Basis problem for analytic multiple gaps

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Hejnice 2014

A. Avilés, S. Todorcevic, *Finite basis for analytic strong n-gaps*, Combinatorica 33(4) 2013, 375-393

Let $\{x_n : n < \omega\}$ be a sequence of objects.
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We can think that it is a sequence of points in a topological space...
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We are going to look at different classes of subsequences of this sequence.
Example 1

Suppose \( \{x_n : n < \omega\} \) is a dense subset of \( \mathbb{R} \).

1. The class \( \Gamma_{\mathbb{Q}} \) are the subsequences converging to a rational.
2. The class \( \Gamma^+ \) are the subsequences converging to a positive irrational.
3. The class \( \Gamma^- \) are the subsequences converging to a negative irrational.

These classes are hereditary, and pairwise disjoint.

The classes \( \Gamma_{\mathbb{Q}} \) and \( \Gamma^+ \) cannot be separated.
Suppose \( \{x_n : n < \omega\} \) is a dense subset of \( \mathbb{R} \).
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These classes are **hereditary**, and **pairwise disjoint**.
Suppose $\{x_n : n < \omega\}$ is a dense subset of $\mathbb{R}$.

1. The class $\Gamma_Q$ are the subsequences converging to a rational.
2. The class $\Gamma^+$ are the subsequences converging to a positive irrational.
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- These classes are hereditary, and pairwise disjoint.
- The classes $\Gamma^+$ and $\Gamma^-$ can be separated through $\{x_n : x_n \geq 0\} \cup \{x_n : x_n < 0\}$. 
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- The classes \( \Gamma_Q \) and \( \Gamma^+ \) cannot be separated.
Example 2

\[
\{ x_n : n < \omega \}
\]

is a sequence of vectors in a Banach space. For every \( 1 \leq p < \infty \), the class \( \Gamma_p \) are the subsequences for which norms of linear combinations are computed as:

\[
\| \sum a_i x_i \| = \left( \sum |a_i|^p \right)^{1/p}
\]

These classes are hereditary, and pairwise disjoint.
Now \( \{x_n : n < \omega \} \) is a sequence of vectors in a Banach space.
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These classes are hereditary, and pairwise disjoint.
Fix a countable set $\mathbb{N}$

**Definition**

An $n$-gap

<table>
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The families $\Gamma_0, \ldots, \Gamma_{n-1}$ are separated if there exists a decomposition $\mathbb{N} = \bigcup_{i < n} N_i$ such that $\Gamma_i \cap P(N_i) = \emptyset$.

Here, disjoint is equivalent to orthogonal: $A \cap B$ is finite whenever $A \in \Gamma_i$, $B \in \Gamma_j$ for $i \neq j$.

The families $\Gamma_i$ live in $P(\mathbb{N}) = 2^\mathbb{N}$, so they might be Borel, analytic, coanalytic, projective, etc.
Fix a countable set $N$

**Definition**

An $n$-gap is a tuple of hereditary families of infinite subsets of $N$.
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An *$n$-gap* is a tuple of hereditary families of infinite subsets of $N$

$$\Gamma = \{\Gamma_0, \ldots, \Gamma_{n-1}\}$$
Gaps and separation

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An *$n$-gap* is a tuple of hereditary families of infinite subsets of $N$

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which are pairwise disjoint and *not separated*. 
Fix a countable set \( N \)

**Definition**

An *\( n \)-gap* is a tuple of hereditary families of infinite subsets of \( N \)

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\Gamma = \{ \Gamma_0, \ldots, \Gamma_{n-1} \}
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which are pairwise disjoint and not separated.

1. The families \( \Gamma_0, \ldots, \Gamma_{n-1} \) are _separated_.

Gaps and separation

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Strong gaps and countable separation

Definition

A strong \( n \)-gap is an \( n \)-gap

\[ \Gamma = \{ \Gamma_0, \ldots, \Gamma_{n-1} \} \]

which is not countably separated.
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$$N = \bigcup_{i<n} N_i^p$$

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$$\forall a_0 \in \Gamma_0, \ldots, a_{n-1} \in \Gamma_{n-1}.$$
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such that

$$\forall a_0 \in \Gamma_0, \ldots, a_{n-1} \in \Gamma_{n-1} \exists p \ |a_i \cap N_i^p| < \aleph_0.$$
An example of a 3-gap

Consider \( N \) the set of successor ordinals below \( \omega^3 \)

\[
\omega \quad \omega \cdot 2 \quad \omega \cdot 3 \quad \omega^2 \quad \omega^2 \cdot 2 \quad \omega^3
\]

\( \in \Gamma_0 \)
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Consider $N$ the set of successor ordinals below $\omega^3$

$$\omega \quad \omega \cdot 2 \quad \omega \cdot 3 \quad \omega^2 \quad \in \Gamma_0 \quad \omega^2 \cdot 2 \quad \omega^3$$

$$\Gamma_0 = \{ A \subset N : \overline{A} \subset \{ \omega^2 \cdot n + \omega \cdot m : n < \omega \} \}$$
An example of a 3-gap

Consider \( N \) the set of successor ordinals below \( \omega^3 \)

\[ \in \Gamma_1 \]

\[ \omega \hspace{1cm} \omega \cdot 2 \hspace{1cm} \omega \cdot 3 \hspace{1cm} \omega^2 \hspace{1cm} \omega^2 \cdot 2 \hspace{1cm} \omega^3 \]

1. \( \Gamma_0 = \{ A \subset N : \overline{A} \subset \{ \omega^2 \cdot n + \omega \cdot m : n < \omega \} \} \)
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Consider $N$ the set of successor ordinals below $\omega^3$

$\in \Gamma_2$

1. $\Gamma_0 = \{ A \subset N : \overline{A} \subset \{ \omega^2 \cdot n + \omega \cdot m : n < \omega \} \}$
2. $\Gamma_1 = \{ A \subset N : \overline{A} \subset \{ \omega^2 \cdot n : n < \omega \} \}$
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\[ \in \Gamma_0 \]

\[
\ldots \omega \ldots \omega \cdot 2 \ldots \omega \cdot 3 \ldots \omega^2 \ldots \omega^2 \cdot 2 \ldots \omega^3 \]

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This is a Borel 3-gap which is not strong.
An example of a 3-gap

Consider $N$ the set of successor ordinals below $\omega^3$

\[ M = \omega \cdot 2 \cdot 3 \cdot \omega^2 \in \Gamma_0 \]

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We can isolate $\Gamma_0$ and $\Gamma_1$ from $\Gamma_2$ by restricting to $M = N \cap \omega^2$. 
Consider $N$ the set of successor ordinals below $\omega^3$

\[ M \]

$ \omega \quad \omega \cdot 2 \quad \omega \cdot 3 \quad \omega^2 \quad \omega^2 \cdot 2 \quad \omega^3$

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We can isolate $\Gamma_0$ and $\Gamma_1$ from $\Gamma_2$ by restricting to $M = N \cap \omega^2$.

Meaning that $\{ \Gamma_0|_M, \Gamma_1|_M \}$ form a 2-gap, but $\Gamma_2|_M = \emptyset$. 
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Meaning that $\{\Gamma_0|_M, \Gamma_1|_M\}$ form a 2-gap, but $\Gamma_2|_M = \emptyset$.

Can we always isolate a part of a gap from the rest?
An example of a 3-gap

Consider $N$ the set of successor ordinals below $\omega^3$.

$$
\begin{array}{c}
\omega & \omega \cdot 2 & \omega \cdot 3 & \omega^2 & \omega^2 \cdot 2 & \omega^3 \\
\end{array}
$$

\[ M \in \Gamma_0 \]

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This is a Borel 3-gap which is not strong.

We can isolate $\Gamma_0$ and $\Gamma_1$ from $\Gamma_2$ by restricting to $M = N \cap \omega^2$. Meaning that $\{ \Gamma_0|_M, \Gamma_1|_M \}$ form a 2-gap, but $\Gamma_2|_M = \emptyset$.

Can we always isolate a part of a gap from the rest? No…
A very exotic example

For each $x \in \mathbb{R}$, fix a sequence of rationals which converges to $x$, $S_x \to x$.
A very exotic example

For each $x \in \mathbb{R}$, fix $S_x \rightarrow x$

Example
If the $\mathbb{Z}_i$ are pairwise disjoint Bernstein sets, then $\Gamma = \{ \Gamma_{\mathbb{Z}_1}, \ldots, \Gamma_{\mathbb{Z}_{n-1}} \}$ is an $n$-gap in which nothing can be isolated.

