h(n)) \to \aleph_\omega$ mod *FIN*.

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This is straightforward. We can readily build a *FIN*-increasing sequence $\vec{g}$ of length $\aleph_{\omega+1}$ such that for every limit $\alpha < \aleph_{\omega+1}$ there is a club set $C \subseteq \alpha$ such that $f_\alpha \geqslant^* \sup_{i \in C} f_i$. There are stationarily many good points in each cofinality, so an eub h exists and $cf(h(n)) \to \aleph_\omega$ mod *FIN*.

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Theorem (Shelah)

Let μ be a singular cardinal of some uncountable cofinality κ, and let $C \subseteq \mu$ be some club set of singular cardinals. Then there are a club subset $D \subseteq C$ and a sequence $\vec{f}$ which is increasing and cofinal in $\prod_{\lambda \in D} \lambda^+$ modulo the NS ideal.

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Proof.

As in the last result it is easy to build a sequence $\vec{g} \in \prod_{\lambda \in C} \lambda^+$ with an eub h such that $cf(h(\lambda)) \to \mu$ mod NS. We claim that $h(\lambda) = \lambda^+$ for NS-almost every λ; for if not then as all λ are singular $cf(h(\lambda)) < \lambda$ on a stationary set, so by Fodor $cf(h(\lambda))$ is bounded on a stationary set of λ, contradiction! □

Let $A \subseteq \omega$ be an infinite set such that there is a scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} \aleph_n$ (there is actually a maximal such A which is well-defined mod finite). Any two such scales are cofinally interleaved, and so the set of good points in such a scale is independent (mod clubs) of the choice of scale. This set can be used to measure how "*L*-like" the universe of set theory is.

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For each n such that $1 < n < \omega$, there are stationarily many good points of cofinality $\aleph_n$. If $\square^*_{\aleph_\omega}$ holds then almost every point is good

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Proof.

First part: Let E be club, and build an IA structure M such that $E \in M \prec H_\theta$ where M has length and cardinality $\aleph_n$ as witnessed by $\langle M_i : i < \aleph_n \rangle$. Now let $\chi_i(m) = \sup(M \cap \aleph_m)$ for $m > n$, let $\gamma = \sup(M \cap \aleph_{\omega+1})$ and note that $\gamma \in E$. The functions χ_i are pointwise increasing and cofinally interleaved with $\vec{f} \restriction \gamma$.

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Second part: $\square^*_{\aleph_\omega}$ implies almost every $\gamma < \aleph_{\omega+1}$ of cofinality $\aleph_n$ is of the form $\sup(M \cap \aleph_{\omega+1})$ for some M as in first part.

□

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Note: Hypotheses imply there is $\langle A_\alpha : \alpha < \mu^+ \rangle$ with A_α unbounded in μ, such that for each proper initial segment we can choose disjoint tails. This implies conclusion.

Lemma (Cummings)

Let $n > 0$. If almost every point of cofinality at least $\aleph_n$ is good, then there is no outer model W such that $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$.

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Proof.

Work in W. Let $\lambda = (\aleph_n)^W$, and write $(\aleph_\omega)^V$ as $\bigcup_{i<\lambda} X_i$ where $|X_i| < \lambda$. Find a stationary set S of points of cofinality λ and a fixed $i < \lambda$ such that $rge(f_\eta) \subseteq X_i$ for every $\eta \in S$. Find $\eta \in S$ such that η is good in V (hence good in W), and get a contradiction as in the discussion of trichotomy. □

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Lemma (Shelah)

If κ is $\kappa^{+\omega+1}$-supercompact then there are unboundedly many inaccessible $\delta < \kappa$ such that there are stationarily many ungood points in $\kappa^{+\omega+1} \cap cof(\delta^{+\omega+1})$.

Proof.

Let j witness that κ is $\kappa^{+\omega+1}$-supercompact, and let $\vec{f}$ be a scale in $\prod_n \kappa^{+n}$ mod finite. Let $\lambda = \sup j``\kappa^{+\omega+1}$, and argue that the function $n \mapsto \sup j``\kappa^{+n}$ is an eub for $j(f) \restriction \lambda$. So on the j-side λ is ungood and $cf(\lambda) = \kappa^{+\omega+1}$, and we finish by standard reflection arguments. □

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Theorem (Cummings, Foreman and Magidor)

If κ is supercompact there is $\delta < \kappa$ such that in the extension by $Coll(\omega, \delta^{+\omega}) \times Coll(\delta^{+\omega+2}, <\kappa)$ there is a co-stationary set of ungood points of cofinality $\aleph_1$.

