Trees, maximality principles, and generic absoluteness

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Definition

A statement φ is generically absolute if its truth is unchanged by forcing:

$$V \models \varphi \iff V[g] \models \varphi,$$

for every generic extension $V[g]$.

Example

For a tree T on Ord ($T \subset \mathrm{Ord}^{<\omega}$) the statement "$T$ is well-founded" is generically absolute.

We will see that this example is typical.

Notation

Given a tree T on $\omega \times \mathrm{Ord}$ ($T \subset \omega^{<\omega} \times \mathrm{Ord}^{<\omega}$) we continuously associate to each real $x \in \omega^\omega$ the tree T_x on Ord:

$$T_x = \{s \in \mathrm{Ord}^{<\omega} : (x \restriction |s|, s) \in T\}.$$

We define the *projection*

$$p[T] = \{x \in \omega^\omega : T_x \text{ is ill-founded}\}.$$

The most generic absoluteness one can prove in ZFC is:

Theorem (Shoenfield)

For every Σ^1_2 formula $\varphi(v)$ then there is a tree T on $\omega \times \mathrm{Ord}$ such that for every generic extension $V[g]$ and every real $x \in V[g]$,

$$V[g] \models \varphi[x] \iff T_x \text{ is ill-founded.}$$

Therefore $\boldsymbol{\Sigma}^1_2$ statements are generically absolute.

Next we consider stronger generic absoluteness hypotheses and their relationship to other hypotheses beyond ZFC.

For a pointclass (take $\underset{\sim}{\mathbf{\Sigma}}^1_3$ for example) we consider two kinds of generic absoluteness.

Definition

- One-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ says for every Σ^1_3 formula $\varphi(v)$, every real x, and every generic extension $V[g]$,
$$V \models \varphi[x] \iff V[g] \models \varphi[x].$$
- Two-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ says that one-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ holds in every generic extension.

Remark

Upward absoluteness (“ $\Longrightarrow$ ”) is automatic by Shoenfield.

Generic absoluteness is related to universally Baire sets.

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. One-step $\underset{\sim}{\Sigma}^1_3$ generic absoluteness holds.
2. Every $\underset{\sim}{\Delta}^1_2$ set of reals is universally Baire.

Here we say a set of reals A is

- λ-universally Baire[1] if $A = p[T]$ for some pair of trees $(T, \tilde{T})$ that is λ-*absolutely complementing*: $p[T] = \omega^\omega \setminus p[\tilde{T}]$ in every $<\lambda$-generic extension.[2]
- universally Baire if it is λ-universally Baire for all λ.

[1] Called $<\lambda$-universally Baire in the original notation.

[2] Meaning a generic extension by a poset of size less than λ.

Generic absoluteness is related to large cardinals.

Theorem (Feng–Magidor–Woodin)

The following statements are equiconsistent.

1. There is a Σ_2-reflecting cardinal.
2. One-step $\underset{\sim}{\mathbf{\Sigma}}^1_3$ generic absoluteness holds.

A cardinal δ is Σ_2-reflecting if

- δ is inaccessible, and
- $V_\delta \prec_{\Sigma_2} V$.

This property is between "inaccessible" and "Mahlo" in consistency strength.

Two-step generic absoluteness is related to stronger large cardinals.

Theorem (Martin–Solovay $\Rightarrow$, Woodin $\Leftarrow$)

The following statements are equivalent:

1. Every set has a sharp.
2. Two-step $\boldsymbol{\Sigma}^1_3$ generic absoluteness holds.

Remark

The reverse direction uses Jensen's covering lemma.

- If $0^\sharp$ does not exist then $\lambda^{+L} = \lambda^+$ where $\lambda = \aleph_\omega$.
- In $V^{\mathrm{Col}(\omega,\lambda)}$ take a real x coding λ.
- "$\omega_1^{L[x]} = \omega_1$" is $\Pi^1_3(x)$ but not generically absolute.

Two-step $\boldsymbol{\Sigma}^1_3$ generic absoluteness, if it holds, must come from trees for Π^1_2 formulas via absoluteness of well-foundedness.

Theorem (Feng–Magidor–Woodin)

The following statements are equivalent:

1. Two-step $\boldsymbol{\Sigma}^1_3$ generic absoluteness holds.
2. For every Π^1_2 formula $\varphi(\vec{v})$ there is a tree $\tilde{T}$ such that for every generic extension $V[g]$ and real $\vec{x} \in V[g]$,

$$V[g] \models \varphi[\vec{x}] \iff T_{\vec{x}} \text{ is ill-founded.}$$

(2) $\implies$ (1) uses absoluteness of well-foundedness and

$$\exists y \in \omega^\omega \, (T_{\vec{x},y} \text{ is ill-founded}) \iff T_{\vec{x}} \text{ is ill-founded}$$

Before going on to higher pointclasses, we consider the following principle.