Formally, if $\{ \Gamma_{\mathbb{Z}_i} | M, \Gamma_{\mathbb{Z}_j} | M \}$ is a 2-gap, then $\Gamma | M$ is an $n$-gap.

Can we always isolate a part of a Borel gap from the rest? Some parts, but not all...
A very exotic example

For each $x \in \mathbb{R}$, fix $S_x \rightarrow x$

Given $Z \subset \mathbb{R}$, let $\Gamma_Z = \{ A \subset \mathbb{Q} : \exists x \in Z : A \subset S_x \}$
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If the $Z_i$ are pairwise disjoint Bernstein sets, then

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is an $n$-gap in which nothing can be isolated.

Formally, if $\{ \Gamma_{Z_i}|_M, \Gamma_{Z_j}|_M \}$ is a 2-gap, then $\Gamma|_M$ is an $n$-gap.
For each $x \in \mathbb{R}$, fix $S_x \rightarrow x$

Given $Z \subset \mathbb{R}$, let $\Gamma_Z = \{ A \subset \mathbb{Q} : \exists x \in Z : A \subset S_x \}$

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**Can we always isolate a part of a Borel gap from the rest?**
For each $x \in \mathbb{R}$, fix $S_x \rightarrow x$

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**Example**

If the $Z_i$ are pairwise disjoint Bernstein sets, then

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Can we always isolate a part of a Borel gap from the rest? Some parts, but not all...
**Theorem**

If $\Gamma_0, \ldots, \Gamma_{n-1}$ is an analytic $n$-gap, then $\exists M \subset N$ and $i < j < n$:

- $\Gamma_i|_M, \Gamma_j|_M$ form a 2-gap.
- $\Gamma_k|_M = \emptyset$ for all other $k$. 

If $\Gamma_0, \ldots, \Gamma_{n-1}$ is an analytic $n$-gap, then $\exists M \subset N$ and $i < j < n$:

- $\Gamma_i|_M, \Gamma_j|_M$ form a 2-gap.
- $\Gamma_k|_M = \emptyset$ for all but at most 58 many of the remaining $k$.

If $\Gamma_0, \ldots, \Gamma_{n-1}$ is a strong analytic $n$-gap, then $\exists M \subset N$:

- $\Gamma_0|_M, \Gamma_1|_M, \Gamma_2|_M$ form a strong 3-gap.
- $\Gamma_k|_M = \emptyset$ for all but at most 6 many of the remaining $k$. 

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If $\Gamma_0, \ldots, \Gamma_{n-1}$ is a strong analytic $n$-gap, then $\exists M \subset N$:
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If \( \Gamma_0, \ldots, \Gamma_{n-1} \) is an analytic \( n \)-gap, then \( \exists M \subset N : \)
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$f(3) = 58$, $f(k) \sim \frac{3.9^k}{8\sqrt{2\pi}k}$

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$f(3) = 6$, $f(k) = k^2 - k$
Part II
The first-move structure of the $n$-adic tree and strong gaps
The $n$-adic tree is the set $n^{<\omega}$ of finite sequences of $0, 1, \ldots, n-1$.
The $n$-adic tree

The $n$-adic tree is the set $n^{<\omega}$ of finite sequences of $0, 1, \ldots, n-1$. 

The 2-adic tree
The $n$-adic tree

The $n$-adic tree is the set $n^{<\omega}$ of finite sequences of $0, 1, \ldots, n-1$.
The first-move structure of $n^{<\omega}$

Relevant characteristics:
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1. The lexicographical order $\prec$

Definition
The meet-closure $\langle\langle A \rangle\rangle$ of a set $A \subset n^{<\omega}$ is the smallest set which contains $A$ and is closed under the meet operation.
The first-move structure of $n^{<\omega}$

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1. The lexicographical order $\prec$

\[
\emptyset \prec 0 \prec 1 \prec 2 \prec 00 \prec 01 \prec 02 \prec 10 \prec 11 \prec 12 \prec \cdots
\]
The first-move structure of $n^{<\omega}$

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2. The tree (partial) order $<$
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**First move**($t, s$) = 3

```
\[
\begin{align*}
\text{First move}(t, s) &= 3 \\
\end{align*}
\]
```
The first-move structure of $n^{<\omega}$

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1. The lexicographical order $\prec$
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Definition

The meet-closure $\langle\langle A\rangle\rangle$ of a set $A \subset n^{<\omega}$ is the smallest set which contains $A$ and is closed under the meet operation.
Let $A$, $B$ be subsets of $\mathbb{n}^{<\omega}$.

A first-move isomorphism between $A$ and $B$ is a bijection $f : A \to B$ which extends to a bijection $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ which preserves all relevant characteristics of the first move structure.
First-move equivalence

Let $A, B$ be subsets of $n^{<\omega}$.

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Ramsey theorem

Theorem

Fix a set \( A \subset n^{<\omega} \), and let \( \mathcal{A} \) be the family of all subsets of \( n^{<\omega} \) first-move isomorphic to \( A \). If \( c : \mathcal{A} \rightarrow \{0,...,m\} \) is measurable, then there exists \( T \subset n^{<\omega} \) such that \( 1^T \) is first-move isomorphic to \( n^{<\omega} \) and \( c \) is constant on the subsets of \( T \).
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This essentially follows from Milliken’s partition theorem.
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1. \( T \) is first-move isomorphic to \( n^{<\omega} \)
2. \( c \) is constant on the subsets of \( T \).
For $i, k < n$, an $(i, k)$-comb is a set that is first-move isomorphic to

$$\{(k), (iik), (iiiik), (i^6k), (i^8k), (i^{10}k), \ldots\}$$
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$$\{(k), (iik), (iiiiik), (i^6 k), (i^8 k), (i^{10} k), \ldots\}$$
An \((i, i)\)-comb is called an \(i\)-chain.
## Properties of combs

1. An \((i, k)\)-comb is first-move equivalent to all of its infinite subsets.
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Let \(S_1, \ldots, S_n\) disjoint subsets of \(m \times m\).
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   We call this a standard strong \(n\)-gap.
Theorem

Let \( \{ \Gamma_i : i < n \} \) be a strong analytic gap on \( \mathbb{N} \). Then there exists a one-to-one map \( u : n^{<\omega} \to \mathbb{N} \) such that

- If \( A \) is an \( i \)-chain, then \( u(A) \in \Gamma_i \).
Finding a standard gap inside

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---

**Diagram:**

- \( n_0 \)
- \( n_1 \)
- \( n_2 \)
- \( n_0, n_1, n_2 \)
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---

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Finite basis for strong $n$-gaps

**Theorem**

Let $\{\Gamma_i : i < n\}$ be a strong analytic gap on $\mathbb{N}$. Then there exists

1. a one-to-one map $u : n^{<\omega} \rightarrow \mathbb{N}$
2. a standard strong $n$-gap $\{\Delta_i : i < n\}$

such that $u(A)$ contains an infinite set from $\Gamma_i$ if and only if $A$ contains an infinite set from $\Delta_i$
Theorem

Let \{\Gamma_i : i < n\} be a strong analytic gap on \(\mathbb{N}\). Then there exists

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• For every \(\Gamma\) there is a standard \(\Delta\) with \(\Delta \leq \Gamma\).
Finite basis for strong $n$-gaps

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- For every $\Gamma$ there is a standard $\Delta$ with $\Delta \leq \Gamma$.
- Inside the standard strong gaps, there are the minimal ones
  - $\Delta$ is minimal if $E \leq \Delta \Rightarrow \Delta \leq E$.
  - Two minimal are equivalent if $\Delta' \leq \Delta$ and $\Delta \leq \Delta'$
Finite combinatorics behind

Problems about general analytic strong gaps are reduced to problems about standard strong gaps,
Finite combinatorics behind

Problems about general analytic strong gaps are reduced to problems about standard strong gaps, which in turn reduce to finite combinatorial problems.
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**Definition**

