The importance of approximate counting in bounded arithmetic

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(based on joint work with Buss-Thapen and Buss-Zdanowski)

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Bounded arithmetic

Bounded arithmetic: collective name for some first-order arithmetic theories with induction only for (some or all) bounded formulas.

Usual language (mostly due to Sam Buss): 
\(+, \cdot, \leq, 0, 1, \log x, 2^{\log x \cdot \log y}, \lfloor x/2^y \rfloor\).

Motivation:

▶ connections to computational complexity,
▶ connections to propositional proof complexity: arithmetical proofs can be translated into short propositional proofs,
▶ foundational concerns: how much “finite mathematics” can be done without the exponential function.
Connection to Mostowski (high-level)

Mostowski did work on first-order arithmetic which spawned bounded arithmetic which is the topic of this talk.
Connection to Mostowski (not-so-high-level)

Mostowski

defined the

Kleene-Mostowski hierarchy

which has an important analogue in

bounded arithmetic

which is the topic of

this talk.
Formula and theory hierarchies

Analogue of Kleene-Mostowski hierarchy:

$\hat{\Sigma}^b_n$ formulas: $\exists x_1 < t_1 \forall x_2 < t_2 \ldots Qx_n < t_n \psi$,

where $\psi$ sharply bounded (only quantifiers of the form $Qx < \log t$).

Full BA: basic axioms + induction for all bounded formulas.

The fragment $T^n_2$: induction only for $\hat{\Sigma}^b_n$ formulas.
(Definition and strength of $T^n_2$ very sensitive to choice of language.)
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**Formula and theory hierarchies**

Analogue of Kleene-Mostowski hierarchy:

\[ \hat{\Sigma}^b_n \text{ formulas: } \exists x_1 < t_1 \forall x_2 < t_2 \ldots Qx_n < t_n \psi, \]
where \( \psi \) sharply bounded (only quantifiers of the form \( Qx < \log t \)).

Full BA: basic axioms + induction for all bounded formulas.

The fragment \( T_2^n \): induction only for \( \hat{\Sigma}^b_n \) formulas.

(Definition and strength of \( T_2^0 \) very sensitive to choice of language.)

Expressive power:

- \( \hat{\Sigma}^b_n \leftrightarrow \Sigma^b_n \), the \( n \)-th level of the polynomial time hierarchy.
  So, for instance, \( \hat{\Sigma}^b_1 \leftrightarrow = \text{NP} \).

- Provably \( \hat{\Delta}^b_n \) in \( T_2^n \leftrightarrow \) polynomial time with \( \Sigma^b_{n-1} \) oracle.
  (Where \( \hat{\Delta}^b_n \) means definable by both \( \hat{\Sigma}^b_n \) and negated \( \hat{\Sigma}^b_n \) flas.)
Witnessing theorems

The connection with computational complexity runs deeper, in the form of witnessing theorems. For example:

- If $T^0_2 \vdash \forall x \exists y \psi(x, y)$ with $\psi \in \hat{\Sigma}^b_0$, then given $x$ as input, $y$ can be found in polynomial time.
- If $T^1_2 \vdash \forall x \exists y \psi(x, y)$ with $\psi \in \hat{\Sigma}^b_0$, then given $x$ as input, $y$ can be found by a polynomial local search procedure.
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Fundamental problem:

“Is (full) BA finitely axiomatizable?”,
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or equivalently,

“Is BA equivalent to one of the theories $T^*_2$?”
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“Is BA equivalent to one of the theories $T_{2}^{n}$?”

or equivalently,

“Does BA prove the collapse of the polynomial-time hierarchy?”
Fundamental problem:

“Is (full) BA finitely axiomatizable?”,

or equivalently,

“Is BA equivalent to one of the theories $T_2^n$?”

or equivalently,

“Does BA prove the collapse of the polynomial-time hierarchy?”

