Descriptive Set Theory on Generalised Baire Space

An exciting area of current research in set theory is *Descriptive Set Theory*, the study of projective sets of reals.

The classical theory focuses on Borel, analytic and sometimes PCA (boldface $\mathbf{\Sigma}^1_2$) relations on reals.

Some of the highlights:

Descriptive Set Theory on Generalised Baire Space

Regularity Properties

Kuratowski-Ulam (Fubini for Category)

Mycielski (Perfectly-many classes for meager equivalence relations)

Analytic sets are $\mathbb{P}$-measurable for many forcings $\mathbb{P}$

Characterisations of $\mathbb{P}$-measurability for Δ^1_2, Σ^1_2 sets (Solovay, Judah-Shelah, Brendle-Löwe, et.al.)

Borel Reducibility

Silver and Harrington-Kechris-Louveau Dichotomies

Countable Borel equivalence relations

The (E_0, E_1) Dichotomy

Isomorphism Relations

The Jump Hierarchy for Borel Isomorphism

Turbulence (Irreducibility to Isomorphism)

Descriptive Set Theory on Generalised Baire Space

Recently there has been a lot of interest in developing Descriptive Set Theory when the *classical Baire space* ω^ω is replaced by the *generalised Baire space* κ^κ for an uncountable regular cardinal κ.

Generalised Baire space κ^κ*:* Basic open sets are of the form $U(\sigma) = \{\eta \in \kappa^\kappa \mid \sigma \subseteq \eta\}$, where σ belongs to $\kappa^{<\kappa}$.

With this topology, closed sets are the sets of κ-branches through a subtree of $\kappa^{<\kappa}$.

We make the *further assumption* $\kappa^{<\kappa} = \kappa$, i.e., there are only κ-many bounded subsets of κ. This implies that there is a dense set of size κ.

Descriptive Set Theory on Generalised Baire Space

Reasons for a Generalised Theory

1. A finer understanding of the classical theory.
For example, Mycielski's Theorem uses the inaccessibility of ω, whereas Kuratowski-Ulam does not.

2. Connections with forcing and combinatorial principles.
Theorems in the classical case sometimes become only consistency results for κ^κ, established using forcing and $\diamondsuit$.

3. Surprises!
Sometimes classical results become provably false for κ^κ, which presents interesting new challenges.

Descriptive Set Theory on Generalised Baire Space

4. Model theory (the original source of my interest).
Shelah's classification theory for first-order theories fits well with descriptive set theory on κ^κ, but not with descriptive set theory on the classical Baire space.

5. Foundational?
Maybe the generalised theory will suggest the "right" axioms to add to ZFC.

Descriptive Set Theory on Generalised Baire Space

In outline, this tutorial looks as follows:

A. Kuratowski-Ulam, Mycielski and the Baire Property.
B. Other regularity properties.
C. Borel reducibility.
D. Isomorphism relations.

And we will see three types of results for uncountable κ:

Type 1: Results from the classical case that provably generalise.
Type 2: Results from the classical case that consistently, but not provably, generalise.
Type 3: Results from the classical case that are false!

And in the case of Type 3, sometimes the results are *better* than in the classical case.

Borel, Meager and Kuratowski-Ulam

The *Borel sets* for κ^κ are obtained by closing the basic open sets under unions of size κ and complements.
Open sets are Borel, thanks to the assumption $\kappa^{<\kappa} = \kappa$.

$X \subseteq \kappa^\kappa$ is *nowhere dense* if its closure contains no nonempty open set and is *meager* if it is the union of κ-many nowhere dense sets.
X has the *Baire property* if its symmetric difference with some open set is meager.

The entire space is not meager, thanks to:

Theorem

(Baire Category Theorem) The intersection of κ-many open dense sets is dense.

Proof. Suppose that D_i, $i < \kappa$ are open dense and let $U(\sigma)$ be a basic open set.

Build a κ-sequence $\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \cdots$ where $U(\sigma_{i+1})$ is contained in D_i and $\sigma_\lambda = \cup_{i<\lambda}\sigma_i$ for limit $\lambda < \kappa$; this is possible as each D_i is open dense.

Then $\eta = \cup_{i<\kappa}\sigma_i$ belongs to each D_i. $\square$

Borel, Meager and Kuratowski-Ulam

Theorem

Borel sets have the Baire property.

Proof. It suffices to show that the collection of sets with the Baire property contains the basic open sets and is closed under size κ unions and complements.

The fact that it contains the basic open sets is trivial and as any closed set differs by a meager set from its interior, it is also closed under complements.

The case of κ-unions follows from the fact that the union of κ-many meager sets is meager. $\square$

Borel, Meager and Kuratowski-Ulam

Theorem

(Kuratowski-Ulam) Let X denote κ^κ and suppose that $A \subseteq X \times X$ has the Baire property. For each $x \in X$ let A_x denote $\{y \mid (x, y) \in A\}$. Then:
(a) $\{x \mid A_x$ has the Baire Property$\}$ is comeager.
(b) A is meager iff $\{x \mid A_x$ is meager$\}$ is comeager (it follows that A is comeager iff $\{x \mid A_x$ is comeager$\}$ is comeager).

The proof is a direct generalisation of the proof in the classical case.

Borel, Meager and Kuratowski-Ulam

Proof. First suppose that A is open dense and we show that A_x is open dense for comeager-many x:
Clearly A_x is open for each x, so we just have to show that A_x is dense for comeager-many x. Let $(V_i \mid i < \kappa)$ be a basis for the topology on X. Then for each i, $U_i = \{x \mid (x, y) \in A$ for some $y \in V_i\}$ is open dense since if W is nonempty and open, $A \cap (W \times V_i)$ is nonempty by the density of A. Thus for $x \in \cap_i U_i$, $A_x \cap V_i$ is nonempty for each i, i.e. A_x is dense.

It follows that if A is meager then A_x is meager for comeager-many x, which is the direction $\rightarrow$ of (b).

To prove (a), choose an open U and meager M so that $A = U \triangle M$. Then for each x, $A_x = U_x \triangle M_x$ and M_x is meager for comeager-many x. It follows that A_x has the Baire property for comeager-many x, which is (a).

Borel, Meager and Kuratowski-Ulam

Finally, suppose that A is not meager; we show that $\{x \mid A_x$ is not meager$\}$ is not meager:
Write $A = U \triangle M$ where U is a nonempty open set and M is meager. U contains $V_0 \times V_1$ where V_0, V_1 are nonempty open sets. For x in V_1, if M_x is meager then $A_x = U_x \triangle M_x$ is comeager on a nonempty open set and therefore not meager. As M_x is meager for comeager-many x, it follows that the set of x such that A_x is not meager is comeager on a nonempty open set and therefore not meager. $\square$

Mycielski's Theorem

A subtree T of $\kappa^{<\kappa}$ is *perfect* if the limit of any increasing sequence of nodes of T of length less than κ is also a node of T (i.e., T is κ-closed) and every node of T has a splitting extension in T.

T is *Sacks-perfect* if in addition the limit of any increasing sequence of splitting nodes of T of length less than κ is a splitting node of T.

A subset of κ^κ is *perfect* (*Sacks-perfect*) if it consists of the κ-branches through a perfect (Sacks-perfect) subtree T of $\kappa^{<\kappa}$.

Mycielski's Theorem

Theorem

(Mycielski for κ^κ) Assume that κ is regular and either $\diamondsuit_\kappa$ holds or κ is strongly inaccessible. Suppose that E is a meager binary relation on κ^κ. Then there is a Sacks-perfect set A such that $E(x,y)$ fails for all distinct x, y in A.

Corollary

Assume κ is a successor cardinal and $\kappa \neq \omega_1$. Then the conclusion of the above Theorem holds.

Proof. Recall that we have assumed $\kappa^{<\kappa} = \kappa$. For $\kappa = \gamma^+$ this means that $2^\gamma = \gamma^+$. It now follows from a theorem of Shelah that $\diamondsuit_\kappa$ holds if γ is uncountable. $\square$

Mycielski's Theorem

Proof of Mycielski for κ^κ. Write E as the union of an increasing κ-sequence $E_0 \subseteq E_1 \subseteq \cdots$ of nowhere dense sets. For each $\sigma \in \kappa^{<\kappa}$ recall that $U(\sigma)$ denotes the basic open set determined by σ, i.e. $\{\eta \in \kappa^\kappa \mid \sigma \subseteq \eta\}$.