A function \( f : n \times n \rightarrow m \times m \) is a morphism if there exists a one-to-one \( u : n^\omega \rightarrow m^\omega \) which takes \((i,j)\)-combs to \( f(i,j)\)-combs.
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- The category formed by sets $n \times n$ and morphisms as above governs the behavior of strong analytic $n$-gaps.
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- The category formed by sets \( n \times n \) and morphisms as above governs the behavior of strong analytic \( n \)-gaps.
- This allows to compute the minimal strong \( n \)-gaps: each of them is given by seven parameters \((A, B, C, D, E, \psi, \gamma)\).
Minimal strong gaps

Minimal analytic strong 2-gaps
Minimal strong gaps

Minimal analytic strong 3-gaps
Part III
The record structure of the $n$-adice tree and general gaps
The set of records from $t$ to $s$

Let $t < s$ be in $n^{<\omega}$,
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The set of records from $t$ to $s$

Let $t < s$ be in $\mathbb{N}^<\omega$, $s = (t_0, \ldots, t_n, r_0, \ldots, r_m)$

**Definition**

A record node from $t$ to $s$ is a node $(t_0, \ldots, t_n, r_0, \ldots, r_{k-1})$ such that $r_k > r_i$ for all $i < k$. 
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Let $t < s$ be in $n^{< \omega}$, $s = (t_0, \ldots, t_n, r_0, \ldots, r_m)$

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**Definition**

A record node from $t$ to $s$ is a node $(t_0, \ldots, t_n, r_0, \ldots, r_{k-1})$ such that $r_k > r_i$ for all $i < k$.

$\text{record}(t, s) = \{t, u, v\}$
The relevant characteristics of the record-structure are the same as for the first-move structure, with the addition of the set of records $\text{record}(t, s)$. 

A record isomorphism between $A$ and $B$ is a bijection $f : A \to B$ which extends to a bijection $f : \langle A \rangle \to \langle B \rangle$ which preserves all relevant characteristics of the record structure.
The relevant characteristics of the record-structure are the same as for the first-move structure, with the addition of the set of records \( \text{record}(t, s) \).

The record-closure \( \langle A \rangle \) of a set \( A \subset n^{<\omega} \) is the smallest set which contains \( A \) and is closed under the meet operation \( t \wedge s \) and under taking \( \text{record}(t, s) \).
The relevant characteristics of the record-structure are the same as for the first-move structure, with the addition of the set of records $record(t, s)$.

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Record equivalence

A set \( \{ t^*, s^* \} \) record-isomorphic to \( \{ t, s \} \) as before:
Record equivalence

A set \( \{ t^*, s^* \} \) record-isomorphic to \( \{ t, s \} \) as before:

\[
\begin{align*}
4 & \leq 4 \\
\leq 6 & \leq 6 \\
\leq 8 & \leq 8 \\
\end{align*}
\]

A set \( \{ t^*, s^* \} \) first-move-isomorphic to \( \{ t, s \} \) as before:
Theorem

Fix a set $A \subset n^{<\omega}$, and let $\mathcal{A}$ be the family of all subsets of $n^{<\omega}$ record isomorphic to $A$. If $c : \mathcal{A} \rightarrow m$ is measurable, then there exists $T \subset n^{<\omega}$ such that

1. $T$ is record isomorphic to $n^{<\omega}$
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This is stronger than the first-move Ramsey theorem.
Ramsey theorem

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1. $T$ is record isomorphic to $n^{<\omega}$
2. $c$ is constant on the subsets of $T$. 

\[
\begin{align*}
&\leq 0 & &\leq 1 \\
&1 & &\leq 1 \\
&\leq 0 & &1 \\
& & &\leq 1 \\
& & &1 \\
\end{align*}
\]
Ramsey theorem

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Fix a set $A \subset n^{<\omega}$, and let $\mathcal{A}$ be the family of all subsets of $n^{<\omega}$ record isomorphic to $A$. If $c : \mathcal{A} \to m$ is measurable, then there exists $T \subset n^{<\omega}$ such that

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\[
\begin{align*}
\leq 0 & \leq 1 \\
\leq 1 & \leq 2
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The role of \((i,j)\)-combs is played now by types.
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There are two kind of types in \(n^{<\omega}\):
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There are two kinds of types in $n^{<\omega}$:

1. Chain-types are given by an increasing sequence of numbers $< n$, like $[1257]$, $[0]$, $[468]$, etc.
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There are two kind of types in \(n^{<\omega}\):

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   \(< n\),
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There are two kind of types in \(n^{<\omega}\):

1. Chain-types are given by an increasing sequence of numbers \(< n\), like \([1257], [0], [468]\), etc.

2. Comb-types are given by two increasing sequences of numbers \(< n\), that we write in two rows, with a global order, which is read from left to right, like \([3^0 5], [1^3 67]\), etc.

   (the rightmost number must always be in the lower row, and the leftmost numbers of each row must be different)
A set \( \{x_0, x_1, x_2, \ldots \} \) of type [468]
A set \( \{x_0, x_1, x_2, \ldots \} \) of type [468]
A set \( \{x_0, x_1, x_2, \ldots \} \) of type \([4 \, \frac{1}{6} \, 7 \, 8]\)
A set \( \{ x_0, x_1, x_2, \ldots \} \) of type \([4 \ 1 \ 6 \ 7 \ 8]\)
A set \( \{x_0, x_1, x_2, \ldots \} \) of type \([4 \quad 1 \quad 6 \quad 7 \quad 8]\)
A set \( \{ x_0, x_1, x_2, \ldots \} \) of type \([4^{\frac{1}{6}}78]\)
A set \( \{x_0, x_1, x_2, \ldots \} \) of type \([\begin{array}{cc} 17 & 68 \end{array}]\)
There are eight types in $2^{<\omega}$:

$[0], [1], [01], [\overline{0} 1], [\overline{1} 0], [01 \overline{1}], [101], [01 1].$
Types

There are eight types in $2^{<\omega}$:


There are 61 types in $3^{<\omega}$,
There are eight types in $2^{<\omega}$:

$[0], [1], [01], [0^1_1], [1^0_1], [0^1_1], [1_0^1], [0^1_1].$

There are 61 types in $3^{<\omega}$,

There are approximately $\sim \frac{3\cdot 9^n}{8\sqrt{2\pi n}}$ types in $n^{<\omega}$
A set of type $\tau$ is record-equivalent to all of its infinite subsets.
# Properties of types

1. A set of type $\tau$ is record-equivalent to all of its infinite subsets.

2. Every infinite set contains a set of type $\tau$ for some $\tau$. 
Properties of types

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   We call this a standard $n$-gap.
Finding a standard gap inside

**Theorem**

Let \( \{ \Gamma_i : i < n \} \) be an analytic gap on \( \mathbb{N} \). Then there exists a one-to-one map \( u : n^{<\omega} \rightarrow \mathbb{N} \) and a permutation \( \varepsilon \) such that

- If \( A \) is an \([i]-\)chain, then \( u(A) \in \Gamma_{\varepsilon(i)} \).
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Theorem

Let \( \{ \Gamma_i : i \leq n \} \) be an analytic gap on \( \mathbb{N} \). Then there exists a one-to-one map \( u : n^\omega \to \mathbb{N} \) and a permutation \( \varepsilon \) such that

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Theorem

Let $\{\Gamma_i : i < n\}$ be an analytic gap on $\mathbb{N}$. Then there exists a one-to-one map $u : n^\omega \to \mathbb{N}$ and a permutation $\varepsilon$ such that

- If $A$ is an $[i]$-chain, then $u(A) \in \Gamma_{\varepsilon(i)}$. 

\[ n_0 \quad n_01 \quad n_{10} \quad n_{11} \]

\[ n_0 \quad n_1 \quad \emptyset \]

\[ [0] \quad [1] \quad [01] \quad [01] \quad [10] \quad [011] \quad [101] \quad [011] \]

$\Gamma_0 \quad \Gamma_1$
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**Theorem**

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Graphical representation of the theorem:

- $n_0, n_1, n_00, n_01, n_10, n_11$
- Arrows indicating the map $u$ and the gaps $\Gamma_0, \Gamma_1$.
Finite basis for strong $n$-gaps

**Theorem**

Let $\{\Gamma_i : i < n\}$ be an analytic gap on $\mathbb{N}$. Then there exists

1. a one-to-one map $u : n^{<\omega} \rightarrow \mathbb{N}$
2. a standard $n$-gap $\{\Delta_i : i < n\}$

such that $u(A)$ contains an infinite set from $\Gamma_i$ if and only if $A$ contains an infinite set from $\Delta_i$
Theorem

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- For every \( \Gamma \) there is a standard \( \Delta \) with \( \Delta \leq \Gamma \).
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such that $u(A)$ contains an infinite set from $\Gamma_i$ if and only if $A$ contains an infinite set from $\Delta_i$.

