The choice of ω_1 and combinatorics at $\aleph_\omega$

Spencer Unger

UCLA

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Outline

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Combinatorial properties

There are three weak square principles that we will be interested in $\square^*_\nu$, $\nu^+ \in I[\nu^+]$ and "All scales of length ν^+ are good". In fact they are strictly decreasing in strength by a theorem of Shelah and some consistency results.

Theorem (Shelah)

For a singular cardinal ν, $\square^*_\nu$ implies $\nu^+ \in I[\nu^+]$ implies "All scales of length ν^+ are good".

Weak square

Weak square was introduced by Jensen who proved that it holds if and only if there is a special Aronszajn tree.

Definition

Let ν be a cardinal. A $\square^*_\nu$-sequence is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ such that

1. for all $\alpha < \nu^+$, $1 \leq |\mathcal{C}_\alpha| \leq \nu$,
2. for all $\alpha < \nu^+$ and for all $C \in \mathcal{C}_\alpha$, C is club in α and $\mathrm{o.t.}(C) \leq \nu$ and
3. for all $\alpha < \nu^+$, $C \in \mathcal{C}_\alpha$ and $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

Approachability

The notion of approachability was introduced by Shelah. In fact there is an ideal of approachable sets.

Definition

Let μ be a regular cardinal and $\langle x_\alpha \mid \alpha < \mu \rangle$ be a sequence of bounded subsets of μ. An ordinal $\gamma < \mu$ is *approachable with respect to* $\vec{x}$ if there is $A \subseteq \gamma$ cofinal such that $\mathrm{o.t.}(A) = \mathrm{cf}(\gamma)$ and for all $\beta < \gamma$, there is $\delta < \gamma$ such that $A \cap \beta = x_\delta$.

Definition

Let μ be a cardinal. A set $S \subseteq \mu$ is in the collection of approachable subsets $I[\mu]$ if and only if there are a club $C \subseteq \mu$ and a sequence $\vec{x}$ such that for all $\gamma \in S \cap C$, γ is approachable with respect to $\vec{x}$.

Fact

For a regular cardinal μ, $I[\mu]$ is an ideal.

If $\mu \in I[\mu]$, then we say that μ has the Approachability property.

Scales

Scales are a notion from Shelah's PCF theory.

- The general setting is a singular cardinal ν with an increasing and cofinal sequence $\langle \nu_i \mid i < \mathrm{cf}(\nu) \rangle$ of regular cardinals less than ν.
- Given members $f, g \in \prod_i \nu_i$ we say that $f <^* g$ if and only if there is a $j < \mathrm{cf}(\nu)$ such that for all $i \geq j$, $f(i) < g(i)$.
- A sequence of functions $\langle f_\alpha \mid \alpha < \nu^+ \rangle$ is a *scale of length* ν^+ *in* $\prod_i \nu_i$ if it is increasing and cofinal in $\prod_i \nu_i$ under the ordering $<^*$.
- A point $\gamma < \nu^+$ with $\mathrm{cf}(\gamma) > \mathrm{cf}(\nu)$ is *good* for a scale $\vec{f}$ of length ν^+ if there are $A \subseteq \gamma$ cofinal and $j < \mathrm{cf}(\nu)$ such that for all $i \geq j$ the sequence $\langle f_\alpha(i) \mid \alpha \in A \rangle$ is strictly increasing.

Scales continued

- A scale $\vec{f}$ is *good* if there is a club $C \subseteq \nu^+$ such that all γ in C of cofinality greater than $\mathrm{cf}(\nu)$ are good for $\vec{f}$.
- A scale $\vec{f}$ is *bad* if it is not good. We say there is a bad scale of length ν^+ if there is a bad scale of length ν^+ in some product $\prod \nu_i$.

Recall

Theorem (Shelah)

For a singular ν, $\square^*_\nu$ implies $\nu^+ \in I[\nu^+]$ implies "All scales of length ν^+ are good".

Stationary reflection

Let μ be a regular cardinal. Recall that a stationary subset S of μ reflects if there is an $\alpha < \mu$ such that $S \cap \alpha$ is stationary.