First suppose that κ is inaccessible. We build the α-th splitting level T_α of T by induction on α. For $\alpha = 0$, T_0 has just the single node $\emptyset$ and for limit α, T_α consists of all limits of branches through the levels T_β, $\beta < \alpha$.

Mycielski's Theorem

Suppose that $\alpha = \beta + 1$. Then we list all pairs $(s * i, t * j)$ where s, t are on level β, i, j are 0 or 1 and $s * i \neq t * j$. As κ is inaccessible there are fewer than κ such pairs. Now choose such a pair $(s * i, t * j)$ and find $(s * i)^1$ extending $s * i$ and $(t * j)^1$ extending $t * j$ so that $U((s * i)^1) \times U((t * j)^1)$ is disjoint from E_β. This is possible as E_β is nowhere dense. Then choose another pair and do the same, repeating this for all pairs and resulting in sequences $(s * i)^1 \subseteq (s * i)^2 \subseteq \cdots$ for each $s * i$. Let $(s * i)^\infty$ be the limit of this sequence and take level T_α to consist of all of these $(s * i)^\infty$'s.

The result is that if x, y are κ-branches through T and extend distinct nodes on level $\beta + 1$ of T then (x, y) does not belong to E_β and therefore (x, y) does not belong E as β can be chosen to be arbitarily large.

Mycielski's Theorem

Now suppose that $\diamondsuit_\kappa$ holds. Fix a $\diamondsuit_\kappa$ sequence $(D_\beta \mid \beta < \kappa)$ that guesses pairs (x, y) in κ^κ, i.e., for such a pair, $\{\beta \mid D_\beta = (x|\beta, y|\beta)\}$ is stationary in κ. Now repeat the above construction except at stage $\beta + 1$ only treat the four pairs $(d_0 * i, d_1 * j)$ if $D_\beta = (d_0, d_1)$ and d_0, d_1 belong to T_β, guaranteeing that if (x, y) extends (d_0, d_1) then (x, y) does not belong to E_β. Other nodes s on level β are simply extended to $s * 0$ and $s * 1$ on level $\beta + 1$. The $\diamondsuit_\kappa$ sequence guarantees that if x, y are distinct branches through the resulting Sacks-perfect tree then (x, y) does not belong to E_β for any β and therefore does not belong to E. $\square$

Open questions. Assume $\kappa^{<\kappa} = \kappa$. Does Mycielski's Theorem hold when κ equals ω_1? Does Mycielski's Theorem hold when κ is weakly (but not strongly) inaccessible?

The Baire Property

Again fix an uncountable κ such that $\kappa^{<\kappa} = \kappa$.

Recall that Borel sets have the Baire property.

Proposition

In L there is a Δ^1_1 set without the Baire property.

Proof. This is because there is a Δ^1_1 wellorder of the subsets of κ: $x \leq y$ iff there exists $\alpha < \kappa^+$ such that L_α models ZFC$^-$ and $L_\alpha \vDash x \leq_L y$ (iff there exists z coding a wellfounded model M of ZFC$^-$ $+$ $V = L$ such that $x, y \in M$ and $M \vDash x \leq_L y$). But "z is a wellfounded binary relation" is Borel (even closed): z is wellfounded iff $z \cap (\alpha \times \alpha)$ is wellfounded for each $\alpha < \kappa$. It follows that $x \leq y$ is a Σ^1_1 and therefore Δ^1_1 wellorder of the subsets of κ.

Now using the Δ^1_1 wellorder it is easy to construct a Δ^1_1 set without the Baire property by diagonalisation. $\square$

The Baire Property

It follows that in L, there are Δ^1_1 sets which are not Borel. Actually this holds in general:

Proposition

There are Δ^1_1 sets which are not Borel.

Proof. Each Borel set B is coded as $B(T)$ where T is a wellfounded, size κ tree of finite sequences whose terminal nodes are labelled with basic open sets and whose nonterminal nodes are labelled with $\sim$ or $\cup$ (use the labels to assign a Borel set to each node of the tree; $B(T)$ is the Borel set assigned to the top node).
The relation $\eta \in B(T)$ is a Δ^1_1 relation of η and T.
It follows that there is a Δ^1_1 set $U(\eta, \nu)$ which is universal for Borel sets in the sense that for each η, U_η is Borel and each Borel set is of this form for some η; but then U is not Borel, else by diagonalisation, $\{\eta \mid \text{not } U(\eta, \eta)\}$ would give a contradiction. $\square$

The Baire Property

Remark. Later we will see a much more concrete example of a Δ^1_1 set that is not Borel.

We have seen that in L there are Δ^1_1 sets without the Baire property.

However, generalising the classical result that it is consistent for all Δ^1_2 sets of reals to have the Baire property, we have:

The Baire Property

Theorem

(???) After forcing over L with Add(κ, κ^+) (the forcing which adds κ^+-many κ-Cohens), every Δ^1_1 set has the Baire property.

Proof. Let G be generic for Add(κ, κ^+) and let X be Δ^1_1 in $V[G]$. Assuming that X is Δ^1_1 with parameter in V we'll show that X has the property of Baire; the general case follows from the fact that any subset of κ belongs to $G \cap$ Add(κ, α) for some $\alpha < \kappa^+$ and Add(κ, κ^+) factors as Add$(\kappa, \alpha) \times$ Add$(\kappa, [\alpha, \kappa^+))$, the second component of this product being isomorphic to Add(κ, κ^+).

The Baire Property

We show that any basic open set $U(\sigma)$ contains a basic open subset $U(\tau)$ on which X is either meager or comeager. Let φ, ψ be Σ^1_1 formulas (with parameters in V) defining X and the complement of X, respectively. We may assume that $G(0)$, the first κ-Cohen added by G, extends σ (if not, then change it below the length of σ so that it does). Suppose that $G(0)$ satisfies φ. Note that $V[G]$ is an extension of $V[G(0)]$ via the κ-closed forcing $\text{Add}(\kappa, [1, \kappa^+))$.

The Baire Property

We claim that $V[G(0)]$ is Σ^1_1 elementary in $V[G]$ and therefore $\varphi(G(0))$ holds in $V[G(0)]$: Indeed, suppose that φ is Σ^1_1 with parameter in $V[G(0)]$ and let T be a tree in $V[G(0)]$ on $\kappa \times \kappa$ such that cofinal branches through T correspond to pairs (x, w) where w witnesses that $\varphi(x)$ holds. Suppose that $\dot{b}$ is an $\mathrm{Add}(\kappa, [1, \kappa^+))$-name for a branch through T; then we can build a branch through T in $V[G(0)]$ by forming a κ-sequence of conditions $p_0 \geq p_1 \geq \cdots$ deciding initial segments of $\dot{b}$. So if φ has a solution in $V[G]$ it also has one in $V[G(0)]$.

The Baire Property

Now let τ be a κ-Cohen condition extending σ which forces $\varphi(\dot{g})$ where $\dot{g}$ denotes the κ-Cohen generic. Let M be a transitive model of ZFC$^-$ of size κ which contains all bounded subsets of κ such that τ forces $\varphi(\dot{g})$ in M. The subsets of κ which are κ-Cohen over M form a comeager set on $U(\tau)$ and if x is κ-Cohen over M extending τ then $M[x]$ and therefore $V[G]$ satisfies $\varphi(x)$. We have shown that X is comeager on $U(\tau)$. If $G(0)$ satisfies ψ, the Σ^1_1 formula that defines the complement of X, then we have shown that X is meager on $U(\tau)$. $\Box$

We turn now to the Baire property for Σ^1_1 sets. Here there is a surprise:

The Baire Property

Theorem

(Halko-Shelah) Let X be the set of $\eta \in \kappa^\kappa$ such that $\eta(i) = 0$ for all i in some closed unbounded subset of κ. Then X does not have the Baire property.