- For every $\Gamma$ there is a standard $\Delta$ with $\Delta \leq \Gamma$.
- Inside the standard $n$-gaps, there are the minimal ones
  - $\Delta$ is minimal if $E \leq \Delta \Rightarrow \Delta \leq E$.
  - Two minimal are equivalent if $\Delta' \leq \Delta$ and $\Delta \leq \Delta'$.
Problems about general analytic gaps are reduced to problems about standard gaps,
Problems about general analytic gaps are reduced to problems about standard gaps, which in turn reduce to finite combinatorial problems.
Let $\mathcal{S}_n$ be the set of types in $n^{<\omega}$.

**Definition**

A function $f : \mathcal{S}_n \longrightarrow \mathcal{S}_m$ is a morphism if there exists a one-to-one $u : n^{<\omega} \longrightarrow m^{<\omega}$ which sends sets of type $\tau$ to sets of type $f\tau$. 
Finite combinatorics behind

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- The category formed by the sets $\mathcal{S}_n$ and morphisms as above governs the behavior of analytic $n$-gaps.
Let $\mathcal{T}_n$ be the set of types in $n^{<\omega}$.

**Definition**

A function $f : \mathcal{T}_n \rightarrow \mathcal{T}_m$ is a morphism if there exists a one-to-one $u : n^{<\omega} \rightarrow m^{<\omega}$ which sends sets of type $\tau$ to sets of type $f\tau$.

- The category formed by the sets $\mathcal{T}_n$ and morphisms as above governs the behavior of analytic $n$-gaps.
- This category is more complex than the one for strong gaps, so we were not able to describe the minimal analytic $n$-gaps.
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Let $\mathcal{G}_n$ be the set of types in $n^{<\omega}$.

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Let $\mathcal{T}_n$ be the set of types in $n^{<\omega}$.

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- We studied some phenomena in this category, so as to find the list of minimals for $n = 2$ and $n = 3$ and to be able to solve the problem at the beginning.
The minimal analytic 2-gaps

There are 9 minimal 2-gaps (5 up to permutation):

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma_0$</th>
<th>$\Gamma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1**</td>
<td>[0]</td>
<td>all other types</td>
</tr>
<tr>
<td>2**</td>
<td>[0]</td>
<td>[1]</td>
</tr>
<tr>
<td>3**</td>
<td>[0]</td>
<td>[1], [01]</td>
</tr>
<tr>
<td>4*</td>
<td>[0], [01]</td>
<td>[1]</td>
</tr>
<tr>
<td>5**</td>
<td>[0]</td>
<td>[1], [01], $[^101]$</td>
</tr>
</tbody>
</table>

**: two permutations
*: equivalent to its permutation
The max function

Definition
If $\tau$ is a type, $\max(\tau)$ is the maximal integer appearing in the type.
The max function

Definition

If \( \tau \) is a type, \( \max(\tau) \) is the maximal integer appearing in the type.

For example, \( \max[3^{17}56] = 7 \).
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If $\tau$ is a type, $\text{max}(\tau)$ is the maximal integer appearing in the type.

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Theorem
For $\tau_0, \ldots, \tau_{n-1} \in \mathcal{T}_m$, TFAE:
The max function

Definition
If $\tau$ is a type, $\max(\tau)$ is the maximal integer appearing in the type.

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Theorem
For $\tau_0, \ldots, \tau_{n-1} \in \mathcal{T}_m$, TFAE:

1. $\max(\tau_0) \leq \max(\tau_1) \leq \cdots \leq \max(\tau_{n-1})$,
The max function

Definition

If $\tau$ is a type, $\text{max}(\tau)$ is the maximal integer appearing in the type.

For example, $\text{max}[3^{17}56] = 7$.

Theorem

For $\tau_0, \ldots, \tau_{n-1} \in \mathcal{T}_m$, TFAE:

1. $\text{max}(\tau_0) \leq \text{max}(\tau_1) \leq \cdots \leq \text{max}(\tau_{n-1})$,
2. There exists a morphism $f : \mathcal{T}_n \rightarrow \mathcal{T}_m$ such that $f[i] = \tau_i$. 
fact provides a nice embedding $u$ such that $\phi u$ satisfies condition (2) of normal embeddings for all 4-families. This finishes the proof by Lemma 1.1.

2. The max function

Given a type $r$, $\max(r)$ denotes the maximal number which appears in $r$. That is,

$$\max(r) = \max(\max(r^0), \max(r^1)).$$

**Theorem 2.1.** For a family $\{r_i : i \in I\} \subseteq \mathbb{Z}_m$ the following are equivalent:

1. There exists a normal embedding $\phi : n^{\infty} \rightarrow m^{\infty}$ such that $\phi(n) = r_i$.
2. $\max(r_0) \leq \cdots \leq \max(r_{n-1}).$

**Proof.** Suppose that item (1) holds, pick $i < j$ and let us check that $\max(r_i) \leq \max(r_j)$. Let $a = \phi(j) \wedge \phi(i)$. Since $\{j, i\}$ are the two first elements of a chain of type $[i]$, it follows that

$$\max(r_i) = \max(\max(\phi(j) \setminus \alpha), \max(\phi(i) \setminus \alpha)).$$

On the other hand, both $\{0, j\}$ and $\{0, i\}$ are the beginnings of chains of type $[i]$, so if $\beta = \phi(0) \wedge \phi(j)$ and $\gamma = \phi(0) \wedge \phi(i)$ we have similar formulas

$$\max(r_j) = \max(\max(\phi(j) \setminus \beta), \max(\phi(i) \setminus \beta)).$$

$$\max(r_i) = \max(\max(\phi(0) \setminus \gamma), \max(\phi(i) \setminus \gamma)).$$

We distinguish three cases. The first case is $\beta = \gamma$, which implies that $\gamma = \beta < \alpha$. By formula (I), it is enough to check that $\max(\phi(j) \setminus \alpha) \leq \max(\phi(i) \setminus \alpha) \leq \max(r_j)$ in this case, $\phi(j) \setminus \alpha = \phi(j) \setminus \beta$ so it is clear that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ by (II). On the other hand, $\phi(i) \setminus \alpha = (\gamma \setminus \alpha)^{-} (\phi(j) \setminus \gamma)$.

$$\max(\gamma \setminus \alpha) = \max(\phi(0) \setminus \beta) \leq \max(r_j)$$

by (II), and on the other side $\max(\phi(j) \setminus \gamma) \leq \max(r_j)$ (III), so we conclude that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$. By formula (I), this finishes the second case.

The third case is that $\beta > \alpha$, which implies that $\gamma = \alpha = \beta$.

By formula (I), it is enough to check that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ and $\max(\phi(i) \setminus \alpha) \leq \max(r_j)$. In this case, $\phi(j) \setminus \alpha = \phi(j) \setminus \beta$ so it is clear that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ by (II). On the other hand, $\phi(i) \setminus \alpha = (\gamma \setminus \alpha)^{-} (\phi(j) \setminus \gamma)$.

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by (II), and on the other side $\max(\phi(j) \setminus \gamma) \leq \max(r_j)$ (III), so we conclude that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$. By formula (I), this finishes the second case.

The third case is that $\beta > \alpha$, which implies that $\gamma = \alpha < \beta$. This is solved in a similar way as in the second case, changing the role of $j$ and $i$. By formula (I), it is enough to check that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ and $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$. Now, $\phi(j) \setminus \alpha = \phi(j) \setminus \gamma$ so it is clear that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ by (III). On the other hand, $\phi(j) \setminus \alpha = (\beta \setminus \alpha)^{-} (\phi(j) \setminus \beta)$.

On one side, $\phi(0) \setminus \beta = (\beta \setminus \gamma)^{-} (\phi(0) \setminus \beta)$ so $\max(\beta \setminus \alpha) = \max(\beta \setminus \gamma) \leq \max(\phi(0) \setminus \gamma) \leq \max(r_j)$ by (III), and on the other side $\max(\phi(j) \setminus \beta) \leq \max(r_j)$ by (II). So we conclude that $\max(\phi(j) \setminus \alpha) \leq \max(r_j)$ and this finishes the third case.