Proof.

Choose δ such that there is a stationary set S of ungood points of cofinality $\delta^{+\omega+1}$ for a scale $\vec{f}$.

Proof.

Choose δ such that there is a stationary set S of ungood points of cofinality $\delta^{+\omega+1}$ for a scale $\vec{f}$. Force to get a generic extension $V[G \times H]$ for the poset $Coll(\omega, \delta^{+\omega}) \times Coll(\delta^{+\omega+2}, < \kappa)$, and note that $\kappa^{+\omega+1} = \aleph_{\omega+1}^{V[G]}$ and $\delta^{+\omega+1} = \aleph_1^{V[G]}$.

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□

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Proof.

Build a scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} \aleph_n$ as before. A closer inspection of the arguments shows that at every γ of cofinality at least $\aleph_4$ there are stationarily many good points, and hence an eub h exists. Now easy to see $cf(h(n)) = cf(\gamma)$ for almost all n, so γ is good. □

A quick review of more PCF theory (without proofs). A *progressive set* is a set A of regular cardinals with $|A|^+ < \min(A)$ and no largest element, and βA is the set of ultrafilters on A. For any singular μ we may choose such an A with $\sup(A) = \mu$, and if $\mu < \aleph_\mu$ we may in addition choose A to be an interval of regular cardinals.

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Remark: clearly $|pcf(A)| \leqslant 2^{2^{|A|}}$, $\sup pcf(A) \leqslant |\prod A|$.

Some basic facts (all due to Shelah)

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9. $|pcf(A)| \leqslant 2^{|A|}$.
10. If A is an interval of regular cardinals, then so is $pcf(A)$.

Given a progressive set A and an elementary substructure $M \prec H_\theta$ for some large θ, the *characteristic function of M on A* is the function given by $\chi_M^A(\kappa) = \sup(\kappa \cap M)$ for all $\kappa \in A$. For certain M these functions are closely tied to PCF.

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For simplicity we focus on the progressive set $A = \{\aleph_n : 1 < n < \omega\}$ and the class of IA elementary substructures M of length and cardinality $\aleph_1$. It is a key fact that for every regular $\kappa > \aleph_1$, if $\kappa \in M$ then $M \cap \kappa$ has a cofinal closed subset of order type ω_1 (with every proper initial segment lying in M).

Theorem (Shelah)

Let M and N be IA elementary substructures of length and cardinality $\aleph_1$, and suppose that $\chi_M^A = \chi_N^A$ where $A = \{\aleph_n : 1 < n < \omega\}$. Then $M \cap \aleph_\omega = N \cap \aleph_\omega$.

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Proof.

We prove by induction that $M \cap \aleph_n = N \cap \aleph_n$, where the base case $n = 1$ is trivial as $\aleph_1 \subseteq M \cap N$. Induction step: let $\delta = \chi_M(\aleph_{n+1}) = \chi_N(\aleph_{n+1})$, so that $M \cap N \cap \aleph_{n+1}$ is cofinal in δ. For each $\gamma \in M \cap N \cap \aleph_{n+1}$ let f be the least injection from γ into $\aleph_n$, then $f \in M \cap N$ and $f``(M \cap \gamma) = M \cap \aleph_n = N \cap \aleph_n = f``(N \cap \gamma)$. So $M \cap \gamma = N \cap \gamma$ for cofinally many $\gamma < \delta$ and so $M \cap \aleph_{n+1} = N \cap \aleph_{n+1}$. □

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- For every λ and every $\alpha \in \lambda \cap cof(\aleph_1)$, $f^\lambda_\alpha(n) = \min\{\sup_{\eta \in C} f_\eta(n) : C \text{ club in } \alpha\}$.

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Theorem (Shelah)

If $M \prec H_\theta$ is IA of length and cardinality $\aleph_1$ and M contains $\langle B_\lambda\rangle$ and $\langle f^\lambda_\alpha\rangle$ (regarded as a function of two variables), then χ_M is the pointwise sup of finitely many functions of the form $f^{\lambda_i}_{\gamma_i}$ where $\lambda_i \in pcf(A) \cap M$ and $\gamma_i = \sup(M \cap \lambda_i)$.

Proof.