Proof. Otherwise choose a basic open set $U(\sigma)$ on which X is either meager or comeager. Suppose that it is comeager on $U(\sigma)$ and choose sets D_i, $i < \kappa$ which are open dense subsets of $U(\sigma)$ with intersection contained in X. But we can build a sequence $\sigma = \sigma_0 \subseteq \sigma_1 \subseteq \cdots$ so that $U(\sigma_{i+1})$ is contained in D_i and for limit λ, σ_λ is an extension of $\cup_{i<\lambda}\sigma_i$ with value 1 at λ. Then the union of the σ_i, $i < \kappa$, clearly does not belong to X but does belong to each D_i, $i < \kappa$, contradiction. If we instead require $\sigma_\lambda = 0$ for limit λ then we obtain something in X belonging to each D_i, verifying that X is not meager on $U(\sigma)$. $\square$

The Baire Property

The Halko-Shelah result is a surprise, as in the classical setting it is consistent that Σ^1_2 and indeed Σ^1_n sets for arbitrary n have the property of Baire. Developing a notion of regularity for Σ^1_n sets for uncountable κ which copes with the Halko-Shelah result remains an interesting challenge in the descriptive set theory of generalised Baire space.

The Halko-Shelah result can also be used to produce examples of Δ^1_1 sets without the Baire property. Indeed, for any regular cardinal κ of L, there is a cardinal-preserving extension of L in which the club filter on κ is Δ^1_1 definable (SDF-Wu-Zdomskyy). However the club filter on κ cannot be Δ^1_1 when κ is weak compact.

Other forms of Regularity

The Baire property is just one example of a regularity property for subsets of generalised Baire space. We now consider other such properties, each associated to a "κ-treelike" forcing in the way that the Baire property is associated to κ-Cohen forcing.

A forcing $\mathcal{P}$ is *κ-treelike* iff it is a κ-closed suborder of the set of subtrees of $\kappa^{<\kappa}$ ordered by inclusion.

Some examples of κ-treelike forcings:

κ-Cohen. These are subtrees of $2^{<\kappa}$ consisting of a stem and all nodes above it.
κ-Sacks. These are κ-closed subtrees of $2^{<\kappa}$ with the property that every node has a splitting extension and the limit of splitting nodes is a splitting node.

Other forms of Regularity

κ-*Miller*. These are κ-closed subtrees of the tree $\kappa_{\uparrow}^{<\kappa}$ of *increasing* sequences in $\kappa^{<\kappa}$ with the property that every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require *continuous club-splitting*, which means that if s is a limit of club-spltting nodes then the club witnessing club-splitting for s is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of s.
κ-*Laver*. These are κ-Miller trees with the property that every node beyond some fixed node (the stem) is club-splitting.

Other forms of Regularity

To define "$\mathcal{P}$-regularity" for the above forcing notions $\mathcal{P}$ we proceed as follows.

A set A is:
Strictly $\mathcal{P}$-null if every tree $T \in \mathcal{P}$ has a subtree in $\mathcal{P}$, none of whose κ-branches belongs to A.
$\mathcal{P}$-null if it is the union of κ-many strictly $\mathcal{P}$-null sets.
$\mathcal{P}$-regular (or *$\mathcal{P}$-measurable*) if any tree $T \in \mathcal{P}$ has a subtree $S \in \mathcal{P}$ such that either all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to A or all κ-branches through S, with a $\mathcal{P}$-null set of exceptions, belong to the complement of A.

Other forms of Regularity

Proposition

A set is κ-Cohen measurable iff it has the property of Baire.

Proof. Let $\mathcal{P}$ denote κ-Cohen forcing. First note that a set is strictly $\mathcal{P}$-null iff it is nowhere dense and therefore is $\mathcal{P}$-null iff it is meager. Now if A is $\mathcal{P}$-measurable it follows that every basic open set has a basic open subset on which A is either meager or comeager. Thus if U is the union of the basic open sets on which A is meager or comeager and U_0 is the union of the basic open sets on which A is comeager, it follows that A differs from U_0 by a meager set, as the complement of U is nowhere dense. So A has the property of Baire.

Other forms of Regularity

Conversely, if $A = U \triangle M$ with U open and M meager, then to verify that A is $\mathcal{P}$-measurable it suffices to show that U is $\mathcal{P}$-measurable. But this is clear, as any basic open set not disjoint from U has a basic open subset that is completely contained in U. $\square$

Now let $\mathcal{P}$ be any of the above κ-treelike forcings.

Proposition

Any Borel set is $\mathcal{P}$-measurable.

Proof. We may assume that $\mathcal{P}$ is not κ-Cohen, as in that case $\mathcal{P}$-measurability is the same as the property of Baire and we know that all Borel sets have the property of Baire. A similar argument applies to κ-Laver which is also κ^+-cc and gives rise to a natural topology, the κ-Laver topology, analagous to the standard topology on κ^κ. So we assume that $\mathcal{P}$ is either κ-Sacks or κ-Miller.

Other forms of Regularity

Note that the collection of $\mathcal{P}$-measurable sets is obviously closed under complements, so it suffices to show that it is closed under κ-unions and that basic open sets are $\mathcal{P}$-measurable.

For the basic open sets, note that for each of the above examples of treelike forcings $\mathcal{P}$, if T belongs to $\mathcal{P}$ then so does $T(\eta)$ for each node η of T (where $T(\eta)$ consists of all nodes in T which are compatible with η). Now if η is an arbitrary element of $\kappa^{<\kappa}$, determining the basic open set $U(\eta)$, and T belongs to $\mathcal{P}$ then either η belongs to T, in which case $T(\eta)$ is a strengthening of T whose κ-branches are all contained in $U(\eta)$, or η does not belong to T, in which case no κ-branch of T belongs to $U(\eta)$. So $U(\eta)$ is $\mathcal{P}$-measurable.

Other forms of Regularity

Suppose that A is the union of A_i, $i < \kappa$ and we know that each A_i is $\mathcal{P}$-measurable. Given $T \in \mathcal{P}$ and $i < \kappa$ we can strengthen T to T_i so that either almost all κ-branches of T_i belong to A_i or almost all κ-branches of T_i do not belong to A_i, where "almost all" refers to a $\mathcal{P}$-null set of exceptions. If the former occurs for some i then almost all κ-branches of T_i also belong to A so we are done. Otherwise we want to strengthen T to T^* so that almost no κ-branch of T^* belongs to any A_i. Of course we can do this for fewer than κ-many A_i's using the κ-closure of the forcing $\mathcal{P}$; to handle κ-many A_i's we use the *fusion property*. This is expressed as follows:

Other forms of Regularity

There are partial orders $\leq_i$ on $\mathcal{P}$ which refine the standard ordering on $\mathcal{P}$ such that:

(a) $i \leq j$, $T^* \leq_j T$ implies $T^* \leq_i T$.
(b) If $(T_i \mid i < \lambda)$, $\lambda \leq \kappa$, belong to $\mathcal{P}$ and for all $i \leq j < \lambda$, $T_j \leq_i T_i$ then there is $T \leq_i T_i$ for all $i < \lambda$.
(c) Suppose that T belongs to $\mathcal{P}$ and D is a set of extensions of T which is dense below T. Then for each $i < \kappa$ there are $T^* \leq_i T$ and $d \subseteq D$ of size at most κ such that each κ-branch through T^* is also a κ-branch through some element of d.

Other forms of Regularity

Now recall that we are given T such that for each i, the set of T^* such that almost no κ-branch through T^* belongs to A_i is open dense below T. Now use fusion to build a sequence $(T_i \mid i < \kappa)$ such that $i \leq j \rightarrow T_j \leq_i T_i$ and each κ-branch through T_{i+1} is a κ-branch through one of κ-many extensions of T_i, almost none of whose κ-branches belong to A_i. If T^* is a lower bound to the sequence of T_i's then almost no κ-branch of T^* belongs to any A_i, so we have verified the $\mathcal{P}$-measurability of A, the union of the A_i's.

