Analogues of properness, and applications

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Forcing axioms

Developed in late 1960s early 1970s, initially to crystalize center points for applications of iterated forcing.

Martin's axiom (MA, for ω_1 antichains): for any c.c.c. poset $\mathbb{P}$ and any collection $\mathcal{A}$ of ω_1 maximal antichains of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every antichain in $\mathcal{A}$.

Obtained through an iteration of enough c.c.c. posets. Can then be used axiomatically as a starting point for consistency proofs that would otherwise require an iteration of c.c.c. posets.

Key points in proving consistency of MA:

(a) Finite support iteration of c.c.c. posets does not collapse ω_1, and in fact the iteration poset is itself c.c.c.

(b) Can "close off", that is reach a point where enough c.c.c. posets have been hit to ensure MA.

Proper forcing

There are classes of posets other than c.c.c. which also preserve ω_1.

Definition

Let $\mathbb{P}$ be a poset. Let κ be large enough that $\mathbb{P} \in H(\kappa)$. $p \in \mathbb{P}$ is a master condition for $M \prec H(\kappa)$ if

1. p forces that every maximal antichain A of $\mathbb{P}$ that belongs to M is met by the generic filter inside M.

Equivalently any of:

2. p forces that $\dot{G} \cap \check{M}$ is generic over M.
3. p forces that $M[\dot{G}] \prec H(\kappa)[\dot{G}]$ and $M[\dot{G}] \cap V = M$.

Definition

$\mathbb{P}$ is proper if for all large enough κ and all countable $M \prec H(\kappa)$, every condition in M extends to a master condition for M.

Proper posets do not collapse ω_1; immediate from (3).

PFA

Proper forcing axiom (PFA): the parallel of MA for proper posets. Again used axiomatically as a starting point for consistency proofs.

Key points in consistency proof of PFA:

- **(a)** Countable support iteration of proper posets does not collapse ω_1, and is indeed proper.
- **(b)** Can close off, assuming a supercompact cardinal.

For (b), fix a supercompact cardinal θ. Iterate up to θ hitting proper posets given by a Laver function. At stage θ, using properties of the Laver function and supercompactness, have covered enough posets to ensure PFA holds.

Obtained in late 1970s, Baumgartner, Shelah.

Consequences (some of many)

Compositions of $\mathrm{Col}(\omega_1, \delta)$ and c.c.c. posets are proper.

Gives: Tree property at ω_2; every tree of size and height ω_1 has at most ω_1 cofinal branches; any two ω_1 dense subsets of $\mathbb{R}$ are order isomorphic; $\square_\kappa$ fails for $\kappa \geq \omega_1$.

Posets using finite sequences of countable models as side conditions to enforce properness.

Gives: P-ideal dichotomy; Open Coloring Axiom; rainbow Ramsey principle on ω_1.

Mapping Reflection Principle (MRP).

Gives: failure of $\square_\kappa$ for $\kappa \geq \omega_1$; SCH; wellordering of $\mathbb{R}$ of ordertype ω_2 definable over $H(\omega_2)$ from parameter contained in ω_1.

Higher analogues?

In the case of MA, the forcing axiom has higher analogues, and in fact strengthenings.

For example it is consistent that for all c.c.c. posets, all maximal antichain in families of size ω_2 can be simultaneously met by a filter.

Initial expectation was that similar analogues should exist for PFA.

Naive attempt: demand existence of master conditions also for models of size ω_1.

Posets in the resulting class preserve ω_1 and ω_2 (certainly a necessary property for a higher analogue).

But preservation under iteration fails.

Search for higher analogues largely dormant.

Two-size nodes

For regular $\theta \geq \omega_2$ and $f \colon H(\theta)^{<\omega} \to H(\theta)$, let $\mathcal{C}(\theta, f)$ consist of M satisfying one of:

1. (Type ω_1.) $|M| = \omega_1$, $M \prec H(\theta)$, internal on a club, closed under f.
2. (Countable type elementary.) $|M| = \omega$, $M \prec H(\theta)$, closed under f.
3. (Countable type tower.) $|M| \leq \omega$, $M \neq \emptyset$, linearly ordered by $\in$, every $N \in M$ satisfies (1), $(\forall N \in M)(M \cap N \in N)$.

Called nodes. Non-tower nodes are elementary.

Easy to check $\mathbb{P}$ proper iff ($\exists$ large enough θ, f) ($\forall$ $\in$-increasing set s of countable elementary nodes) ($\forall Q \in s$) every $p \in \mathbb{P} \cap Q$ which is a master condition for all $M \in s \cap Q$ extends to a master condition for all $M \in s$.

Right-to-left direction immediate. Left-to-right by iterated applications of condition defining properness.