Fix $\langle M_i \rangle$ witnessing M is IA. Let $\lambda_0 = max\ pcf(A)$ and $\gamma_0 = \sup(M \cap \lambda_0)$. Find $C_0 \subseteq M \cap \gamma_0$ club so that $f^{\lambda_0}_{\gamma_0} = \sup_{\eta \in D} f^{\lambda_0}_{\eta}$ for all club $D \subseteq C_0$, and observe $f^{\lambda_0}_{\gamma_0} \leqslant \chi_M^{B_{\lambda_0}}$.

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Key point: There is $A_0 \in M \cap J_{<\lambda_0}$ such that $f^{\lambda_0}_{\gamma_0}$ and $\chi^{B_{\lambda_0}}_{M}$ agree on $B_{\lambda_0} \setminus A_0$. Proof: find $i < \omega_1$ such that for all κ $f^{\lambda_0}_{\gamma_0}(\kappa) < \chi^{B_{\lambda_0}}_{M}(\kappa) \implies f^{\lambda_0}_{\gamma_0}(\kappa) < \chi^{B_{\lambda_0}}_{M_i}(\kappa)$, find $\alpha_0 \in M \cap C_0$ such that $\chi^{B_{\lambda_0}}_{M_i} <_{J_{<\lambda_0}} f^{\lambda_0}_{\alpha_0}$, and set $A_0 = \{\kappa : \chi^{B_{\lambda_0}}_{M_i}(\kappa) \geqslant f^{\lambda_0}_{\alpha_0}(\kappa)\}$.

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Fix $\langle M_i \rangle$ witnessing M is IA. Let $\lambda_0 = max\ pcf(A)$ and $\gamma_0 = \sup(M \cap \lambda_0)$. Find $C_0 \subseteq M \cap \gamma_0$ club so that $f^{\lambda_0}_{\gamma_0} = \sup_{\eta \in D} f^{\lambda_0}_{\eta}$ for all club $D \subseteq C_0$, and observe $f^{\lambda_0}_{\gamma_0} \leqslant \chi^{B_{\lambda_0}}_M$.

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If A_0 is nonempty let $\lambda_1 < \lambda_0$ be least such that $A_0 \notin J_{<\lambda_1}$, so that $\lambda_1 \in M$ and $A_0 \setminus B_{\lambda_1} \in J_{<\lambda_1}$. Let $\gamma_1 = \sup(M \cap \lambda_1)$. As above find $A_1 \in M \cap J_{<\lambda_1}$ such that $A_0 \setminus B_{\lambda_1} \subseteq A_1$, and $f^{\lambda_1}_{\gamma_1}$ and $\chi^{B_{\lambda_1}}_M$ agree on $B_{\lambda_1} \setminus A_1$. Repeat till done. □

Corollary

There is a stationary set in $[\aleph_\omega]^{\aleph_0}$ *of size* $\max\ pcf(A)$.

Proof.

The set of traces of suitable IA structures of size $\aleph_1$ is stationary and has size at most $\max\ pcf(A)$. Decompose each trace as a continuous union of ω_1 countable structures to get a stationary set of the same size in $[\aleph_\omega]^{\aleph_0}$. Note that no cofinal set X in $[\aleph_\omega]^{\aleph_0}$ can have size less that $\lambda = \max pcf(A)$; otherwise we form $\{\chi_x^{B_\lambda} : x \in X\}$, find α such that f_α^λ dominates them all mod $J_{<\lambda}$, and observe that no $x \in X$ can contain $rge(f_\alpha^\lambda)$. □

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Proof.

By strong limit hypothesis $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$, and if we fix a cofinal set C in $[\aleph_\omega]^{\aleph_0}$ then easily $\aleph_\omega^{\aleph_0} \leqslant 2^{\aleph_0}|C|$. Now use previous result and the easy observation that since $pcf(A)$ is an interval of cardinals and $|pcf(A)| \leqslant 2^{|A|} = 2^{\aleph_0}$, max $pcf(A) < \aleph_{(2^{\aleph_0})^+}$. □

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By another deep result of Shelah, for any interval A of regular cardinals we have $|pcf(A)| < |A|^{+4}$, which yields his celebrated result that if $\aleph_\omega$ is strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

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Theorem (Shelah)

If μ is singular there is a colouring of $[\mu^+]^2$ in $cf(\mu)$ colours such that every unbounded subset of μ^+ contains pairs of every colour.