Now, suppose that (2) holds. For every $i$ fix $(u_i, v_i)$ a rung of type $r_i$ and write $u_i = \llav{\lessdot U_i}$ in such a way that $|U_i| = |r_i|$. When $r_i$ is a chain type, $v_i = U_i$ and $U_i = u_i$. When $r_i$ is a comb type we can make the additional assumption that the last integer of $U_i$ and the first integer of $U_i$ are both equal to 0. We shall construct an embedding $\phi : n^{\infty} \rightarrow m^{\infty}$ together with auxiliary functions $\phi_1 : n^{\infty} \rightarrow m^{\infty}$ for $i = 0, \ldots, n - 1$. All of them will be defined by induction on the $\prec$-order of $n^{\infty}$. We first choose $\phi(0), \phi(\emptyset), \phi'(\emptyset)$. Let $\{j_1, \ldots, j_n\}$ be an enumeration of all indices $i$ such that $r_i$ is a comb type and such that

$$\max(\tau_j) \geq \max(\tau_j^1) \geq \cdots \geq \max(\tau_j^k),$$

and moreover, if $\max(\tau_j^r) = \max(\tau_j^1)$, then $j_r < j_r$ if and only if $r > s$.

We define

$$\phi_1(0) = \emptyset,$$

$$\phi_{i+1}(0) = v_{j_1} \cdots v_{j_{i+1}}$$

$$\phi(0) = v_{j_1} \cdots v_{j_{n-1}}$$

$$\phi'(0) = \phi_1(0) \setminus \llav{U_i} \setminus 0_i$$

if $r_i$ is a comb type.

$$\phi'(\emptyset) = \phi'(0)$$

if $r_i$ is a chain type.

The number $l_i$ of $0_i$'s added to construct $\phi'(\emptyset)$ is chosen so that $\phi'(\emptyset)$ has length strictly larger than $\phi(\emptyset)$. Figure 1 represents how $\phi(0), \phi(\emptyset)$ and $\phi'(\emptyset)$ look like in the tree. The pattern reflected in this picture will be repeated for $\phi(x), \phi'(x)$ and $\phi'(x)$ for any $x$. It is natural to make the notational convention that $\phi_{i+1} = \emptyset$ and this will avoid repeating some arguments along the proof.

---

1The proof of later Lemma 5.5 may be enlightening about the necessity of constructing $\phi$ in such a complicated way.

2The aim of this assumption is to make sure that the critical nodes of $u_i$ are far away from the splitting between $U_i$ and $U_i$, and to avoid in this way peculiar situations.
We shall see how to define all these functions on \( x^{-k} \) once they are defined on all \( y^{-k} \), in particular on \( y = x \). We consider

\[
q = q(k) = \min\{ r : \max(\tau_i^r) < \max(\tau_i^k) \text{ or } j_r \leq k \}
\]

(If there is no \( r \) like that we may assign the value \( q = p + 1 \)). The definition of the functions is then made as follows:

\[
\begin{align*}
\phi(x^{-k}) &= \phi^k(x^{-k}) \bar{w} v_{j_1} \bar{w} v_{j_2} \cdots \bar{w} v_{j_p}, \\
\phi_j(x^{-k}) &= \phi_j(x) \text{ if } r < q, \\
\phi_j(x^{-k}) &= \phi^k(x^{-k}) \bar{w} v_{j_1} \bar{w} v_{j_2} \cdots \bar{w} v_{j_{q-1}} \text{ if } r = q, \\
\phi_j(x^{-k}) &= \phi_j(x) \text{ if } r \geq q, \\
\phi^k(x^{-k}) &= \phi(x^{-k}) = \phi(x^{-k}) \text{ if } r \geq q.
\end{align*}
\]

Now, the number \( l \) of 0's added to construct \( \phi^k(x^{-k}) \) is chosen so that \( \phi^k(x^{-k}) \) has length larger than \( \phi(x^{-k}) \) but also larger than all \( \phi(x) \), \( \phi(y) \), \( \phi^j(y) \) that have been already constructed for \( y < x^{-k} \). A picture of what is going on is given by Figure 2. The point is that both sets \{ \phi(x), \phi_j(x), \phi^k(x), k < \omega \} and \{ \phi(x^{-k}), \phi_k(x^{-k}), \phi^k(x^{-k}), k < \omega \} must follow the pattern provided by Figure 1, but we make \( \phi_j(x^{-k}) \) to stay the same as \( \phi_j(x) \) for \( r < q \), while \( \phi_j(x) \) is moved above \( \phi^k(x^{-k}) \) for \( r \geq q \).

**Claim 1**: For every \( x \in n^{<\omega} \) and \( r = 1, \ldots, p \),

\[(*) \quad \phi_j(x^{-k}) = \phi_j(x) \bar{w} v_{j_r} \bar{w} \text{ for some } w \text{ such that } \max(w) \leq \max(v_{j_r}).\]

Proof of Claim 1: This holds when \( x = 0 \). We suppose that it holds for \( x \) and we prove it for \( x^{-k} \). For \( r < q \equiv q(k) \) we have that \( \phi_j(x^{-k}) = \phi_j(x) \) while for \( r \geq q \) we have that

\[
\phi_j(x^{-k}) = \phi^k(x^{-k}) \bar{w} v_{j_1} \bar{w} v_{j_2} \cdots \bar{w} v_{j_{r-1}}.
\]

---

\(^3\)When we say *following the same pattern*, we mean up to equivalence. Looking at Figure 2, one may wonder if the long path from \( \phi_j(x^{-k}) \) till \( \phi_j(x^{-k}) \) is really equivalent to \( v_{j_r-1} \) as Figure 1 suggests. This is the content of Claim 1.

\(^4\)Remember our convention that \( \phi_{j_r-1}(x) = \phi(x) \)

---

4. Working in the \( m \)-adic Tree

---

**Figure 2**: Passing from \( x \) to \( x^{-k} \)

Thus, we have \( \phi_j(x^{-k}) = \phi_j(x) \bar{w} v_{j_r} \) when either \( r < q - 1 \) or \( r \geq q \). Only the case when \( r = q - 1 \) deserves special attention. In this case

\[
\phi_j(x^{-k}) = \phi_{j_r-1}(x) \bar{w} v_{j_r} = \phi_{j_r-1}(x) \bar{w} v_{j_r}.
\]

Either \( \tau_k \) is a chain type (in which case \( \phi_k(x) = \phi(x) \)) or \( k = j_i \) for some \( i \) which must satisfy \( l \geq q \) by the definition \(^5\) of \( q \). In either case the inductive hypothesis implies that \( \phi_{j_{k-1}}(x) \bar{w} v_{j_{k-1}} \bar{w} v_{j_{k-2}} \bar{w} v_{j_{k-3}} \cdots \bar{w} v_{j_0} \) where \( \max(v_{j_0}) \leq \max(v_{j_{k-1}}) \). If \( \tau_k \) is a chain type, then \( \phi^k(x^{-k}) = \phi_k(x) \), so

\[
\phi_j(x^{-k}) = \phi^k(x^{-k}) \bar{w} v_{j_1} \bar{w} v_{j_2} \cdots \bar{w} v_{j_{q-1}} \bar{w} v_{j_q} \bar{w} \text{ and this is what we were looking for because}\n\]

\[
\max(v_k) \leq \max(\tau_k) \leq \max(v_{j_{k-1}}) = \max(v_{j_{k-1}}).
\]

On the other hand, if \( \tau_k \) is a comb type, then \( \phi^k(x^{-k}) = \phi_k(x) \bar{w} v_{j_1} \bar{w} \), so

\[
\phi_j(x^{-k}) = \phi^k(x^{-k}) \bar{w} v_{j_1} \bar{w} v_{j_2} \cdots \bar{w} v_{j_{q-1}} \bar{w} v_{j_q} \bar{w} \text{ and this is again what we were looking for, because}\n\]

\[
\max(v_k) \leq \max(\tau_k) \leq \max(v_{j_{k-1}}) = \max(v_{j_{k-1}}) \text{ similarly as in the previous case. This finishes the proof of Claim 1.}
\]

---

\(^5\)If \( j_i = k \) then in particular \( j_i \leq k \) so by the minimality of \( q \) in its definition, \( q \leq i \).