Finally we verify the fusion property for the κ-Sacks and κ-Miller forcings:

Other forms of Regularity

κ-Sacks:
If T is a condition then let $f_T : 2^{<\kappa} \to T$ be the natural order-preserving bijection between the full tree $2^{<\kappa}$ and the set of splitting nodes of T. Then define $T^* \leq_i T$ iff $f_{T^*}(s) = f_T(s)$ for all $s \in 2^{<\kappa}$ of length at most i. Then property (a) is clear. Note that for limit i, this is the same as requiring this just for s of length less than i, as the limit of spltting nodes is a splitting node; this gives property (b). For (c), for each $s \in 2^{<\kappa}$ of length i and $j \in \{0, 1\}$ we choose $T_{s*j} \leq T(f_T(s) * j)$ in D, let d be the set of such T_{s*j}'s and let T^* the the union of the T_{s*j}'s. As $\kappa^{<\kappa} = \kappa$, there are only κ-many such $s * j$'s.

Other forms of Regularity

κ-Miller:
If T is a condition then let $f_T : \kappa_{\uparrow}^{<\kappa} \to T$ be the natural order-preserving bijection between the full tree $\kappa_{\uparrow}^{<\kappa}$ and the set of splitting nodes of T. Define $T^* \leq_i T$ iff $f_{T^*}(s) = f_T(s)$ for all $s \in 2^{<\kappa}$ such that $s(\alpha) \leq i$ for all $\alpha < |s|$. Property (a) is clear and property (b) follows using (diagonal) intersections when λ equals κ. (c) is verified as for κ-Sacks. $\square$

Thus we know that Borel sets are $\mathcal{P}$-measurable for our 4 standard examples of κ-treelike forcings $\mathcal{P}$. However as in the specific case of κ-Cohen forcing:

Other forms of Regularity

Theorem

Not every Σ^1_1 set is $\mathcal{P}$-measurable.

Proof. First we verify this for κ-Sacks. Let A consist of all $x \in 2^\kappa$ such that $\{i \mid x(i) = 0\}$ contains a club. Suppose that T is a κ-Sacks tree. Then there are κ-branches of T in A and also κ-branches of T in the complement of A: For the former simply choose a κ-branch x through T as the union of splitting nodes s_i of T of lengths α_i such that for each i, $s_{i+1}(\alpha_i) = 0$; this is possible as the limit of splitting nodes of T is also a splitting node of T. For the latter do the same, but with $s_{i+1}(\alpha_i) = 1$ for limit i.

To handle the other cases we prove the following general fact, patterned after work of Brendle-Löwe, Khomskii and Laguzzi in the classical case.

Other forms of Regularity

Lemma

Let Γ be a pointclass closed under continuous pre-images (like Δ^1_n, Σ^1_n or Π^1_n). Let $\Gamma(\mathcal{P})$ be the statement that every set in Γ is $\mathcal{P}$-measurable. Then:

$\Gamma(\kappa\text{-Cohen}) \to \Gamma(\kappa\text{-Miller})$
$\Gamma(\kappa\text{-Laver}) \to \Gamma(\kappa\text{-Miller})$
$\Gamma(\kappa\text{-Miller}) \to \Gamma(\kappa\text{-Sacks})$.

Proof of Lemma. For the first implication, first note the following:

Other forms of Regularity

Fact 1. $\Gamma(\kappa\text{-Cohen})$ $(= \Gamma(2^{<\kappa}\text{-Cohen}))$ implies $\Gamma(\kappa_{\uparrow}^{<\kappa}\text{-Cohen})$.

Proof of Fact 1. Note that there is $D \subseteq 2^\kappa$ which is the κ-intersection of open dense subsets of 2^κ (and therefore comeager) such that D is homeomorphic to $\kappa_{\uparrow}^{\kappa}$. We may choose D to consist of all $x \in 2^\kappa$ such that $x(i) = 1$ for cofinally many $i < \kappa$; the homeomorphism sends x to $y \in \kappa_{\uparrow}^{\kappa}$ where $x = 0^{y(0)} * 1 * 0^{y(1)} * \cdots$. If $A \subseteq \kappa_{\uparrow}^{\kappa}$ belongs to Γ then the $\kappa_{\uparrow}^{<\kappa}$-measurability of A follows from that of its pre-image under this homeomorphism, which in turn follows from $\Gamma(\kappa\text{-Cohen})$, as D is comeager in 2^κ. $\square$ (*Fact 1*)

Other forms of Regularity

Now let A belong to Γ and let T be a κ-Miller tree. Under the assumption $\Gamma(\kappa\text{-Cohen})$ we want to find a κ-Miller subtree of T, all of whose κ-branches belong to A or all of whose κ-branches belong to the complement of A.
Let φ be the natural order-preserving bijection between the full tree $\kappa_{\uparrow}^{<\kappa}$ (of increasing $<\kappa$-sequences through κ) and the splitting nodes of T. Also let φ^* denote the induced homeomorphism between $\kappa_{\uparrow}^{\kappa}$ and $[T]$, the set of κ-branches through T. Let A' be $(\varphi^*)^{-1}[A]$, which belongs to Γ as by assumption Γ is closed under continuous pre-images. Apply $\Gamma(\kappa\text{-Cohen})$ to get a basic open set $U(\eta)$ such that A' is either meager or comeager on $U(\eta)$. Without loss of generality assume the latter. Now build a κ-Miller tree S' such that $[S']$ is contained in $U(\eta) \cap A'$: assume that $A' \cap U(\eta)$ contains the intersection of U_i, $i < \kappa$, where each U_i is open dense on $U(\eta)$ and ensure that any $x \in \kappa^\kappa$ extending a node on the i-th

Other forms of Regularity

splitting level of S' belongs to U_i. We can also require that splitting nodes μ of S' are full-splitting, in the sense that if $\mu * \alpha$ belongs to S' for all $\alpha < \kappa$. Then $\varphi[S']$ consists of the splitting nodes of a κ-Miller tree S contained in T with the property that $[S]$ is contained in A.

Other forms of Regularity

For the second implication (from κ-Laver to κ-Miller), note that like κ-Cohen forcing, κ-Laver forcing is κ^+-cc and we can form a topology, which we call the Laver topology, whose basic open sets are of the form $[T]$ for T a κ-Laver tree. Then in analogy to κ-Cohen forcing we have:

Fact 2. A is κ-Laver measurable iff A is of the form $O \triangle M$ where O is open in the Laver topology and M is meager in the Laver topology.

Now we use *Fact 2* to prove the second implication, by imitating the argument used for the first implication. Let A belong to Γ and let T be a κ-Miller tree. Under the assumption $\Gamma(\kappa\text{-Laver})$ we want to find a κ-Miller subtree of T, all of whose κ-branches belong to A or all of whose κ-branches belong to the complement of A.

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We "collapse" T into a κ-Laver tree T' as follows: Define a function ψ from the splitting nodes of T to nodes of the full tree $\kappa_{\uparrow}^{<\kappa}$ by induction as follows. If η is a splitting node of T which is not the limit of splitting nodes of T then write η as $\eta_0 * \alpha * \eta_1$ where η_0 is the longest splitting node of T properly contained in η (or $\emptyset$ if η is the least splitting node of T) and set $\psi(\eta) = \psi(\eta_0) * \alpha$. If η is a limit of splitting nodes of T then set $\psi(\eta) =$ the union of the $\psi(\eta_0)$ for η_0 a splitting node of T properly contained in η. Let φ be the inverse of ψ, mapping the κ-Laver tree T' onto the splitting nodes of T, and let φ^* be the induced homeomorphism between $[T']$ and $[T]$, the sets of κ-branches of T' and T, respectively.

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Now let A' be $(\varphi^*)^{-1}[A]$, which belongs to Γ as by assumption Γ is closed under continuous pre-images. Apply $\Gamma(\kappa$-Laver) to get a κ-Laver subtree of T' such that A' is either Laver-meager or Laver-comeager on $[T]$. Without loss of generality assume the latter. Now build a κ-Miller tree S' such that $[S']$ is contained in $[T] \cap A'$: assume that $A' \cap [T]$ contains the intersection of U_i, $i < \kappa$, where each U_i is Laver-open dense on $[T]$ and ensure that any $x \in \kappa^\kappa$ extending a node on the i-th spitting level of S' belongs to U_i. Then $\varphi[S']$ consists of the splitting nodes of a κ-Miller tree S contained in T with the property that $[S]$ is contained in A.