Theorems from supercompactness

It turns out that all of the weak square principles above fail in the presence of supercompactness.

Theorem

Let κ be a supercompact cardinal and $\nu > \kappa$ be singular with $\mathrm{cf}(\nu) < \kappa$, every scale of length ν^+ is bad.

- Let $\langle f_\alpha \mid \alpha < \nu^+ \rangle$ be a scale in the product $\prod_i \nu_i$.
- Let $j : V \to M$ witness that κ is ν^+-supercompact. In particular $\mathrm{crit}(j) = \kappa$ and ${}^{\nu^+}M \subseteq M$.

A proof continued

- Let $\gamma = \sup j``\nu^+$ and note that $\gamma < j(\nu^+)$.
- It is not hard to show that the function, $i \mapsto \sup j``\nu_i$ is a so called exact upper bound for $\langle j(f)_\alpha \mid \alpha < \gamma \rangle$.
- It follows that γ is bad for $j(\vec{f})$.
- Standard reflection arguments show that there is a stationary set $S \subseteq \nu^+$ of bad points for $\vec{f}$. Moreover S concentrates on cofinalities μ^+ where μ is a singular cardinal of cofinality $\mathrm{cf}(\nu)$.

Theorems from supercompactness continued

Theorem

Suppose that ν is a singular limit of supercompact cardinals, then every stationary subset of ν^+ reflects.

Proof.

- Suppose that $S \subseteq \nu^+$ is stationary. There is a regular cardinal $\delta < \nu$ such that $S' =_{def} S \cap \mathrm{cof}(\delta)$ is stationary. It is enough to show that S' reflects.
- Let κ be a supercompact cardinal greater than δ and let $j : V \to M$ witness that κ is ν^+-supercompact.
- Now since $\delta < \kappa$ we can show that $j(S') \cap \sup j``\nu^+$ is stationary in M.
- It follows that S' reflects in V.

□

Bringing results down to $\aleph_\omega$

Theorem

Assuming there is a supercompact cardinal it is relatively consistent that there is a bad scale of length $\aleph_{\omega+1}$, in particular $\square^*_{\aleph_\omega}$ fails.

Ideas of the proof:

- Fix a scale of length $\kappa^{+\omega+1}$ in the product $\prod \kappa^{+n}$.
- By our previous argument the scale has stationarily many bad points of some cofinality $\mu^+ < \kappa$.
- Force with $\mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^{++}, < \kappa)$.
- Show that the scale is still a scale.
- Show that the stationary set of bad points is preserved.
- Show that it is still a set of bad points, but now concentrating on cofinality ω_1.

Bringing results down continued

Theorem (Magidor)

Assume that there are infinitely many supercompact cardinals, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

Ideas of the proof:

- Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals.
- Iterate Levy Collapses with full support starting with $\mathrm{Coll}(\omega < \kappa_0)$ and in general $\Vdash_n \dot{\mathbb{Q}}_n = \mathrm{Coll}(\kappa_n, < \kappa_{n+1})$.
- Try to repeat the proof we sketched above. Now that there is forcing involved, we must use *generic* elementary embeddings.
- The proof is mostly the same except that we need to force to see that the generic embeddings exist.
- The key point in the proof is to show that the forcing that adds some elementary embedding cannot destroy the stationarity of some set.
- To prove this fact we actually show that in the final model $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$.

The choice of ω_1

Some Remarks:

- In the proof that it is consistent that there is bad scale of length $\aleph_{\omega+1}$, we chose μ^+ which was a successor of a singular cardinal of cofinality ω to become ω_1.
- In the proof of Magidor's theorem above we made a supercompact cardinal into ω_1.
- Recall that by the theorem of Shelah at the beginning, there is a bad scale of length $\aleph_{\omega+1}$ is incompatible with the approachability property at $\aleph_{\omega+1}$.
- We will see later Magidor's theorem requires a formerly supercompact ω_1.