Definition (Hamkins)

The *boldface maximality principle* $\underset{\sim}{\mathrm{MP}}$ says that for every formula $\varphi(v)$ and real x, if there is a generic extension such that $\varphi[x]$ holds in every further generic extension, then $\varphi[x]$ holds in V.

Remark

We cannot allow uncountable parameters because $\varphi(v)$ could say "v is countable."

Definition

The *necessary boldface maximality principle* $\Box\underset{\sim}{\mathrm{MP}}$ says that $\underset{\sim}{\mathrm{MP}}$ holds in every generic extension.

Theorem (Hamkins)

$\underset{\sim}{\mathrm{MP}}$ is equiconsistent with a (Σ_ω-)reflecting cardinal.

Remark

$\underset{\sim}{\mathrm{MP}}$ implies one-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$:
Given a Σ^1_3 formula $\varphi(v)$, a real x, and a generic extension satisfying $\varphi[x]$,

- every further generic extension satisfies $\varphi[x]$ by Shoenfield absoluteness, so
- V satisfies $\varphi[x]$ by $\underset{\sim}{\mathrm{MP}}$.

Remark

Therefore $\Box\underset{\sim}{\mathrm{MP}}$ implies two-step generic absoluteness for $\underset{\sim}{\mathbf{\Sigma}}^1_3$.
Unfortunately, $\Box\underset{\sim}{\mathrm{MP}}$ is not known to be consistent.

Next we consider an similar situation “higher up” in terms of large cardinals and descriptive set theory.

- Let λ be a limit of Woodin cardinals.
- Let uB_λ be the pointclass of λ-universally Baire sets.

Analogy:

$$\Sigma^1_2 \rightsquigarrow (\Sigma^2_1)^{\mathrm{uB}_\lambda}$$
$$\Pi^1_2 \rightsquigarrow (\Pi^2_1)^{\mathrm{uB}_\lambda}$$
$$\Sigma^1_3 \rightsquigarrow \exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$$

Definition

A formula $\varphi(\vec{v})$ is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ if, for some formula $\theta(\vec{v})$, it has the form

$$\exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \theta(\vec{v}).$$

Example

The formula $\varphi(v)$ saying "the real v is in a mouse with a uB_λ iteration strategy" is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$.

The following theorem is analogous to Shoenfield absoluteness.

Theorem (Woodin)

If λ is a limit of Woodin cardinals and $\varphi(\vec{v})$ is a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula, then there is a tree T such that for every $<\lambda$-generic extension $V[g]$ and every real $\vec{x} \in V[g]$,

$$V[g] \models \varphi[\vec{x}] \iff T_{\vec{x}} \text{ is ill-founded.}$$

Therefore $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ statements are generically absolute below λ.

Definition

A formula $\varphi(\vec{v})$ is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ if, for some formula $\theta(u, \vec{v})$, it has the form

$$\exists u \in \omega^\omega \, \forall B \in \mathrm{uB}_\lambda \, (\mathrm{HC}; \in, B) \models \theta(u, \vec{v}).$$

Example

The sentence φ saying "there is a real that is not in any mouse with a uB_λ iteration strategy" is $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$.

Remark

Generic absoluteness for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ is interesting because it is not known to follow from any large cardinal hypothesis.

Any large cardinal hypothesis that implied $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness could not have a conventional inner model theory.

- A canonical inner model with a limit of Woodins λ should satisfy the $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ sentence φ:
  "Every real is in a mouse with a uB_λ iteration strategy."
- This sentence becomes false after adding a Cohen real.

So how do we get generic absoluteness at this level?
As for $\underset{\sim}{\Sigma}^1_3$, maximality principles provide an easy way.

Remark

If λ is a limit of Woodin cardinals and $V_\lambda \models \utilde{\mathrm{MP}}$, then one-step $\exists^{\mathbb{R}}(\utilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ:
For a $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula $\varphi(v)$, real x, and $<\lambda$-generic extension satisfying $\varphi[x]$,

- every further $<\lambda$-generic extension satisfies $\varphi[x]$ by $(\utilde{\boldsymbol{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness, so
- V satisfies $\varphi[x]$ by $\utilde{\mathrm{MP}}$ in V_λ, using $\mathrm{uB}_\lambda = (\mathrm{uB})^{V_\lambda}$.