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For the third implication (κ-Miller to κ-Sacks), let A belong to Γ and let T be a κ-Sacks tree. Under the assumption $\Gamma(\kappa$-Miller) we want to find a κ-Sacks subtree S of T such that $[S]$ is either contained in or disjoint from A.

Define an injection φ_0 from the full tree $\kappa_{\uparrow}^{<\kappa}$ into $2^{<\kappa}$ as follows:

$\varphi_0(\emptyset) = \emptyset$
$\varphi_0(\eta) = (\bigcup_{\alpha<|\eta|} \varphi_0(\eta|\alpha))$, if $|\eta| =$ the length of η is a limit ordinal
$\varphi_0(\eta * \alpha) = \varphi_0(\eta) * 0^{\alpha-|\eta|} * 1$, where 0^β denotes a β-sequence of 0's.

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And let φ_0^* be the injection from $\kappa^\kappa_\uparrow$ into 2^κ induced by φ_0. Also let ψ be the natural bijection between $2^{<\kappa}$ and the splitting nodes of T and ψ^* the induced bijection between 2^κ and $[T]$. Define $\varphi = \psi \circ \varphi_0$ and $\varphi^* = \psi^* \circ \varphi_0^*$.

As φ^* is continuous, $A' = (\varphi^*)^{-1}[A]$ belongs to Γ. Apply Γ(κ-Miller) to obtain a κ-Miller tree S' such that $[S']$ is either contained in or disjoint from A'. Thin S' to guarantee that if η is a splitting node of S' then the length $|\eta|$ of η is the sup of its range and $\eta * |\eta|$ belongs to S'. Then $\varphi[S'] = S$ generates a κ-Sacks subtree S of T such that $[S]$ is either contained in or disjoint from A. $\square$ (*Lemma*)

Using the Lemma, we conclude that Σ^1_1 measurability fails for κ-Miller, κ-Cohen and κ-Laver. $\square$

Other forms of Regularity

We have seen that $\Delta^1_1(\kappa\text{-Cohen})$ is consistent, from which it follows by the above Lemma that $\Delta^1_1(\kappa\text{-Miller})$ and $\Delta^1_1(\kappa\text{-Sacks})$ are consistent. What about $\Delta^1_1(\kappa\text{-Laver})$? For this we can imitate the proof for the κ-Cohen case. First we need a lemma.

Lemma

Let M be a transitive model of ZFC$^-$ containing κ and all bounded subsets of κ which is elementary in $H(\kappa^+)$. Then $x \in \kappa^\kappa$ is κ-Laver generic over M iff x belongs to every Borel set coded in M which is open dense in the Laver topology of M (equivalently, open dense in the Laver topology of V).

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Proof. Of course when we say that x is κ-Laver generic over M we mean that $G_x = \{T \in M \mid T$ is a κ-Laver tree of M and $x \in [T]\}$ is κ-Laver generic over M in the strict sense. If this holds and B is a Borel set coded in M which is open dense in the Laver topology of M then the set of T in M such that $M \vDash [T] \subseteq B$ is open dense in the κ-Laver forcing of M and therefore there is such a T in G_x; by the elementarity of M in $H(\kappa^+)$, $V \vDash [T] \subseteq B$ and therefore as x belongs to $[T]$ it also belongs to B. Conversely, suppose that x belongs to every Borel set coded in M which is open dense in the Laver topology of M and that $D \in M$ is open dense on the κ-Laver forcing of M. Let $X \in M$ be a maximal antichain contained in D; then X has size at most κ and $B =$ the union of the $[T]$ for T in X is a Borel set coded in M which is open dense in the κ-Laver topology of M. By hypothesis x belongs to B and therefore to some $[T]$ where T belongs to X; so G_x meets D. $\square$ (*Lemma*)

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Theorem

After forcing with Laver(κ, κ^+) (the iteration of κ^+-many κ-Laver forcings with support of size $< \kappa$), every Δ^1_1 set is κ-Laver measurable.

Proof. Note that the forcing Laver(κ, κ^+) is κ^+-cc; this follows using a Δ-system argument and the fact that κ-Laver forcing is both κ-closed and κ-centered.

Let G be generic for Laver(κ, κ^+) and let X be Δ^1_1 in $V[G]$. We'll show that any κ-Laver tree T contains a κ-Laver subtree S such that $[S]$ is either contained in or disjoint from X modulo a Laver-null set. We may assume that the defining parameter for X and the tree T belong to V (otherwise factor over $V[G|\alpha]$ for some large enough $\alpha < \kappa^+$). Let φ, ψ be Σ^1_1 formulas (with parameters in V) defining X and the complement of X, respectively.

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Let M be a transitive elementary submodel of $H(\kappa^+)^V$ of size κ which contains all bounded subsets of κ and T. Then by the κ^+-cc of Laver(κ, κ^+), $M[G]$ is elementary in $H(\kappa^+)^V[G] = H(\kappa^+)^{V[G]}$. If α is $M \cap \kappa^+$ then $M[G] = M[G|\alpha]$; we may assume that $G(\alpha)$, the κ-Laver generic added by G at stage α, belongs to $[T]$, as it is dense to force this for some M. Note that $G(\alpha)$ is also κ-Laver generic over $M[G|\alpha]$ as this model is Σ_1 elementary in $H(\kappa^+)^V[G|\alpha]$ (and the property of being a maximal antichain is Π_1). Without loss of generality assume that $\varphi(G(\alpha))$ holds in $V[G]$ and therefore also in $V[G|\alpha][G(\alpha)]$ (as the former is a κ-closed forcing extension of the latter).

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Now let S be a κ-Laver condition in $M[G|\alpha]$ extending T which forces $\varphi(\dot{g})$ where $\dot{g}$ denotes the κ-Laver generic. Using the Lemma, the set of $x \in \kappa^{\kappa}_{\uparrow}$ which are Laver-generic over $M[G|\alpha]$ is Laver-comeager in $V[G]$ as it is the intersection of κ-many Borel sets, each of which is open dense in the Laver topology of $M[G|\alpha]$ and therefore in the Laver topology of $V[G]$. And if x is a κ-branch through S which is Laver-generic over $M[G|\alpha]$ then $M[G|\alpha][x]$ and therefore $V[G]$ satisfies $\varphi(x)$. We have shown that $[S]$ is contained in X modulo a Laver-null set and therefore X is Laver-measurable in $V[G]$. $\square$

Remark. In the classical case one can similarly obtain the Laver-measurability of all Δ^1_2 sets by iterating Laver forcing ω_2 times over L; but Shoenfield absoluteness makes the argument easier.

Borel Reducibility

If E and F are equivalence relations on κ^κ then we say that E *is Borel reducible to* F, written $E \leq_B F$, if there is a Borel function f such that for all x, y: $E(x, y)$ iff $F(f(x), f(y))$. The relation $\leq_B$ is reflexive and transtive and we write $\equiv_B$ for the equivalence relation it induces.

For Borel equivalence relations E, F with at most κ-many equivalence classes Borel reducibility is quite trivial: $E \equiv_B F$ iff E and F have the same number of equivalence classes. This is because if E and F have the same number of classes we may choose sets X_E and X_F of the same size selecting one element from each equivalence class of E, F respectively and then extend any bijection between X_E and X_F to a Borel reduction of E to F (and similarly obtain a Borel reduction of F to E).

Borel Reducibility: Dichotomies

In the classical setting one has two important Dichotomies:

Silver Dichotomy. Suppose that E is a Borel (or even Π^1_1) equivalence relation on ω^ω with uncountably many classes. Then id is Borel (even continuously) reducible to E, where id is the equivalence relation of equality on 2^ω.

Harrington-Kechris-Louveau Dichotomy. Suppose that E is a Borel equivalence relation. Then either E is Borel reducible to id or E_0 is Borel reducible to E, where E_0 is the equivalence relation of equality mod finite.