\(^6\)Just apply the formula (\(*)\) repeatedly for \( r = q - 1, q, \ldots \) till arriving at \( \phi_k(x) \).

\(^7\)The central inequality \( \max(\tau_k) \leq \max(v_{j_{k-1}}) \) follows from the definition of \( q \).
2. THE MAX FUNCTION

Proof of Claim 2: We proceed by induction on the length of w. Together with
the statement of the claim, we shall also prove that for every i = 0, ..., k, we can
write \( \phi_i(x^-w) = \phi_i(x^-u_k^-w'_i) \) where max(w') \leq max(\( \tau_k \)). The first case is that
w = (k). Remember that

\[
\phi(x^-k) = \phi(x^-k) - u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^- 
\]

and since \( \tau_k \) is a chain type, \( \phi(x^-k) \) is \( \phi(x^-k) \) and there is nothing to
prove. The other case is that i = j_r for some r. Then, by the definition of q, \( r \geq q \)
so the expression above is as desired, and the claim is proven for w = (k).
Concerning \( \phi_i(x^-k) \), if \( \tau_i \) is a chain type, \( \phi_i(x^-k) \) is \( \phi(x^-k) \) and there is nothing to
prove. The other case is that i = j_r for some r. Then, by the definition of q, \( r \geq q \)
so the expression above is as desired, and the claim is proven for w = (k).

Now we assume that our statement holds for w = (k). We fix \( \xi \in \{0, ..., k\} \) and
we shall prove that the statement holds for w = \( \xi \xi \). First,

\[
\phi(x^-w^-\xi) = \phi(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^-) 
\]

Notice that max(w') \leq max(\( \tau_k \)), and in the same way as we had the expression (**) the defining formulas of q(\( \xi \)) implies that

\[
(****) \text{ max}(v_{j_p}) \leq \cdots \leq \text{ max}(v_{j_1}) \leq \text{ max}(\xi) 
\]

so all vectors \( v_{j_p} \) appearing in the expression (*** above are bounded by max(\( \tau_k \)).
Hence, the expression (*** above can be rewritten as

\[
\phi(x^-w^-\xi) = \phi(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^-) 
\]

If \( \tau_k \) is a chain type, then \( \phi(x^-w^-w) = \phi(x^-w^-w) \) and we are done, by the inductive
hypothesis. If \( \tau_k \) is a chain type, then

\[
\phi(x^-w) = \phi(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^- 0^0) 
\]

This provides the desired form because max(\( u_k^- v_{j_1}^- 0^0 \)) \leq max(\( \tau_k \)) and we can apply the inductive hypothesis to \( \phi(x^-w^-w) \).

Finally, we fix i \( \in \{0, ..., k\} \) and we prove that also \( \phi_i(x^-w^-\xi) \) is of the form

\[
\phi(x^-w^-\xi) = \phi(x^-w^-u_k^-) \text{ with max}(w') \leq \text{ max}(\tau_k) 
\]

If \( \tau_i \) is a chain type, there is nothing to prove because \( \phi_i = \phi \).
Otherwise \( \phi_i \) is a comb type, and i = j_r for some r. If \( r < q(\xi) \) then \( \phi(x^-w^-\xi) = \phi(x^-w) \) and we apply directly the inductive hypothesis. If \( r \geq q(\xi) \), then

\[
\phi(x^-w^-\xi) = \phi(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^- 0^0) 
\]

By the definition of q, either max(v_{j_k}) < max(\( \tau_k \)) or j_k \leq k. In the latter case, max(v_{j_k}) \leq max(\( \tau_k \)) by the statement (2) of Theorem 2.1 that we are assuming.

4. WORKING IN THE n-ADIC TREE

By the expression (***) above, all vectors to the right of \( \phi(x^-w^-w) \) are bounded by

\[
\phi(x^-w^-w) = \phi(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_p}^- 0^0) 
\]

of the form \( \phi(x^-w^-u^- w') \) with max(w') \leq max(\( \tau_k \)), by the inductive hypothesis.
This finishes the proof of Claim 2.

Claim 3: Suppose that \( \tau_k \) is a comb type, \( x \in \mathbb{N}^\omega \) and \( w \in W_k \). Then

\[
\phi_k(x^-w) = \phi_k(x^-w^-u_k^-) 
\]

where max(w') \leq max(\( \tau_k \)).

Proof of Claim 3: Since \( \tau_k \) is a comb type, \( k = j_r \) for some r. We proceed
by induction on the length of w. The first case is that w = (k). Notice that

\[
r \geq q \text{ because } j_r - k \leq r \text{ (by the definition of q), hence } \phi_k(x^-w) = \phi_k(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_r}^-) 
\]

It is enough to show that all vectors to the right of \( u_k^- \) in the expression above are bounded by max(\( \tau_k \)). This is equivalent to show that either \( r = q \) or \( \text{ max}(v_{j_1}^-) \leq \text{ max}(\tau_k) \). Remember that max(\( v_{j_i}^- \)) = max(\( \tau_k \)) for any \( \xi \).

From the definition of q, one of the following two cases must hold:

Case 1: max(\( \tau_k \)) \leq max(\( \tau_k \)). In this case, since \( k = j_r \) and \( q \leq r \) we have that

\[
\text{ max}(\tau_k) \geq \text{ max}(\tau_k) \geq \text{ max}(\tau_k) 
\]

From the two inequalities above we conclude that max(\( \tau_k \)) \leq max(\( \tau_k \)), hence

\[
\text{ max}(\tau_k) = \text{ max}(\tau_k) 
\]

Therefore max(\( \tau_k \)) \leq max(\( \tau_k \)) as we wanted to prove.

Case 2: max(\( \tau_k \)) \geq max(\( \tau_k \)) and \( j_r - k \leq r \). Now, \( j_r - k \leq r \) implies that

\[
\text{ max}(\tau_k) \leq \text{ max}(\tau_k) 
\]

hence actually max(\( \tau_k \)) = max(\( \tau_k \)). If max(\( \tau_k \)) \leq max(\( \tau_k \)) then we are done, so we

suppose that max(\( \tau_k \)) = max(\( \tau_k \)) > max(\( \tau_k \)). We combine the two previous equations we get that

\[
\text{ max}(\tau_k) = \text{ max}(\tau_k) = \text{ max}(\tau_k) 
\]

but this implies (by the way in which chose the order of the enumeration \( \{j_1, ..., j_r\} \) and the fact that \( j_r - k \leq r \)) assumed in Case 2) that \( r \leq q \), hence \( r = q \) as we wanted to prove. This finishes Case 2, and finishes the proof of initial case \( w = (k) \) as well.

Now we suppose that Claim 3 holds for \( w \), we fix \( \xi \leq k \) and we shall prove that Claim 3 holds for \( w \xi \) as well. If \( r < q(\xi) \) then\( \phi_k(x^-w^-\xi) = \phi_k(x^-w^-w) \) and we apply directly the inductive hypothesis. Hence, we suppose that \( r \geq q(\xi) \) and therefore

\[
(*** \text{ for } \xi) \phi_k(x^-w^-w) = \phi_k(x^-w^-u_k^- v_{j_1}^- v_{j_2}^- \cdots v_{j_r}^- v_{j_{r+1}}^- \cdots v_{j_{r+1}}^-) 
\]

On the other hand,"
so applying the inductive hypothesis to $\phi_k(x \sim w')$, we get that

$$\phi^k(x \sim w') = \phi^k(x) \sim \alpha_k \sim w'$$

with $\max(w') \leq \max(\tau_k^t)$. Looking back at the expression (1) above, it is enough to show that all members of that expression to the right of $\phi^k(x \sim w')$ are bounded by $\max(\tau_k^t)$. This is equivalent to prove that either $q = q(\xi)$ or $\max(\tau_k^t) = \max(\tau_k^t) \leq \max(\tau_k^t)$. Let now $q = q(\xi)$. We distinguish two cases:

Case 1: $\max(\tau_k^t) < \max(\tau_k^t)$. In this case, since $k = j$, and we supposed that $q \leq r$ we have that

$$\max(\tau_k^t) \geq \max(\tau_k^t) = \max(\tau_k^t)$$

From the two inequalities above we conclude that $\max(\tau_k^t) \leq \max(\tau_k^t)$, hence $\max(\tau_k^t) = \max(\tau_k^t)$. Therefore $\max(\tau_k^t) < \max(\tau_k^t) = \max(\tau_k^t)$ as we wanted to prove.

