New examples of small Polish groups

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Abstract

We answer some questions from [4] by giving suitable examples of small Polish groups. First, we present a class of small Polish group structures without generic elements. Next, we construct a first example of a small non-zero-dimensional Polish G-group.

Introduction

In [4], Krupiński defined and investigated Polish structures by methods motivated by model theory.

Definition 0.1. A Polish structure is a pair \((X,G)\), where \(G\) is a Polish group acting faithfully on a set \(X\) so that the stabilizers of all singletons are closed subgroups of \(G\). We say that \((X,G)\) is small if for every \(x\in X\), there are only countably many orbits under the action of \(G\) (we call the orbit of \(x\) the orbit of \((X,G)\) over \(x\)).

A particularly interesting situation is when the underlying set \(X\) is a group itself. Throughout this paper, we follow the terminology from [4].

Definition 0.2. Let \(G\) be a Polish group:

(i) A Polish group structure is a Polish structure \((X,G)\) such that \(X\) is a group and \(G\) acts as a group of automorphisms of \(X\).

(ii) A topological Polish structure is a Polish structure \((X,G)\) such that \(X\) is a topological group and the action of \(G\) on \(X\) is continuous.

(iii) A Polish [compact] group is a topological Polish group \((X,G)\), where \(X\) is a Polish [compact] group.

Let \((X,G)\) be a Polish structure. For any finite \(C\subseteq X\), \(G\) denotes the pointwise stabilizer of \(C\) in \(G\), and for a finite tuple \(x\) of elements of \(X\), \(b\in(G\cup\{1\})\) we denote the orbit of \(b\) under the action of \(G\) (we call it the orbit of \((X,G)\) over \(b\)).

A fundamental concept for [4] is the notion of \(\sigma\)-independence in an arbitrary Polish structure.

Definition 0.3. Let \(X\) be a finite tuple and \(A\) an infinite subset of \(X\). For any \(a\in(A\setminus A)\), \(G\) is defined by \((X,G)\) to be \(\sigma\)-independent if \(\forall x\in(A\setminus A)\) \(G\) is non-meager in \(\langle\langle\langle X,G\rangle\rangle\rangle\).

It is a generalization of \(\sigma\)-independence, which was introduced by Weiss for profinite structures. Under the assumptions of our paper, \(\sigma\)-independence has similar properties to those of forking independence in standard model theory, and hence it allows to transfer some ideas and techniques from stability theory to small Polish structures (which are purely topological objects). The investigation of Polish structures has been undertaken in [5] and [1]. For example, in [5], some structural theorems about compact Polish groups were proved, and in [1], dendrites were considered as Polish structures, and some properties introduced in [4] were examined for them.

The class of Polish structures contains many more interesting examples from classical mathematics than the class of profinite structures. For example, for any compact metric space \(P\), we can consider the group Hom\(\langle P,\mathbb{P}\rangle\) of all homeomorphisms of \(P\) equipped with the compact-open topology, then \((P,\text{Hom}\langle P,\mathbb{P}\rangle)\) is a Polish structure (examples of small Polish structures of this form were investigated in [4] and in [5]). Also, if \(X\) is a compact metrizable group, then \((H,\text{Aut}(H))\) is a Polish group structure. However, in the class of small Polish structures, it is more difficult to construct interesting examples in this way. In the present paper, we answer some questions from [4] by constructing suitable examples of small Polish group structures.

The following is [4, Question 5.1] (see Definition 1.3 for the notion of \(\sigma\)-independent group).

Question 0.4. Let \((G,H)\) be a Polish group structure. Does \(H\) possess an \(\sigma\)-generic orbit?

Proposition 5.5 from [4] gives us a positive answer to Question 0.4 in the full generality, i.e. in the class of all Polish group structures. Recall that Fact 1.4 tells us that the answer is positive for small Polish G-groups.

Corollary 2.2. Let \((G,H)\) be a small Polish structure, where \(H\) is \(\sigma\)-independent. Then, \((G,H)\) is a Polish group structure, and it has no generic orbit (neither left nor right).

Proof. Take any \(h\in H\) and a finite subset \(A\subseteq H\). We will show that \(h\) is not a generic orbit. Take any \(h\in H\) and \((X,G)\) such that \((X,G)\) be a Polish structure, and for any finite \(A\subseteq H\), we see that \((X,G)\) by Proposition 2.2. Hence, \((X,G)\) is a Polish group structure, and it has no generic orbit (neither left nor right).

The above corollary and Fact 1.4, we get in particular:

Corollary 2.4. If \((G,H)\) is an uncountable small Polish group structure, and \(H\) is a \(\sigma\)-independent countable group, then there is no Polish topology on \(H\) such that the action of \(G\) on \(H\) is continuous.

In this section, we construct a first example of a small non-zero-dimensional Polish G-group.

3. A non-zero-dimensional small Polish G-group

In this section, we construct a first example of a small non-zero-dimensional Polish G-group.

We define a structure of a group on the complete Erdös space as in [3, Proposition 4.3] (with the only difference that we do not choose a particular \(q_0\), which is done as follows. Fix any \(p\in\mathbb{N}\). Let \(S\subseteq B_\mathbb{R}\) be the ternary Cantor set, and \(X=C^{\mathbb{N}}\langle B_\mathbb{R}\rangle\). By results of [2] and [3], \(X\) is a Polish group with the topology induced from \(\mathbb{R}\) is homeomorphic to the complete Erdös space. Then, \((X,G)\) is a Polish group structure, and it has no generic orbit (neither left nor right).

We define a metric \(d(x,x')\) on \(X\) as \(d(x,y) = 2^{-n}\), where \(n\) is the characteristic of \((A,B)\) and the action of \(G\).

In this section, we construct a first example of a small non-zero-dimensional Polish G-group (in this paper we skip the proof).

Proposition 3.1. \((G,H)\) is a Polish group structure, and \((G,H)\) is a small Polish group structure.

One can check that \((X,G)\) by the previous corollary, and the latter \((X,G)\) by Proposition 4.3. Thus, for every finite \(A\subseteq H\), we have that \((X,G)\) is a Polish group structure, and it has no generic orbit (neither left nor right).

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In this section, we construct a first example of a small non-zero-dimensional Polish G-group (in this paper we skip the proof).

References