(No matter how you state it, the question is apparently out of reach.)
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The relativized setting

To get some unprovability results, it helps to consider relativized BA, with a new “oracle” predicate $\alpha$ (which leads to $\hat{\Sigma}^b_n(\alpha)$, $T^m_2(\alpha)$ etc.).

For instance:

- $T^0_2(\alpha) \subsetneq T^1_2(\alpha) \subsetneq T^2_2(\alpha) \ldots$
  (Krajíček-Pudlák-Takeuti 1991),

- $BA(\alpha) \not\vdash PHP(\alpha)$,
  viz. that for all $a$, $\alpha$ is not an injective function from $a + 1$ to $a$
Two problems from the research frontier

1. Can the theories $T^n_2(\alpha)$ be separated by a $\forall \hat{\Sigma}^b_1(\alpha)$ sentence?
   
   - only $T^0_2(\alpha) \not\leq_{\forall \hat{\Sigma}^b_1(\alpha)} T^1_2(\alpha) \not\leq_{\forall \hat{\Sigma}^b_1(\alpha)} T^2_2(\alpha)$ known.

2. An “interesting” independence result for BA(\alpha) with a parity quantifier, “there is an odd number of $x < t$ such that”.
   
   - e.g. for PHP(\alpha), or maybe for the counting principle mod 3: there is no partition of $\{0, \ldots, 3n\}$ into 3-element sets.

Both problems have very closely corresponding versions in propositional proof complexity.
The weak pigeonhole principle

\( \text{iWPHP}(\mathcal{F}) \) says:
no function \( f \in \mathcal{F} \) is an injection from \( 2a \) into \( a \) (where \( a > 0 \)).

Paris, Wilkie and Woods 1988:

- \( \text{BA}(\alpha) \vdash \text{iWPHP}(\alpha) \),
- \( \text{BA} \vdash \text{if there are no primes between } a \text{ and } a^{11}, \text{then there is a bounded-definable injection from } 9a \log a \text{ into } 8a \log a \). (So, in particular, there are primes between \( a \) and \( a^{11} \).

)
WPHP in the bounded arithmetic hierarchy

Theorem (essentially Maciel-Pitassi-Woods 2002)
\[ T_2^2(\alpha) \vdash \text{iWPHP}(\alpha). \]

Theorem (Chiari-Krajíček 1998)
\[ T_2^1(\alpha) \not\vdash \text{iWPHP}(\alpha). \]
The surjective WPHP and approximate counting

$s_{\text{WPHP}}(\mathcal{F})$ says:
no function $f \in \mathcal{F}$ is a surjection from $a$ onto $2a$ (where $a > 0$).

**Theorem (Jeřábek 2009)**

The theory $T^1_2 + s_{\text{WPHP}}(\hat{\Delta}^b_2)$ can perform approximate counting of $\hat{\Sigma}^b_1$-definable sets: given bounded and $\hat{\Sigma}^b_1$-definable $X$ it finds surjections witnessing $s \leftarrow X \leftarrow s + s/\text{polylog}(s)$ for some $s$.

- J. actually needs to prohibit surjections from $a$ onto $a + a/\log(a)$, but this is to a large extent conservative over $s_{\text{WPHP}}$.
- A weak form of aprx. counting is available in $T^0_2 + s_{\text{WPHP}}(\hat{\Delta}^b_1)$. The two theories are sometimes called APC$_1$ and APC$_2$. 
Approximate counting and the research frontier

The two problems from a few slides back:

1. Can the theories $T_2^n(\alpha)$ be separated by a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence?
   - only $T_2^0(\alpha) \not\leq_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^1(\alpha) \not\leq_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^2(\alpha)$ known.

2. An “interesting” independence result for BA(\alpha) with a parity quantifier, “there is an odd number of $x < t$ such that”.

APC$^2(\alpha)$ plays a major role in both problems!
(Note that $T_1^2(\alpha) \subseteq APC^2(\alpha) \subseteq T_2^3(\alpha)$.)