Case 2: $\max(\tau_k^t) \geq \max(\tau_k^t)$. Since $\xi \leq k$ this implies that $\max(\tau_k^t) \geq \max(\tau_k^t) \geq \max(\tau_k^t)$. By the definition of $q = q(\xi)$, this further implies that $j_\xi \leq \xi$. Now, $j_\xi \leq k$ implies that

$$\max(\tau_k^t) \leq \max(\tau_k^t) \leq \max(\tau_k^t)$$

hence actually $\max(\tau_k^t) = \max(\tau_k^t)$. If $\max(\tau_k^t) = \max(\tau_k^t)$ then we are done, so we suppose that $\max(\tau_k^t) = \max(\tau_k^t) > \max(\tau_k^t)$. We combine the previous equations and we get that

$$\max(\tau_k^t) = \max(\tau_k^t) = \max(\tau_k^t)$$

but this implies (by the way in which we chose the order of the enumeration $\{j_1, \ldots, j_k\}$ and the fact that $j_\xi \leq \xi \leq j_\xi$, that we noticed above) that $r \leq q$, hence $q = r$ as we wanted to prove. This finishes Case 2, and finishes the proof of Claim 3 as well.

We fix $k < n$ and we shall prove that if $Y \subset n^\circ\omega$ is a set of type $[k]$, then $\phi(Y)$ is a set of type $\tau_k$. This will finish the proof of the theorem because, if $\phi$ was not a normal embedding, we can get a normal embedding by composing with a nice embedding using Theorem 1.3.

If $\tau_k$ is a chain type, then the fact that $\phi(Y)$ has type $\tau_k$ follows immediately from Claim 2. So suppose that $\tau_k$ is a comb type, $k = j$, and $Y = \{y_1, y_2, y_3, \ldots\}$. If we look at the inductive definition of $\phi$, and consider the case when $z = x \sim w$ and $k = j$, notice that then $r \geq q$ by the definition of $q$ since $j_\xi \leq j \leq k$, and we can write

$$\phi(z) = \phi_k(z \sim y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1})$$

where max($y_{j_\xi-1}$) \leq \max($y_{j_\xi-1}$) = \max($y_{j_\xi-1}$) for all $r = 1, \ldots, p$. If we apply this to $z = y_\xi$, we can write that

$$\phi(y_\xi) = \phi_k(y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1})$$

where max($w_\xi$) \leq \max($w_\xi$). On the other hand, Claim 3 provides the fact that

$$\phi_k(y_{j+1}) = \phi_k(y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1}) = \phi_k(y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1})$$

where max($w_\xi$) \leq max($w_\xi$). Remember that in the inductive definition of $\phi$, the number $\zeta$ of 0’s above was chosen so that the length of $\phi_k(y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1}$ is larger than the length of $\phi_k(y_\xi \sim y_{j_\xi-1} \sim \cdots \sim y_{j_\xi-1})$. The expressions (1) and (1*) together yield that $\phi(Y)$ is a set of type $\tau_k$ with underlying chain $\{\phi_k(y_i) : i \leq \omega\}$, as it is shown in Figure 3.

**Corollary 2.2.** If $\phi : n^{<\omega} \to m^{<\omega}$ is a normal embedding, then $\max(r) \leq \max(\sigma r) \leq \max(\sigma r)$.

**Corollary 2.3.** If $\{S_i : i \in n\}$ are pairwise disjoint sets of types in $m^{<\omega}$, then $\{\Gamma_{S_i} : i \in n\}$ is an n-gap.

**Proof.** The intersection of two sets of different types is finite, so it is clear that the ideals are mutually orthogonal. We have to prove that they cannot be separated. After reordering if necessary, we can find types $\tau_i \in S_i$ such that $\max(\tau_i) \leq \max(\tau_i) \leq \cdots \leq \max(\tau_{i-1})$. By Theorem 2.1, there is a normal embedding $\phi : n^{<\omega} \to m^{<\omega}$ such that $\phi[\tau] = \tau_i$. Finally, use Lemma 0.23.

We can now prove our first example of minimal analytic $n$-gap:

**Corollary 2.4.** Let $\mathcal{M}_1$ be the set of all types $\tau$ in $n^{<\omega}$ such that $\max(\tau) = i$. The $n$-gap $\mathcal{M}_1 = \{\Gamma_{S_i} : i < n\}$ in $n^{<\omega}$ is a minimal $n$-gap.

**Proof.** Suppose that $\Gamma \leq \mathcal{M}$ and we must show that $\mathcal{M} \leq \Gamma$. By Theorem 0.25, we can suppose that $\Gamma = \{\Gamma_{S_i} : i < n\}$ is a standard gap in $n^{<\omega}$. That is, there is a permutation $\varsigma : n \to n$ such that $\varsigma(i) \in S_{\varsigma(i)}$. By Theorem 1.3, there is a normal embedding $\phi : n^{<\omega} \to m^{<\omega}$ such that $\phi[\tau] = \tau_i$. Since $\varsigma(i) \in S_{\varsigma(i)}$, $\phi[\tau] = \tau_i$. Therefore, $\phi[\tau] = \tau_i$. Since $\varsigma(0) \leq \varsigma(1) \leq \cdots \leq \max(\tau_i - 1)$, so $\varsigma(0) \leq \varsigma(1) \leq \cdots \leq \max(\tau_i - 1)$, which implies that $\varsigma$ is the identity permutation. Moreover, we claim that $\Gamma = \mathcal{M}_1$. For pick $\tau \in \mathcal{M}_1$. Then $\max(\tau) = \max(\phi[\tau]) = \max(\phi[\tau]) = \tau_i$ which implies that $\phi[\tau] \in \mathcal{M}_1$, hence $\tau \in S_i$. This shows that $\mathcal{M}_1 \subset S_i$ for every $i$. Since the union of the sets $S_i$ gives all types in $n^{<\omega}$, this actually implies that $\mathcal{M}_1 = S_i$ for every $i < n$.

For a permutation $\delta : n \to n$, let us denote by $\mathcal{M}^\delta = \{\Gamma_{\mathcal{M}_i} : i < n\}$ the $\delta$-permutation of $\mathcal{M}$. The minimal gaps $\mathcal{M}^\delta$ are characterized by their extreme asymmetry in the following sense:

**Corollary 2.5.** The minimal $n$-gap $\mathcal{M}^\delta$ has the following two properties:

1. $\mathcal{M}$ is dense.
Domination

**Definition**

We say that the type $\tau$ dominates the type $\sigma$ if

1. the second integer from the right in $\tau$ is in the upper row
2. and it is greater or equal than $\max(\sigma)$
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Definition
We say that the type \( \tau \) dominates the type \( \sigma \) if
1. the second integer from the right in \( \tau \) is in the upper row
2. and it is greater or equal than \( \max(\sigma) \)

Examples:
- \([02, 1, 3, 2] \) dominates \([02]\)
- \([02, 3, 1, 2] \) does not dominate \([02]\)
- \([1, 5] \) does not dominate \([0, 2]\).
Definition

We say that the type $\tau$ dominates the type $\sigma$ if
1. the second integer from the right in $\tau$ is in the upper row
2. and it is greater or equal than $\max(\sigma)$

Theorem

For $\sigma, \tau$ types in $m^{<\omega}$, TFAE
1. $\tau$ dominates $\sigma$,
2. There exists a morphism $f : \mathcal{T}_2 \to \mathcal{T}_m$ such that
   - $f[0] = \sigma$,
   - $f \nu = \tau$ for all other $\nu \in \mathcal{T}_2$. 
case: it can be taken a top-comb type with $\max((\tau^1)) = k - 1$. In this way we reduce the general case to the first case.

If $\phi$ satisfies the conditions of Lemma 3.3 we shall say that $\phi$ collapses below $k$ (or that $\phi$ collapses up to $k - 1$) into a chain of type $\sigma$. The fact that in condition (1) of Lemma 3.3 the maximum of $\tau$ is attained in $\tau^1$ is important, for consider the following example: We can construct a normal embedding $\phi: 3^{\omega_1} \rightarrow 2^{\omega_1}$ such that for every $x$, $\phi(x^{\omega_1}) 2^\omega \leq \phi^1(x^{\omega_1})$, and $\phi(x^{\omega_1})$ equals $\phi(x)$ followed by a finite sequence of $0$'s when $i = 0, 1$. Such an embedding can be constructed inductively so that $x < y$ implies $|\phi(x)| < |\phi(y)|$. Notice that $\phi[0, 1] = [0, 1]$ but $\phi$ does not collapse below $3$.