Terminology: If id Borel reduces to E we say that E *has a perfect set of classes* and if E Borel reduces to id we say that E is *smooth*.

Borel Reducibility: Dichotomies

The Silver Dichotomy for κ^κ fails in L:

Theorem

(SDF-Hyttinen-Kulikov) Assume $V = L$. Then there are Borel equivalence relations E with more than κ classes which are strictly below id with respect to Borel reducibility.

Proof. If $V = L$ then there is a *weak Kurepa tree* on κ, a tree T of height κ with κ^+ many branches such that the α-th splitting level of T has size at most card(α) for stationary-many $\alpha < \kappa$.

There can be no continuous injection from 2^κ into $[T]$, the set of κ-branches through T, because this would yield a club of $\alpha < \kappa$ such that the α-th splitting level of T has 2^α many nodes. In fact there cannot be such an injection which is Borel, as any Borel function is continuous on a comeager set and any comeager set contains a copy of 2^κ.

Now define xE_Ty iff x, y are not branches through T or $x = y$. Then E_T is a Borel equivalence relation with κ^+ classes yet id cannot Borel reduce to E_T for the reasons given above. Clearly E_T is Borel reducible to id. $\square$

Remark. This can be improved to get (assuming $V = L$) 2^κ Borel Reducibility Degrees below id as well as Borel equivalence relations which are incomparable with id with respect to Borel reducibility.

Borel Reducibility: Dichotomies

One might hope that if a Borel equivalence relation has not just κ^+ many classes but a large number of classes then it must have a perfect set of classes. But also this can consistently fail:

Theorem

Let κ be regular and uncountable in L. Then in a cardinal-preserving forcing extension of L, $2^\kappa = \kappa^{+++}$ and there is a Borel equivalence relation on κ^κ with exactly κ^{++} classes. (The same holds with $\kappa^{+++}, \kappa^{++}$ replaced by any pair of cardinals $\lambda_1 \geq \lambda_0$ of cofinality greater than κ.)

Proof. Add a (weak) Kurepa tree T on κ with κ^{++} branches. The forcing for doing this is κ-closed and κ^+-cc and therefore preserves cardinals. Then follow this by adding κ^{+++} many κ-Cohen sets (by a product with supports of size less than κ). Again cardinals are preserved. But notice that the second forcing does not add branches to T as it is κ-closed. Now (as before) take the equivalence relation E_T defined by xE_Ty iff x, y are not κ branches through T or $x = y$. $\square$

We'll return to the Silver Dichotomy later, but now turn to the Harrington-Kechris-Louveau Dichotomy. Recall that in the classical case, E_0 is defined by: xE_0y iff $x \triangle y$ is finite.

The first question to resolve is: How shall we define E_0 on κ^κ? The next result answers this question:

Borel Reducibility: Dichotomies

Theorem

For λ an infinite cardinal $\leq \kappa$ define $E_{0,\lambda}$ by $xE_{0,\lambda}y$ iff $x \triangle y$ has size less than λ. Then id $\leq_B E_{0,\lambda}$, $E_{0,\lambda}$ is Borel and:
$(*)$ $E_{0,\kappa}$ is not Borel reducible to id but $E_{0,\lambda}$ is Borel reducible to id for $\lambda < \kappa$.

In light of this result we take E_0 to be $E_{0,\kappa}$.

Proof. To prove $(*)$, first suppose that λ is less than κ. For each $\alpha < \kappa$ use the axiom of choice to choose a function $f_\alpha : 2^\alpha \to 2^\alpha$ such that for x, y in 2^α, $x \triangle y$ has size less than λ iff $f_\alpha(x) = f_\alpha(y)$. Then for x, y in 2^κ, $x \triangle y$ has size less than λ iff $f_\alpha(x|\alpha) = f_\alpha(y|\alpha)$ for all $\alpha < \kappa$ (here we use $\lambda < \kappa$). So we obtain a reduction of $E_{0,\lambda}$ to id by sending x to $(f_\alpha(x) \mid \alpha < \kappa)$.

Borel Reducibility: Dichotomies

The proof that $E_{0,\kappa}$ is *not* Borel reducible to id is just as in the classical case: Suppose that f were a reduction and let x be sufficiently κ-Cohen (i.e., κ-Cohen over a transitive model of ZFC$^-$ of size κ containing all bounded subsets of κ as well as the parameter for this reduction). Define $\bar{x}(i) = 1 - x(i)$ for $i < \kappa$. As $\sim xE_0\bar{x}$ we can choose $\sigma \subseteq x$, $i < \kappa$ and $j \in \{0, 1\}$ such that for sufficiently κ-Cohen y, $f(y)(i) = j$ if y extends σ and $f(y)(i) = 1 - j$ if y extends $\bar{\sigma}$. But $y = \bar{\sigma} * (x \text{ above } \bar{\sigma})$ is E_0 equivalent to x yet $f(y) \neq f(x)$, contradiction.

Borel Reducibility: Dichotomies

Unfortunately the Harrington-Kechris-Louveau Dichotomy is provably false for κ^κ, κ uncountable:

Theorem

(SDF-Hyttinen-Kulikov) There is a Borel equivalence relation E_0' which is strictly above id and strictly below E_0 with respect to Borel reducibility.

Proof. We define E_0' on 2^κ as follows:

Borel Reducibility: Dichotomies

$xE_0'y$ iff
xE_0y and $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals.

Claim 1. id $\leq_B E_0' \leq_B E_0$.
For the first reduction use $f(x) =$ the set of codes for proper initial segments of x; then $x = y \rightarrow f(x)E_0'f(y)$ and
$x \neq y \rightarrow \sim f(x)E_0f(y) \rightarrow \sim f(x)E_0'f(y)$.
For the second reduction: for each $\alpha < \kappa$ choose $f_\alpha : 2^\alpha \rightarrow 2^\alpha$ such that for $x, y \in 2^\alpha$, $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals iff $f_\alpha(x) = f_\alpha(y)$ and for $x \in 2^\kappa$ define $f(x) =$ the set of codes for the pairs $(f_\alpha(x|\alpha), x(\alpha))$, $\alpha < \kappa$; then
$xE_0'y \rightarrow f(x)E_0f(y)$ and $\sim xE_0'y \rightarrow \sim f(x)E_0f(y)$.

Claim 2. $E_0' \not\leq_B \mathrm{id}$.
Otherwise let M be a transitive model of ZFC^- of size κ containing all bounded subsets of κ as well as a code for the Borel reduction f. Let $x \in 2^\kappa$ be κ-Cohen generic over M and define $\bar{x}(i) = 1 - x(i)$ for each $i < \kappa$.
Then as $\sim x E_0 \bar{x}$ there is $\alpha < \kappa$ such that $f(x) \neq f(y)$ whenever y is κ-Cohen generic over M and extends $\bar{x}|\alpha$. But then $f(x) \neq f((\bar{x}|\alpha) * (x|[\alpha, \kappa)))$, contradicting $x E_0' ((\bar{x}|\alpha) * (x|[\alpha, \kappa)))$.

Borel Reducibility: Dichotomies

Claim 3. $E_0 \not\leq_B E_0'$.

As in the previous argument choose a reduction f, a transitive model M of ZFC$^-$ of size κ containing all bounded subsets of κ as well as a Borel code for f and $x \in 2^\kappa$ which is κ-Cohen over M. Choose α_0 so that for some ordinal $i_0 < \alpha_0$, $f(x)(i_0) \neq f(y)(i_0)$ whenever y is κ-Cohen over M and extends $\bar{x}|\alpha_0$; this is possible as $\sim xE_0\bar{x}$ and therefore $\sim f(x)E_0'f(\bar{x})$. Then choose $\alpha_1 > \alpha_0$ so that for some ordinal $i_1 \in [\alpha_0, \alpha_1)$, $f(x)(i_1) = f(y)(i_1)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha) * (x|[\alpha_0, \alpha_1))$; this is possible as $xE_0((\bar{x}|\alpha) * (x|[\alpha_0, \kappa)))$ and therefore $f(x)E_0'f((\bar{x}|\alpha) * (x|[\alpha_0, \kappa)))$. After ω steps we obtain $\sim f(x)E_0'f(y)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha_0) * (x|[\alpha_0, \alpha_1)) * (\bar{x}|[\alpha_1, \alpha_2)) * \cdots$, contradicting the fact that there is such a y which is E_0 equivalent to x. $\square$

Borel Reducibility: Dichotomies

In summary: Even for Borel equivalence relations, the Silver Dichotomy can consistently fail and the Harrington-Kechris-Louveau Dichotomy is provably false.