Remark

Similarly if $V_\lambda \models \Box\utilde{\mathrm{MP}}$ we get two-step $\exists^{\mathbb{R}}(\utilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness, but $\Box\utilde{\mathrm{MP}}$ is not known to be consistent.

Along the lines of Feng–Magidor–Woodin for $\underset{\sim}{\mathbf{\Sigma}}^1_3$ we may ask:
How is generic absoluteness for $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ related to

- the extent (or closure properties) of the pointclass of (λ-)universally Baire sets?
- large cardinals?
- the absoluteness of well-foundedness for trees?

Generic absoluteness for $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ is related to closure properties of the uB_λ sets.

Proposition

For a limit λ of Woodin cardinals, the following statements are equivalent:

1. One-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ.
2. Every $(\undertilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

Proof idea

(1) $\implies$ (2): Similar to Feng–Magidor–Woodin.
(2) $\implies$ (1): If a $\forall^{\mathbb{R}}(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ statement holds in V then we can pick witnesses in a $(\undertilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ way. If we can pick witnesses in a uB_λ way, this fact is absolute to $<\lambda$-generic extensions.

Generic absoluteness for $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ is related to large cardinals.

Proposition

Con(1) $\implies$ Con(2), where

1. There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is Σ_2-reflecting in V_λ.
2. There is a limit λ of Woodin cardinals such that one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness holds below λ.

In consistency strength, (1) is between an inaccessible limit of Woodin cardinals and a Mahlo limit of Woodin cardinals.

Question 1

Con(2) $\implies$ Con(1)?

If two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds, must it come from trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas via absoluteness of well-foundedness? More precisely,

Question 2

For a limit of Woodin cardinals λ, are the following statements equivalent?

1. Two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ.
2. For every $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula $\varphi(v)$ there is a tree $\tilde{T}$ such that for every $<\lambda$-generic extension $V[g]$ and every real $x \in V[g]$ we have $V[g] \models \varphi[x] \iff T_x$ is ill-founded.

Again (2) $\implies$ (1) is by absoluteness of well-foundedness.

What is the consistency strength of two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness?
If the answer to Question 2 is "yes" then the following statements are equiconsistent:

1. There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is $<\lambda$-strong
2. There is a limit λ of Woodin cardinals such that two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness holds below λ.

Because by results of Woodin, (1) is equiconsistent with

- There is a limit λ of Woodin cardinals such that:
  For every $(\Pi^2_1)^{\mathsf{uB}_\lambda}$ formula $\varphi(v)$ there is a tree $\tilde{T}$ such that for every $<\lambda$-generic extension $V[g]$ and every real $x \in V[g]$ we have $V[g] \models \varphi[x] \iff T_x$ is ill-founded.

We will give a partial "yes" answer to Question 2.

Remark

- To get trees for Π^1_2 formulas from two-step $\underset{\sim}{\mathbf{\Sigma}}^1_3$ generic absoluteness, one uses Jensen's covering lemma to get sharps (by Woodin's argument) and then the Martin–Solovay construction to get the trees.
- To get trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas from two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ, we need a higher covering lemma.
- Our "covering lemma" will bypass the inner model theory step and directly construct the trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$.

Lemma

Let λ be a measurable cardinal with a normal measure μ. Let T be a tree on $\omega \times \text{Ord}$. Assume that for μ-almost every $\alpha < \lambda$ we have

$$|\mathcal{P}(V_\alpha) \cap L(T, V_\alpha)| = \alpha. \tag{*}$$

Then in some $<\lambda$-generic extension, T has an λ-absolute complement $\tilde{T}$.

Remark

In our application, T will be a tree for a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula and the "failure of covering" (*) will come from $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness in $V^{\mathrm{Col}(\omega,\alpha)}$ applied to the statement "$L[T, x] \cap \mathbb{R}$ is countable" for a real x coding V_α.

A partial answer to Question 2:

Theorem

Let λ be a measurable cardinal that is a limit of Woodin cardinals. Assume two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ. Then in some $<\lambda$-generic extension we have:
For every $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula $\varphi(v)$ there is a tree $\tilde{T}$ such that for every further $<\lambda$-generic extension $V[g]$ and every real $x \in V[g]$ we have $V[g] \models \varphi[x] \iff \tilde{T}_x$ is ill-founded.

Question 2a

Can we prove the theorem without assuming that our limit λ of Woodin cardinals is measurable? If so, then the following statements are equiconsistent.

- There is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is $<\lambda$-strong
- There is a limit λ of Woodin cardinals such that two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathsf{uB}_\lambda}$ generic absoluteness holds below λ.

Question 2b

Can we get the trees $\tilde{T}$ in V? If so, then the trees would fully explain the generic absoluteness hypothesis.