One more theorem for background

Theorem (Cummings and Foreman)

Assuming there are infinitely many supercompact cardinals, it is consistent that the tree property holds at $\aleph_n$ *for* $2 \leq n < \omega$.

Some new theorems

Let $\mathbb{R}_\omega$ be the Cummings-Foreman iteration. For each of the following theorems we assume that there are infinitely many supercompact cardinals.

Theorem (U)

Suppose that V is a model obtained by forcing with $\mathrm{Coll}(\omega, < \kappa)$ for some supercompact κ, then in $V[\mathbb{R}_\omega]$

1. $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ *and*
2. *for every $n < \omega$, every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_n)$ reflects at a point of cofinality $\aleph_{n+1}$.*

Theorem (U)

There are a $\mu < \kappa_0$ and a generic object $d * G_\omega$ for $\mathrm{Coll}(\omega, \mu) * (\mathbb{R}_\omega)_{V[\mathrm{Coll}(\omega,\mu)]}$ such that in $V[d * G_\omega]$, there are a bad scale of length $\aleph_{\omega+1}$ and a non-reflecting stationary subset of $\aleph_{\omega+1}$.

Another theorem

Theorem (U and Fontanella independently)

In the Cummings-Foreman model, $\mathrm{ITP}(\aleph_n, \lambda)$ *holds for all n with* $1 < n < \omega$ *and all* $\lambda \geq \aleph_n$.

An explanitory theorem

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Let $\mathbb{C}$ be the direct sum of $\mathrm{Coll}(\omega, \mu)$ for $\mu < \kappa_0$ a singular cardinal of cofinality ω. Let $\dot{\mathbb{R}}$ be a $\mathbb{C}$-name for the full support iteration of Levy collapses to make κ_n into $\aleph_{n+1}$ for all $n < \omega$.

Theorem

There is a generic object $c * G$ for $\mathbb{C} * \dot{\mathbb{R}}$ such that in $V[c * G]$ there are a bad scale of length $\aleph_{\omega+1}$ and a non reflecting stationary subset of $\aleph_{\omega+1}$.

Ideas from the proof

- Let V_0 be a model of GCH with infinitely many supercompact cardinals κ_n for $n < \omega$ and let V be the extension of V_0 by the Laver preparation for κ_0.
- Let $\vec{f}$ be a scale in V_0 in $\prod \kappa_n$. Note that $\vec{f}$ is a bad scale in V_0 and in V.
- It is not hard to show that $\vec{f}$ remains a scale in $V[\mathbb{C} * \dot{\mathbb{R}}]$.
- It is also not hard to show if α is a bad point of $\vec{f}$ in V_0, then it is still a bad point of $\vec{f}$ in $V[\mathbb{C} * \dot{R}]$.
- The key difficulty is to show that the stationary set of bad points from V_0 is preserved in the extension by some generic for $\mathbb{C} * \dot{\mathbb{R}}$.
- Recall that $\mathbb{C}$ chooses ω_1.

A useful lemma

Lemma

Suppose that V is obtained from V_0 by forcing with the Laver preparation for some supercompact cardinal κ. Further assume that $\vec{f} \in V_0$ is a bad scale of length ν^+ for singular ν with $\mathrm{cf}(\nu) < \kappa$ with set of bad points S. If $\mathbb{P}$ is κ-directed closed forcing which preserves ν^+, then in $V[\mathbb{P}]$ S contains a stationary set of bad points.

Ideas continued

- To show that there is a non-reflecting stationary set we note that actually the stationary set of bad points of cofinality ω_1 does not reflect.
- To see this note that the set of bad points cannot reflect at a good point.
- Then we just show that all points of cofinality $\aleph_n$ for $n > 1$ are good for our scale.
- It follows that the set of bad points does not reflect.

Remarks

Extra difficulties when the forcing is not just iterated Levy Collapses.

- The generic embeddings are added my much more complex forcing.
- It is no longer true that the forcing can be written as (κ_n-cc forcing) * (κ_n-closed forcing).
- We can replace κ_n-closed with $< \kappa_n$-distributive and this is enough.

The paper

A model of Cummings and Foreman revisited, Submitted