But there is still some hope for the Harrington-Kechris-Louveau Dichotomy and also some good news for the Silver Dichotomy.

Regarding the Harrington-Kechris-Louveau Dichotomy: Recall that we found a Borel equivalence relation E_0' strictly between id and E_0 with respect to Borel reducibility.

Open question. Suppose that a Borel equivalence relation E is not Borel reducible to id. Then is E_0' Borel reducible to E?

This seems unlikely, but so far has not been ruled out as a possible valid generalisation of the Harrington-Kechris-Louveau Dichotomy for κ^κ.

Borel Reducibility: Dichotomies

Regarding the Silver Dichotomy, first consider one more negative result:

Theorem

There is a Δ^1_1 equivalence relation with κ^+ classes but no perfect set of classes. So the Silver Dichotomy provably fails for Δ^1_1.

Proof. The relation is $xE^{\mathrm{rank}}y$ iff x, y do not code wellorders or x, y code wellorders of the same length. This has exactly κ^+ classes and is Δ^1_1. Suppose T were a perfect tree whose distinct κ-branches were E^{rank}-inequivalent. Now let x be a generic branch through T (treating T as a version of κ-Cohen forcing) and let $p \in T$ be a condition forcing that x codes a wellorder of some rank $\alpha < \kappa^+$. Then any sufficiently generic branch through T extending p codes a wellorder of rank α, which contradicts the fact that there are distinct such branches in V. $\square$

Borel Reducibility: Dichotomies

So a first step toward obtaining the consistency of Silver's Dichotomy for κ^κ is the following.

Theorem

The relation E^{rank} of the previous theorem is not Borel.

Proof. For $\alpha < \kappa$ let $\mathcal{L}_\alpha$ denote the forcing to Lévy collapse α to κ (using conditions of size less than κ). If g is $\mathcal{L}_\alpha$-generic then g^* denotes the subset of κ defined by $i \in g^*$ iff $g((i)_0) \leq g((i)_1)$ where $i \mapsto ((i)_0, (i)_1)$ is a bijection between κ and $\kappa \times \kappa$.

By induction on Borel rank we show that if B is Borel then there is a club C in κ^+ such that:

$(*)$ For $\alpha \leq \beta$ in C and (p_0, p_1) a condition in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces that (g_0^*, g_1^*) belongs to B iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ forces that (g_0^*, g_1^*) belongs to B.

Borel Reducibility: Dichotomies

If B is a basic open set then we may take C to consist of all ordinals greater than κ in κ^+.

Inductively, suppose that B is the intersection of Borel sets B_i, $i < \kappa$, of smaller Borel rank. By intersecting clubs obtained by applying $(*)$ to the B_i's we obtain a club C ensuring the desired conclusion for B.

Finally if B is the complement of the Borel set B_0 then by induction we have a club C_0 such that for $\alpha \leq \beta$ in C_0 and $(p_0, p_1) \in \mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces $(g_0^*, g_1^*) \in B_0$ iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces this. Now thin out the club C_0 to a club C so that for α in C, if (p_0, p_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ and there is some β in C_0 and some (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ below (p_0, p_1) which $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B_0 then there is such a (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$. Then (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces for any $\alpha \leq \beta$ in this thinner club, completing the induction.

Borel Reducibility: Dichotomies

It follows that E^{rank} is not Borel, as otherwise we have $g_0^* E^{\text{rank}} g_1^*$ where g_0, g_1 are sufficiently generic Lévy collapse generics for ordinals $\alpha < \beta$. $\square$

Borel Reducibility: Dichotomies

Now using an analogous argument we have:

Theorem

Suppose that $0^{\#}$ exists, κ is regular in L and λ is the κ^+ of V. Then after forcing over L with the Lévy collapse turning λ into κ^+, the Silver Dichotomy holds for κ^κ.

Proof Sketch. Suppose that p is a condition forcing that $(\sigma_i \mid i < \lambda)$ are pairwise E-inequivalent (where E is a Borel equivalence relation with parameter in L). Assuming that E does not have a perfect set of classes we may assume that the class of σ_i does not depend on the choice of Lévy generic. Let I denote the Silver indiscernibles $i < \lambda$ such that p belongs to L_i. For $i < j$ in I let π_{ij} be an elementary embedding from L to L with critical point i, sending i to j. Also for each $i \in I$ let $f(i)$ denote the L-rank of the name σ_i. Then in analogy to the previous proof, show that as E

Borel Reducibility: Dichotomies

is Borel there is a club C contained in I such that for $i \leq j$ in C and (p_0, p_1) in $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$ below (p, p), $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-forces $\sigma_i^{g_0} E \sigma_i^{g_1}$ iff $(p_0, \pi_{ij}(p_1))$ $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(j)}$-forces $\sigma_i^{g_0} E \sigma_j^{g_1}$. But (p_0, p_1) does $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-force $\sigma_i^{g_0} E \sigma_i^{g_1}$ as the class of σ_i is independent of the choice of generic; it follows that for $i < j$ in C some condition forces $\sigma_i E \sigma_j$, contradicting our assumption that σ_i, σ_j are forced to be pairwise E-inequivalent. $\square$

Borel Reducibility: Size $\leq \kappa$ Borel Equivalence Relations

I'll now discuss some recent work regarding the analogue of the countable Borel equivalence relations for κ^κ, i.e., those Borel equivalence relations whose classes have size at most κ.

An *orbit equivalence relation* is one induced by a Borel action of a Polish group G: xEy iff $g \cdot x = y$ for some $g \in G$. Two important facts about countable equivalence relations in the clasical setting are:

E_∞. Among orbit equivalence relations induced by a Borel action of a *countable* group, there is one of maximum complexity, called E_∞.
Feldman-Moore. In fact *any* countable Borel equivalence relation is the orbit equivalence relation induced by a Borel action of a countable group.

The first of these facts holds true for κ^κ, but in a surprising way:

Borel Reducibility: Size $\leq \kappa$ Borel Equivalence Relations

Theorem

(SDF-Hyttinen-Kulikov) If E is the orbit equivalence relation of a Borel action of a group of size at most κ then E is Borel reducible to E_0.

Proof. The key observation is this: Let F_κ denote the free group on κ generators. Then F_α has cardinality less than κ for $\alpha < \kappa$ (this fails when κ equals ω). Using this one shows that the shift action of F_κ (sending (g, X) in $G \times \mathcal{P}(F_\kappa)$ to $\{g \cdot x \mid x \in X\}$) reduces to E_0: Map $X \subseteq F_\kappa$ to the sequence $f(X) = (<_\alpha$-least element of $\{g_\alpha \cdot (X \cap F_\alpha) \mid g_\alpha \in F_\alpha\} \mid \alpha < \kappa)$. If X, Y are equivalent under shift then it is easy to check $f(X) E_0 f(Y)$; the converse uses Fodor's theorem. $\square$

Borel Reducibility: Size $\leq \kappa$ Borel Equivalence Relations

The Feldman-Moore Theorem however consistently fails for κ^κ:

Theorem

(SDF-Hyttinen-Kulikov) Assume $V = L$. Then there is a Borel equivalence relation with classes of size 2 which is Borel reducible to id but which is not the orbit equivalence relation of any Borel action of a group of size at most κ.