The importance of approximate counting in bounded arithmetic
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The two problems from a few slides back:

1. Can the theories \( T_2^m(\alpha) \) be separated by a \( \forall \hat{\Sigma}_1^b(\alpha) \) sentence?
   - only \( T_2^0(\alpha) \not\equiv_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^1(\alpha) \not\equiv_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^2(\alpha) \) known.

2. An “interesting” independence result for \( BA(\alpha) \) with a parity quantifier, “there is an odd number of \( x < t \) such that”.

\( APC_2(\alpha) \) plays a major role in both problems!
(\( \text{Note that } T_2^1(\alpha) \subseteq APC_2(\alpha) \subseteq T_2^3(\alpha). \))
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2. An “interesting” independence result for $\text{BA}(\alpha)$ with a parity quantifier, “there is an odd number of $x < t$ such that”.

$\text{APC}_2(\alpha)$ plays a major role in both problems!
(Note that $T_2^1(\alpha) \subseteq \text{APC}_2(\alpha) \subseteq T_2^3(\alpha)$.)
$\forall \hat{\Sigma}^b_1(\alpha)$ principles separating low levels of the hierarchy

- iWPHP($\alpha$), with “$x$ maps to $y$” formalized not by $\alpha(x, y)$, but by: $\forall i < \log a \ [\alpha(x, i) \equiv (\text{bit}(y, i) = 1)]$.
- Ramsey’s principle: the graph determined by $\alpha$ on $[0, b)$ has a homogeneous set of size $(\log b)/2$.
- Herbrandized ordering principle HOP: if $\preceq_{[0, c)}$ is a linear ordering, then $h$ cannot be the associated total predecessor function. (Here $\preceq$ and $h$ given by the oracle $\alpha$.)
The importance of approximate counting in bounded arithmetic

$\forall \hat{\Sigma}_1^b(\alpha)$ principles separating low levels of the hierarchy

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All three of these (and more) are provable in $\text{APC}_2(\alpha)$!
Status of HOP

Proposition
Both $T^2_2(\alpha)$ and $\text{APC}_2(\alpha)$ prove HOP.

Proof.
In $T^2_2$, prove "$\leq \restriction_{[0,x)}$ has a least element" by induction on $x$. 
Status of HOP

Proposition
Both $T^2_2(\alpha)$ and $APC^2_2(\alpha)$ prove HOP.

Proof.
In $T^2_2$, prove “$\preceq\upharpoonright_{[0,x)}$ has a least element” by induction on $x$.

$APC^2_2$ is known to prove the tournament principle: given a tournament, there is a log-sized dominating set. Apply this to the tournament given by $\preceq$.
Finding the least element of the log-sized set can be done in $T^0_2$. □
Fragments of APC$_2(\alpha)$

Open problem:

Is there a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence separating APC$_2(\alpha)$ from full BA($\alpha$)?
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So what about mild weakenings of APC$_2(\alpha)$?
Fragments of $\text{APC}_2(\alpha)$

Open problem:
Is there a $\forall \hat{\Sigma}^b_1(\alpha)$ sentence separating $\text{APC}_2(\alpha)$ from full $\text{BA}(\alpha)$?

So what about mild weakenings of $\text{APC}_2(\alpha)$?

- $T^1_2(\alpha) + \text{sWPHP}(\hat{\Delta}^b_1(\alpha))$.
- $T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}^b_1(\alpha))$.
- $T^0_2(\alpha) + \text{sWPHP}(\hat{\Delta}^b_2(\alpha))$. 
Fragments of APC\(_2(\alpha)\)

Open problem:
Is there a \(\forall \hat{\Sigma}_1^b(\alpha)\) sentence separating APC\(_2(\alpha)\) from full BA(\(\alpha\))?

So what about mild weakenings of APC\(_2(\alpha)\)?

- \(T^1_2(\alpha) + \text{sWPHP}(\hat{\Delta}_1^b(\alpha))\).