Two-size side conditions

A two-size side condition is a set of nodes, $\in$-increasing (each node belongs to its successor), and closed under intersections in the sense:

- If $N \in M$ of type ω_1 and countable elementary both in s, then $M \cap N$ in s.
- If $N \in M$ of type ω_1 and tower both in s, and $M \cap N \neq \emptyset$, then there is tower $\bar{M} \supseteq M \cap N$ occurring in s before N.

Ordered in the natural way, reverse inclusion as sets.

For elementary $Q \in s$, the residue of s in Q is $s \cap Q$. Denoted $\mathrm{res}_Q(s)$. Is itself a two-size side condition.

Lemma

If $Q \in s$ elementary and $t \in Q$ extends $\mathrm{res}_Q(s)$, then s and t are compatible.

Gives strong properness for poset of two-size side conditions. Poset preserves ω_1, ω_2, collapses $H(\theta)$ to ω_2.

Two-size properness

Recall $\mathbb{P}$ proper iff ($\exists$ large enough θ, and f) ($\forall$ $\in$-increasing set s of countable elementary nodes) ($\forall Q \in s$) every $p \in \mathbb{P} \cap Q$ which is a m.c. for all $M \in s \cap Q$ extends to a m.c. for all $M \in s$.

Two-size properness (1st approx.): ($\exists$ large enough θ, f) ($\forall$ two-size side condition s) ($\forall Q \in s$ elementary) every $p \in \mathbb{P} \cap Q$ which is a m.c. for all $M \in \mathrm{res}_Q(s)$ extends to a m.c. for all $M \in s$.

(By m.c. for tower M means m.c. for all $N \in M$.)

For added generality, replace "m.c. for M" with "$\in \mathrm{mc}(M)$", where $M \mapsto \mathrm{mc}(M)$ abstracts essential properties of the function $M \mapsto \{$master conditions for $M\}$.

Some essential properties: every $q \in \mathrm{mc}(M)$ is a m.c. for M; $\mathrm{mc}(M \cap N) \supseteq \mathrm{mc}(M) \cap \mathrm{mc}(N)$; $\mathrm{mc}(M)$ open in $\mathbb{P}$; $\mathrm{mc}(M) \subseteq \mathrm{mc}(M')$ for $M' \subseteq M$ both tower.

Posets satisfying this (for some mc) are two-size proper.

Two-size proper forcing axiom

Two-size proper posets admit master conditions for countable models and models of size ω_1. (But definition requires more.) Preserve ω_1 and ω_2.

Two-size proper forcing axiom: For every two-size proper $\mathbb{P}$, every collection $\mathcal{A}$ of ω_2 maximal antichains of of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every antichain in $\mathcal{A}$.

Theorem (N.) (2012 as stated, 2010 finer tower nodes)
Suppose θ is supercompact. Then there is a forcing extension satisfying the two-size proper forcing axiom.

Covers posets of two-size side conditions, in particular posets which collapse arbitrary $\delta \geq \omega_2$ to ω_2.

Covers c.c.c. posets.

Class is closed under compositions.

Similar to classes for initial uses of PFA.

Relaxing

Recall two-size properness: ($\exists$ large enough θ, f) ($\forall$ two-size side condition s) ($\forall Q \in s$ elementary) every $p \in Q \cap \bigcap_{M \in \mathrm{res}_Q(s)} \mathrm{mc}(M)$ extends to $q \in \bigcap_{M \in s} \mathrm{mc}(M)$.

Relax the extension condition by placing restrictions on the configuration of s and Q.

Only require condition to hold in following instances:

- $Q \in s$ and $\mathrm{res}_Q(s) = \emptyset$.
- $Q \in s$ countable, $p \in \mathrm{mc}(U) \cap Q$ for some tower $U \in Q$ which subsumes $\mathrm{res}_Q(s)$.

By U subsumes r mean every $M \in r$ is either contained in U or belongs to U. For tower U, in particular implies r has only tower and type ω_1 nodes.

Resulting class is relaxed two-size proper.

Theorem (N.)(2013)

Suppose θ is supercompact. There is a forcing extension satisfying the relaxed two-size proper forcing axiom.

Some words on the proof

Lifts new method for PFA consistency using finite support.

Method relies on two-type side conditions (ctbl elem.; transitive) to preserve properness. N., building on Mitchell-Friedman posets for adding clubs in ω_2 with finite conditions.

To generalize need three-type side conditions, preserve ω_1, ω_2, supercompact θ (which becomes ω_3).

Requires introduction of non-elementary nodes, which give rise to tower node in two-size properness.

Initial version with fine, very technical, notion of non-elementary nodes 2010.

Around the same time, independently, Aspero-Mota used finite side condition with ctbl models to show weakenings of PFA for ω_2-c.c. posets consistent with large continuum.

Aspero-Mota class subsumed in ω_2-c.c. relaxed two-size proper.