Before the proof, some preliminaries. Ideas we have seen already show that if μ is singular, there exist an increasing sequence $\langle \mu_i : i < cf(\mu) \rangle$ of regular cardinals which is cofinal in μ, and a sequence $\vec{f}$ of length μ^+, such that $\vec{f}$ is increasing and cofinal in $\prod_i \mu_i$ under eventual domination. We may assume that $cf(\mu) < \mu_0$.

Proof.

Sufficient to build a colouring c such that on each unbounded set c takes on all large values. Let $c(\alpha, \beta)$ be the largest j such that $f_\alpha(j) > f_\beta(j)$ (when it exists).

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Fix $A \subseteq \mu^+$ which is unbounded. There is $i < cf(\mu)$ such that for every $h \in \prod_i \mu_i$ there is $\gamma \in A$ such that $h(j) < f_\gamma(j)$ for all $j \geqslant i$; otherwise choose a counterexample h_i for each i and find $\gamma \in A$ such that $\sup_i h_i <^* f_\gamma$ to get an immediate contradiction.

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Let $j \geqslant i$. Let $M \prec H_\theta$ for some large θ with $cf(\mu) \cup \{A, \vec{f}, i\} \subseteq M$ and $|M| < \mu_j$. Find $\beta \in A$ such that $\sup(M \cap \mu_k) < f_\beta(k)$ for $k \geqslant j$, set $N = Hull(M \cup \mu_j)$ and note $\sup(N \cap \mu_k) = \sup(M \cap \mu_k)$ for $k > j$. Now use elementarity to choose $\alpha \in A \cap N$ such that $f_\alpha(j) > f_\beta(j)$; for $k > j$ we have $f_\alpha(k) \in N$ so that $f_\alpha(k) < \sup(N \cap \mu_k) = \sup(M \cap \mu_k) < f_\beta(k)$. So $c(\alpha, \beta) = j$. □

Now for some application to *Jonsson cardinals*. Recall that κ is Jonsson if every algebra on κ (that is to say a structure for a countable FOL with underlying set κ) has a proper elementary substructure of size κ.

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It is open whether successors of singulars can ever be Jonsson. PCF gives information about this question: the idea is that scales of length μ^+ in a singular cardinal μ create some connection between μ^+ and regular cardinal below μ. We will give a couple of illustrations of this idea.

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If κ is a regular Jonsson cardinal then every stationary subset of κ reflects.

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Let S be non-reflecting and appeal to Jonsson property to get $\kappa, S \in M \prec H_\theta$ such that $|M \cap \kappa| = \kappa$ and $\kappa \not\subseteq M$.

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Let S be non-reflecting and appeal to Jonsson property to get $\kappa, S \in M \prec H_\theta$ such that $|M \cap \kappa| = \kappa$ and $\kappa \not\subseteq M$. Use Solovay in M to find a partition lying in M of the set S into κ stationary pieces S_α, and observe that if $\alpha \notin M$ then $S_\alpha \cap M = \emptyset$. In particular $S \setminus M$ is stationary.

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For every regular κ, the set $\kappa^+ \cap cof(\kappa)$ does not reflect, so κ^+ is not Jonsson. What about $\aleph_{\omega+1}$? A result by Magidor shows that consistently every stationary subset of $\aleph_{\omega+1}$ reflects.

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$\aleph_{\omega+1}$ *is not Jonsson.*

For every regular κ, the set $\kappa^+ \cap cof(\kappa)$ does not reflect, so κ^+ is not Jonsson. What about $\aleph_{\omega+1}$? A result by Magidor shows that consistently every stationary subset of $\aleph_{\omega+1}$ reflects.

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$\aleph_{\omega+1}$ is not Jonsson.

Proof.

Let $\vec{f}$ be a scale of length $\aleph_{\omega+1}$ in $\prod_{n\in A}\aleph_n$ for some infinite A and let $M \prec H_\theta$ with $\vec{f} \in M$ and $M \cap \aleph_{\omega+1}$ unbounded. $M \cap \aleph_n$ is unbounded for all large $n \in A$, otherwise we can find $\gamma \in M$ such that $\sup(M \cap \aleph_n) < f_\gamma(n) \in M$ for some n. The $\aleph_n$'s are not Jonsson and M sees witnesses to this so $\aleph_\omega \subseteq M$, hence $\aleph_{\omega+1} \subseteq M$. □

Further reading:

Abraham and Magidor, Cardinal arithmetic, in the Handbook of Set Theory

Kojman, The abc of pcf, available on the web at http://www.cs.bgu.ac.il/ kojman/ABCI.pdf

Shelah, Cardinal arithmetic, Oxford University Press