- \(T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))\).
  Doesn’t prove HOP (Buss-K.-Thapen).

- \(T^0_2(\alpha) + \text{sWPHP}(\hat{\Delta}_2^b(\alpha))\).
Fragments of APC\(_2(\alpha)\)

Open problem:
Is there a \(\forall \hat{\Sigma}_1^b(\alpha)\) sentence separating APC\(_2(\alpha)\) from full BA\((\alpha)\)?

So what about mild weakenings of APC\(_2(\alpha)\)?

- \(T^1_2(\alpha) + s\text{WPHP}(\hat{\Delta}_1^b(\alpha))\).
- \(T^1_2(\alpha) + i\text{WPHP}(\hat{\Delta}_1^b(\alpha))\).
- Doesn’t prove HOP (Buss-K.-Thapen).

- \(T^0_2(\alpha) + s\text{WPHP}(\hat{\Delta}_2^b(\alpha))\).
- Doesn’t prove HOP (Buss-K.-Thapen).
Fragments of \( \text{APC}_2(\alpha) \)

Open problem:
Is there a \( \forall \hat{\Sigma}^b_1(\alpha) \) sentence separating \( \text{APC}_2(\alpha) \) from full \( \text{BA}(\alpha) \)?

So what about mild weakenings of \( \text{APC}_2(\alpha) \)?

- \( T^1_2(\alpha) + \text{sWPHP}(\hat{\Delta}^b_1(\alpha)) \).
  Breaking news! Doesn’t prove HOP (Atserias-Thapen).

- \( T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}^b_1(\alpha)) \).
  Doesn’t prove HOP (Buss-K.-Thapen).

- \( T^0_2(\alpha) + \text{sWPHP}(\hat{\Delta}^b_2(\alpha)) \).
  Doesn’t prove HOP (Buss-K.-Thapen).
Fragments of $\text{APC}_2(\alpha)$

$T_2^1(\alpha) + \text{iWPHP}(\Delta^b_1(\alpha))$. 
Independence for $T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))$

Theorem

$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha)) \not\vdash \text{HOP}$. 
Independence for $T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))$

**Theorem**
$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha)) \not\vdash \text{HOP}.$

**Proof sketch**
Assume $T_2^1(\preceq, h) + \text{iWPHP}(\hat{\Delta}_1^b(\preceq, h)) \vdash \text{HOP}.$ Then there exist:

- a polytime $f^{\preceq, h}(\cdot, \cdot)$ such that $f(a, \cdot) : [0, a^2) \to [0, a)$ for each $a$,
- a *polynomial local search* procedure with oracles $\preceq, h, r_1, r_2$ that takes input $c$ and finds either a witness to $\text{HOP} \upharpoonright [0,c)$ or some $a \in [c, t(c))$ such that $r_1(a) = r_2(a)$ or $f(a, r_1(a)) \neq f(a, r_2(a))$.

Let’s pretend that the PLS procedure is just a polytime function $g$, which runs for polylog$(c)$ steps and asks queries: “$x_1 \preceq x_2?$”, “$h(x) = ?$”, “$r_1(y) = ?, r_2(y) = ?$”
$T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling $g$

We gradually define $\preceq \upharpoonright [0,c)$ so as to answer the queries without revealing a witness to HOP. For queries about $\preceq$ and $h$, this is easy. We define $\preceq$ on one or two more points.
$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))$: fooling $g$

For queries about $r_1$ and $r_2$, we have to extend $\preceq$ by polylog($c$) more points so that $r_1(a)$ and $r_2(a)$ give a collision in $f(a, \cdot)$. 
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$T^1_2(\alpha) + iWPHP(\Delta^b_1(\alpha))$: fooling $g$

We do this in stages. At each stage, we think of $\preceq$ as defined on all of $[0, c)$, but only part is settled; the rest is tentative. Also, at each stage $\gg a$ of the $a^2$ pigeons are still active; the rest have been discarded.

```
pigeons
  ○ ● ●
  ● ● ●
  ○ ● ●

holes
  ●
  ●
  ●
```

$p \quad ?$
$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))$: fooling $g$

At a stage, if $> a$ active pigeons make it through the computation of $f(a, \cdot)$ without querying $h$ of the currently $\preceq$-smallest point $p$, there is a collision in $f(a, \cdot)$ that we can use to define $r_1(a), r_2(a)$. 