Proof. Let X be the Borel set of Master Codes for initial segments of L of size κ and $\sim X$ its complement. Define a bijection $f :\sim X \to X$ with Borel graph and define $E(x, y)$ iff $y = f(x)$ or $x = f(y)$. Then E is smooth. If it were induced by a Borel action of a group of size at most κ then f would be Borel on a non-meager set, which is impossible. $\square$

Borel Reducibility: Size $\leq \kappa$ Borel Equivalence Relations

Questions. (1) Are all Borel equivalence relations with classes of size at most κ Borel reducible to E_0? (2) Is the Feldman-Moore Theorem for κ^κ consistent?

Borel Reducibility: The (E_0, E_1) Dichotomy

Another interesting dichotomy from the classical case, due to Kechris-Louveau, is:

(E_0, E_1) *Dichotomy.* There is no Borel equivalence relation strictly between E_0 and E_1 with respect to Borel reducibility.

Theorem

(SDF-Hyttinen-Kulikov) For κ^κ there is a Borel equivalence relation strictly between E_0 and E_1 with respect to Borel reducibility.

This counterexample to the (E_0, E_1) Dichotomy is defined analagously to the counterexample to the Harrington-Kechris-Louveau Dichotomy; for $x = (x_\alpha \mid \alpha < \kappa)$ and $y = (y_\alpha \mid \alpha < \kappa)$ where $x_\alpha, y_\alpha \in 2^\kappa$:

$xE_1'y$ iff xE_1y and $\{\alpha < \kappa \mid x_\alpha \neq y_\alpha\}$ is a finite union of intervals and $x_\alpha \neq y_\alpha \rightarrow x_\alpha(i) \neq y_\alpha(i)$ for all $i < \kappa$.

Borel Reducibility: Isomorphism Relations

An important class of Σ^1_1 equivalence relations is the class of *isomorphism relations*. View the elements of κ^κ as codes for structures with universe κ (for a language of size at most κ). An *isomorphism relation* is given by specifying a sentence φ of the infinitary logic $L_{\kappa^+\kappa}$ and defining:
$xE_\varphi y$ iff x, y do not code models of φ or x, y code isomorphiic models of φ.
We can eliminate the logic using the following result:

Theorem

(Vaught) X is the set of codes for models of a sentence of $L_{\kappa^+\kappa}$ iff X is Borel and invariant: if x belongs to X and y codes a model isomorphic to the model coded by x then y also belongs to X.

These relations need not be Borel and there is one of maximum complexity, the relation of isomorphism of graphs.

Borel Reducibility: Isomorphism Relations

As in the classical case the Borel isomorphism relations are classified using a version of the Friedman-Stanley jump. Define:

xE^+y iff $\{[(x)_i]_E \mid i < \kappa\} = \{[(y)_i]_E \mid i < \kappa\}$

where $((x)_i \mid i < \kappa)$ is a canonical decomposition of $x \in \kappa^\kappa$ into a κ-sequence of elements of κ^κ.

One defines transfinite iterates $E^{+\alpha}$ of the jump in the natural way. Then imitating the proof from the classical case one has:

Theorem

The relations $id^{+\alpha}$ for $\alpha < \kappa^+$ are Borel bireducible to Borel isomorphism relations and any Borel isomorphism relation is Borel reducible to one of these.

Borel Reducibility: Isomorphism Relations

Adapting some of the theory of Hjorth's turbulence to κ^κ we have:

Theorem

(Hyttinen-Kulikov-Schlicht) E_1 is not Borel reducible to id$^+$ nor to any of its iterates id$^{+\alpha}$, $\alpha < \kappa^+$.

Corollary

E_1 is not Borel reducible to any Borel isomorphism relation.

But in the classical case one has more: E_1 is not Borel reducible to any isomorphism relation (nor to any orbit equivalence relation of a Polish group action). Unfortunately Hjorth's turbulence theory does not fully adapt to κ^κ (due to failures of the Baire property) and indeed:

Borel Reducibility: Isomorphism Relations

Theorem

(SDF-Hyttinen-Kulikov) Assume $V = L$ and let κ be the successor of a regular cardinal. Then all Σ^1_1 equivalence relations (including E_1) are Borel reducible to isomorphism.

I give a hint of the proof. Write $\kappa = \lambda^+$ where λ is regular, let $\mathcal{Q}$ be a λ-saturated dense linear order without endpoints and let $\mathcal{Q}_0$ be $\mathcal{Q}$ together with a least point. For any subset S of κ let $\mathcal{L}(S)$ be obtained from the natural order on κ by replacing α by $\mathcal{Q}_0$ if α is a limit ordinal in S and by $\mathcal{Q}$ otherwise.

Borel Reducibility: Isomorphism Relations

Fact. $\mathcal{L}(S)$ is isomorphic to $\mathcal{L}(T)$ iff $S \triangle T$ is nonstationary in κ.

Now the key Lemma is that in L, any Σ^1_1 set X is Borel reducible to the collection (ideal) of nonstationary sets in the sense that there is a Borel function f such that $x \in X$ iff $f(x)$ is nonstationary. One strengthens this to show that in fact any Σ^1_1 equivalence relation is Borel reducible to equality modulo a nonstationary set and therefore by the above *Fact* to isomorphism of dense linear orders.

Borel Reducibility: Isomorphism Relations

More Questions. Is it consistent that isomorphism is *not* complete for Σ^1_1 equivalence relations under Borel reduciblity? Is it consistent that E_1 is *not* Borel reducible to isomorphism? Is the Friedman-Stanley jump strict in the sense that E^+ is never Borel reducible to E? Does this at least hold for isomorphism relations E?

Borel Reducibility: First-order Isomorphism Relations

Another interesting aspect of Descriptive Set Theory on κ^κ is its connection with Shelah's stability theory. The basic question is:

Question. How does the model-theoretic complexity of a countable first-order theory T compare to the complexity in terms of Borel reducibility of the equivalence relation of isomorphism on the models of T?

Koerwien looked at this question in terms of the *countable* models of T and discovered a surprising discrepancy in these two notions of complexity:

Borel Reducibility: First-order Isomorphism Relations

Theorem

There is a countable first-order theory which is ω-stable of depth 2 and NDOP such that isomorphism on its countable models is not Borel.

I.e., the above first-order theory is very simple model-theoretically but rather complicated in terms of classical descriptive set theory.

Conversely, Dense Linear Orders is a simple example of a first-order theory which is complex model-theoretically (it is unstable) but is trivial in terms of classical descriptive set theory (it is ω-categorical).

This discrepancy is eliminated by turning to Descriptive Set Theory on κ^κ:

Borel Reducibility: First-order Isomorphism Relations

Theorem

(SDF-Hyttinen-Kulikov)
(a) Suppose that $\kappa = \kappa^{<\kappa}$ is a successor cardinal bigger than $2^{\aleph_0}$. Then a first-order theory is classifiable and shallow iff the isomorphism relation on its models of size κ is Borel.
(b) Suppose in addition that $\kappa = \lambda^+$ where $\lambda^{<\lambda} = \lambda$. Then T is classifiable iff equality modulo a μ-nonstationary set is not Borel reducible to the isomorphism relation on its models of size κ for all regular $\mu < \kappa$.

Borel Reducibility: First-order Isomorphism Relations

On the other hand, Hyttinen-Kulikov have shown that if $V = L$ then there is a theory which is stable with NDOP and NOTOP such that the isomorphism relation on its models of size κ (where $\kappa = \lambda^+$, $\lambda^{\aleph_0} = \lambda$) is complete as a Σ^1_1 equivalence relation.

Thus if $V = L$ we again lose some of the correlation between model-theoretic and descriptive set-theoretic complexity.

The Right Axioms?

The above results strongly suggest that $V = L$ is an unsatisfying hypothesis both for the Descriptive Set Theory of κ^κ and for its connections with model-theoretic complexity.

Rather it seems that a hypothesis which ensures maximum regularity for Δ^1_1 sets, the Silver Dichotomy for Borel equivalence relations and a restored correlation of the complexity of isomorphism relations with first-order stability theory is desired.

I don't yet know what this hyothesis would be. But surely this question, together with the other open questions mentioned earlier in this tutorial, suggest that much interesting work remains to be done in order to gain a full understanding of the Descriptive Set theory on Generalised Baire Space.

Thanks for listening.