Square at ω_1

Lemma (independently Krueger, N.)

There is a finite conditions poset, strongly proper for countable and size ω_1 nodes, forcing $\square_{\omega_1}$.

Earlier work on forcing $\square_{\omega_1}$ with finite conditions by Dolinar-Dzamonja, but clubs for the square sequence added with ctbl fragments. Not strongly proper.

Poset in lemma not relaxed two-size proper; extension condition fails. Variant (N.) for $\square_{\omega_1,\mathrm{fin}}$ is.

Corollary

The relaxed two-size proper forcing axiom implies $\square_{\omega_1,\mathrm{fin}}$.

Not necessarily a good thing; may create too much structure on ω_2. (Non-relaxed) two-size proper forcing axiom does not imply $\square_{\omega_1,\omega}$. Suggests some applications may require restricting forcing class—seems to weaken axiom, but may give extra preservation on ω_2.

Analogue of MRP

Fix X. $\Sigma \subseteq \mathcal{P}(X)$ is open if for every $A \in \Sigma$ there is finite $a \subseteq \Sigma$ so that $a \subseteq B \subseteq A \to B \in \Sigma$.

$\Sigma \subseteq \mathcal{P}(X)$ is N-stationary on size κ if $\forall f\colon X^{<\omega} \cup \kappa \to X$ in N, there is $A \in N \cap \Sigma$ closed under f and containing $f''\kappa$.

Map Σ into $\mathcal{P}(X)$ is open, κ-stationary if for every $N \in \mathrm{dom}(\Sigma)$, $\Sigma(N)$ open, N-stationary on size κ.

Work with sequences $\langle M_\xi \mid \xi < \kappa^+\rangle$, $\in$-linear, continuous, M_ξ of size κ, $\kappa \subseteq M_\xi$.

$\alpha < \kappa^+$ is a Σ reflection point if ($\forall$ large enough $\xi < \alpha$ of cofinality κ) $M_\xi \cap X \in \Sigma(M_\alpha)$.

Mapping Reflection Principle (Moore): for ω-stationary open map Σ on club of ctbl $N \prec H(\theta)$, exists $\langle M_\xi \mid \xi < \omega_1\rangle$ with club of Σ reflection points.

Follows from PFA. Foundationally important consequences: $\neg\square_\lambda$ for $\lambda \geq \omega_1$; wo of $\mathbb{R}$ of ordertype ω_2, definable over $H(\omega_2)$ from parameter $\subseteq \omega_1$; SCH.

Analogue of MRP (cont.)

For ctbl P, $\mathrm{fatten}(P) = P \cup \bigcup\{Z \in P \mid |Z| = \omega_1\}$.

$\Sigma \subseteq \mathcal{P}(X)$ is N-amenable, for N of size ω_1 internal on club, if for club of ctbl $P \subseteq N$, $\Sigma(N) \cap \mathrm{fatten}(P) \in N$.

Map Σ is amenable of if $\forall N \in \mathrm{dom}(\Sigma)$, Σ is N-amenable.

Let $\mathcal{P}_{\text{ic-}\omega_1}(H) = \{N \subseteq H \mid |N| = \omega_1, N \text{ internal on club}\}$.

Lemma (N.)

Let Σ be amenable open ω_1-stationary map, with $\mathrm{dom}(\Sigma)$ containing a club relative to $\mathcal{P}_{\text{ic-}\omega_1}(H(\theta))$. Then there is a relaxed two-size forcing adding $\langle M_\xi \mid \xi < \omega_2\rangle$ with stationary set of Σ reflection points.

Corollary

Consistent that for every amenable open ω_1-stationary map Σ with domain containing a club relative to $\mathcal{P}_{\text{ic-}\omega_1}(H(\theta))$, exists $\langle M_\xi \mid \xi < \omega_2\rangle$ with stationary set of Σ reflection points.

Analogue of MRP (cont.)

Enough to imply failure of $\square_\lambda$ at $\lambda \geq \omega_2$, through direct analogue of the MRP antithreading argument.

Not enough for analogue of coding of reals, in MRP argument for wellordering of $\mathbb{R}$.

In MRP argument, given real coded by $\sup(\mathrm{Ord} \cap \bigcup M_\xi)$. Here there is also a dependence on a stationary $S \subseteq \omega_2$.

If $\langle M_\xi \mid \xi < \omega_2 \rangle$ actually generic (not pseudo generic), outside S behavior is generic and does not code any real. So x uniquely determined from $\sup(\mathrm{Ord} \cap \bigcup M_\xi)$.

Possible that by restricting forcing class, can preserve "non-coding" through an iteration.

Would allow strengthening theorem to add this property to S.

Would then get wellordering of $\mathbb{R}$ of ordertype ω_3 definable over $H(\omega_3)$ from parameter $\subseteq \omega_2$.