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**Diagram:**

- **Pigeons**:
  - Two empty circles (○) and two filled circles (●) are shown.

- **Holes**:
  - Two filled circles (●) are shown.

- **Mappings**:
  - Arrows indicate the mapping from pigeons to holes.

- **Red Arrow**:
  - A red arrow points to a question mark (?) next to the pigeon labeled 'p'.
$T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling $g$

Otherwise, we find a tentative point $q$ which is queried by few of the active pigeons and move it below $p$. Discard the pigeons which do not query $p$ or do query $q$. 
$T_2^1(\alpha) + i\text{WPHP}(\hat{\Delta}_1^b(\alpha))$: fooling $g$

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Diagram:

- Pigeons:
  - $\circ$ (open circle)
  - ● (solid circle)

- Holes:
  - ●

- Arrows:
  - q (query at point $q$)
$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha))$: fooling $g$

Otherwise, we find a tentative point $q$ which is queried by few of the active pigeons and move it below $p$. Discard the pigeons which do not query $p$ or do query $q$. 

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In this way, all remaining active pigeons make it one step further in the computation of $f(a, \cdot)$ without querying the $\preceq$-least point. After polylog($c$) many stages, we find a “safe” collision in $f(a, \cdot)$.

$$T_2^1(\alpha) + \text{iWPHP}(\hat{\Delta}_1^b(\alpha)) : \text{fooling } g$$
Approximate counting and the research frontier

The two problems from a few slides back:

1. Can the theories $T_m^2(\alpha)$ be separated by a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence?
   - only $T_2^0(\alpha) \not\leq_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^1(\alpha) \not\leq_{\forall \hat{\Sigma}_1^b(\alpha)} T_2^2(\alpha)$ known.

2. An “interesting” independence result for $BA(\alpha)$ with a parity quantifier, “there is an odd number of $x < t$ such that”.

$APC_2(\alpha)$ plays a major role in both problems!
(Nota that $T_2^1(\alpha) \subseteq APC_2(\alpha) \subseteq T_2^3(\alpha)$.)
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Theories with the $\oplus$ quantifier

$\oplus x < y := \text{“there is an odd number of } x < y \text{ such that”}$.

$\text{BA}^{\oplus}$: induction for bounded formulas in the language with $\exists, \forall, \oplus$.

$\hat{\Sigma}_n^{b, \oplus}$ formulas: $\exists x_1 < t_1 \forall x_2 < t_2 \ldots \mathcal{O} x_n < t_n \psi$, where $\psi$ open except for perhaps $\oplus$ in front of $\Sigma^b_0$ formulas.

$T_{2}^{n, \oplus}$: induction for $\hat{\Sigma}_n^{b, \oplus}$. Note that $\bigcup_n T_{2}^{n, \oplus} \neq \text{BA}^{\oplus}$.

This all relativizes smoothly to $\alpha$. 
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**Collapse for theories with ⊕**

**Theorem**

$\text{BA}^{⊕}$ is conservative over $\text{APC}_2^{⊕P}$, also in the relativized setting.

**Proof sketch:**

- The idea is to formalize *Toda’s Theorem*: a known collapse result for bounded formulas involving $\exists, \forall, ⊕$.

- We inductively assign to each bounded formula $ϕ(x)$ with $⊕$ a $\hat{Σ}_0^b$ formula $ψ(x, y, r)$ such that

  - $ϕ(x) \Rightarrow \text{w.h.p. over } r, \bigoplus y < t ψ(x, y, r),$
  - $¬ϕ(x) \Rightarrow \text{w.h.p. over } r, \neg \bigoplus y < t ψ(x, y, r).$

- On a bounded interval, “w.h.p. over $r$, $\bigoplus y < t ψ(x, y, r)$” is $\hat{Δ}_1^{b,⊕P}$, and already $T_2^{0,⊕P}$ has induction for it.

□
The collapse result: comments on proof

- A crucial step in the induction is dealing with ∃, based on the so-called Valiant-Vazirani Lemma: given $\hat{\Sigma}^b_1$ formula $\varphi(x)$, there is a $\hat{\Sigma}^b_0$ formula $\psi(x, y, r)$ such that
  $$\varphi(x) \Rightarrow \Pr_r[\exists ! y < t \psi(x, y, r)] > 1/t(x) \text{ for some term } t,$$
  $$\neg \varphi(x) \Rightarrow \Pr_r[\exists y < t \psi(x, y, r)] = 0.$$  

- To get this, we need to know things like: given a propositional formula in $n$ variables, there is $k \leq n$ such that the formula has between $2^{k-2}$ and $2^k$ satisfying assignments. This seems to engage the full power of the approx. counting.

- The other inductive steps are more or less natural, but we must make sure that correctness of the translation can be verified in $\text{APC}_2^\oplus P$, particularly in the case of nested $\oplus$’s.
The collapse result: propositional consequences

Theorem

Any simple enough (say, DNF) formula which has a proof of size $s$ in the system with connectives $\land, \lor, \neg, \oplus$ and formulas of depth $\leq d$, has a proof of size at most $s^{\log c(d)}$ with formulas of depth $\leq 3$.

Proof sketch:

- $\text{BA}^{\oplus}(\alpha)$ proves a reflection principle for the depth $d$ system.
- So, $\text{APC}^{P}_2(\alpha)$ and hence $T^{3,\oplus P}_2(\alpha)$ proves it too.
- Proofs in $T^{3,\oplus P}_2(\alpha)$ translate into short proofs in the depth 3 system (Paris-Wilkie translation from arithm. to prop. logic).
- So, the depth 3 system proves reflection for the depth $d$ system.
- The simulation follows.
The research frontier, revisited

1. Can the theories $T_n^2(\alpha)$ be separated by a $\forall \hat{\Sigma}^b_1(\alpha)$ sentence?

2. An “interesting” independence result for $\text{BA}^\oplus(\alpha)$.
The research frontier, revisited

1. Can $\text{APC}_2(\alpha)$ be separated from $\text{BA}(\alpha)$ by a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence?

2. An “interesting” independence result for $\text{BA}^\oplus(\alpha)$. 

Some form of the relativized WPHP for functions involving $\oplus$. The best we can do is $T_1^{\oplus} \oplus P_2(\alpha) + \text{sWPHP}(\hat{\Delta}_2^b(\alpha))$. Maybe the model-theoretic properties of WPHP could help?
The research frontier, revisited

1. Can $\text{APC}_2(\alpha)$ be separated from $\text{BA}(\alpha)$ by a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence?

2. An “interesting” independence result for a theory containing some form of the relativized WPHP for functions involving $\oplus$. (The best we can do is $T_{2,\oplus}^1(\alpha) + s\text{WPHP}(\hat{\Delta}_2^b(\alpha))$.)

Maybe the model-theoretic properties of WPHP could help?
The research frontier, revisited

1. Can $\text{APC}_2(\alpha)$ be separated from $\text{BA}(\alpha)$ by a $\forall \hat{\Sigma}_1^b(\alpha)$ sentence?

2. An “interesting” independence result for a theory containing some form of the relativized WPHP for functions involving $\oplus$. (The best we can do is $T_{2,1}^{1,\oplus^P}(\alpha) + \text{sWPHP}(\hat{\Delta}_2^b(\alpha))$.)

Maybe the model-theoretic properties of WPHP could help?