4. Domination

The notion of top-comb introduced in Definition 3.2 and illustrated in Figure 4 is going to be crucial in this section. The key property now will be the following:

**Lemma 4.1.** Let $\tau$ be a top-comb type and let $(n, v)$ be a rung of type $\tau$. If $w$ is such that $\max(w) \leq \max(\tau^1)$ and $|v^{-\omega}w| < |v|$, then $(u, v - w)$ is also a rung of type $\tau$.

**Proof.** Straightforward. Just look at the left-hand side of Figure 4.

**Definition 4.2.** We say that a type $\tau$ dominates another type $\sigma$, and we will write $\tau \triangleright \sigma$, if $\tau$ is a top-comb type and $\max(\tau^1) \geq \max(\sigma)$.

**Lemma 4.3.** Let $\phi: n^{\omega_1} \rightarrow m^{\omega_1}$ be a normal embedding, and let $\tau \in \Sigma_m$ be a type that dominates $\phi\sigma$ for all $\sigma \in \Sigma_n$. Then, there exists a normal embedding $\psi: (n + 1)^{\omega_1} \rightarrow m^{\omega_1}$ such that $\psi\sigma = \phi\sigma$ if $\max(\sigma) < n$, and $\psi\sigma = \tau$ if otherwise $\max(\sigma) = n$.

**Proof.** Let $m_0 = \max(\tau^1) + 1$. Without loss of generality we will suppose that $m = m_0$. We can do this because the domination hypothesis implies that all types $\phi\sigma$ live in $m_0^{\omega_1}$, and therefore we can find $\phi_0: n^{\omega_1} \rightarrow m_0^{\omega_1}$ such that $\phi_0\sigma = \phi\sigma$ for all $\sigma$. Let $Y = \{y_0, y_1, \ldots\}$ be an infinite subset of $m_0^{\omega_1}$ of type $\tau$, and let $z: n^{\omega_1} \rightarrow \{1, 2, 3, \ldots\}$ be a bijection such that $z < y$ if and only if $b(z) < b(y)$. If $x \in (n + 1)^{\omega_1} \setminus n^{\omega_1}$ there is a unique way to write $x$ in the form $x = u^{-\omega}v$ with $u \in (n + 1)^{\omega_1}$ and $v \in n^{\omega_1}$, by splitting $x$ at the position of the last coordinate equal to $n$. Using this, we can define $\psi: (n + 1)^{\omega_1} \rightarrow m^{\omega_1}$ as

$$
\psi(v) = y_n^{-\omega}(v),
$$

$$
\psi(u^{-\omega}v) = y_{k_0}(u)(v)
$$

where $v \in n^{\omega_1}$, $u \in (n + 1)^{\omega_1}$.

Claim 1: If $X \subset (n + 1)^{\omega_1}$ is a set of type $\sigma$ with $\max(\sigma) < n$, then $\psi(X)$ is a set of type $\sigma$.

Proof of Claim 1: This is clear, because $X$ must be either contained in either $n^{\omega_1}$, in which case $\psi(X) = \phi(X)$, or $X$ is contained in a set of the form $\{u^{-\omega}v : v \in n^{\omega_1}\}$ for some $v \in n^{\omega_1}$, in which case $\psi(X) = \{y_{k_0}(u) - x : x \in X\}$.

**Theorem 4.4.** For $(\tau_i : i \in n) \subset \Sigma_m$ pairwise different, the following are equivalent:

1. $(\tau_i)_{i=1}^n$ dominates $(\tau_{i-1})_{i=1}^n$ for every $k = 1, \ldots, n - 1$.
2. There exists a normal embedding $\phi: n^{\omega_1} \rightarrow m^{\omega_1}$ such that $\phi\sigma = \tau_{\max(\sigma)}$ for every $\sigma \in \Sigma_n$.

**Proof.** That (2) implies (1) follows from repeated application of Lemma 4.3. We prove that (2) implies (1). As a first case, we prove the implication when $n = 2$ and $k = 1$. Thus, we have $\tau_1 \neq \tau_2$ and a normal embedding $\phi: 2^{\omega_1} \rightarrow m^{\omega_1}$ such

1One way to do this is to define $\phi_0(t) = (s_0, \ldots, s_i)$, where $\phi(t) = (n_0, \ldots, n_k), s_i = \min(s_i, m_0 - 1)$.

11If we had $u_i = u_j$ for $i < j$, then the set $\{x_i, x_j\}$ would be equivalent to $\{u_i, u_j\} \subset n^{\omega_1}$, but being $X$ of type $\sigma$, it is also equivalent to $\{x, u^{-\omega}v\}$ for a rung $(u, v)$ of type $\sigma$, and $\max(\sigma) = n$. 

\[\text{Figure 5. The set } \psi(X)\]

\[\text{Figure 6. The set } \psi(X) \text{ after passing to a subsequence.}\]
Proof of the domination theorem

4. DOMINATION

Proof of the domination theorem

Now, for $p, q < \omega$ let $z_{pq} = \max\{t \in B : t < \phi(x_{pq})\}$. We distinguish two cases:

Case 1: There exists $p < \omega$ and $q_1 < q_2 < \cdots$ such that $z_{pq_1} < z_{pq_2} < z_{pq_3} < \cdots$. In this case, $(x_{pq_1}, x_{pq_2}, \ldots)$ has type $\emptyset$, hence $Z = \{\phi(x_{pq_1}), \phi(x_{pq_2}), \ldots\}$ has type $\tau_0$. But each $\phi(x_{pq_i})$ goes out from the chain $B$ at the node $x_{pq_i}$, so these nodes $\phi(x_{pq_i})$ of the set $Z$ are displayed exactly in the same way as shown in Figure 8 (with now $p = p_1 = p_2 = \cdots$). We argue now that actually $Z$ contains a subsequence of type $\tau_1$, and this derives a contradiction since we said that $Z$ has type $\tau_0$ and we supposed that $\tau_0 \neq \tau_1$. The point is that each node $x_{pq_i}$ is a member of some sequence $(x_{pq_i}, q_{ij})$ having property $(*)$, so each node $\phi(x_{pq_i})$ is a node of some set of type $\tau_1$ with underlying branch $B$. Thus, for high enough $t \in B$, the pair $(t \setminus z_{pq_i}, \phi(x_{pq_i}))$ is a rung of type $\tau_1$. In this way, we can construct a subsequence of $Z$ of type $\tau_1$ as desired.

Case 2: For every $p$ there exists an infinite set $Q_p \subset \omega$ such that $z_{pq} = z_{pq_p}$ for all $q, q' \in Q_p$. We denote $z_p = z_{pq_p}$. We can also suppose that $\phi(x_{pq}) > z_p$ for all $q \in Q_p$. The set $Y_p = \{\phi(x_{pq}) : q \in Q_p\}$ is now a set of type $\tau_0$ because it is the image under $\phi$ of a set of type $\emptyset$. Moreover, all elements of $Y_p$ are above $z_p$. The situation is illustrated in Figure 9. Similarly as in Case 1, we know that each $\phi(x_{pq})$ is an element of a set of type $\tau_1$ with underlying branch $B$, so $(t \setminus z_p, \phi(x_{pq}) \setminus z_p)$ is a rung of type $\tau_1$ for every $p, q$ and high enough $t \in B$. We prove now that $\tau_1$ dominates $\tau_0$. Pick $q_1 < q_2 < q_3 < \cdots$ in $Q_p$. We have that $\max(\tau_1^{(3)}) = \max(\phi(x_{pq_1}) \setminus z_p)$, but since $Y_p$ is of type $\tau_0$,

$$\max(\phi(x_{pq_1}) \setminus z_p) \geq \max(\phi(x_{pq_2}) \setminus z_p) = \max(\tau_0),$$

which proves that $\max(\tau_1^{(3)}) \geq \max(\tau_0)$. Finally, we prove that $\tau_1$ is a top-comb type. We know that $(u, v) = (t \setminus z_p, \phi(x_{pq}) \setminus z_p)$ is a rung of type $\tau_1$ for some high enough $t$. Let $k$ be the length of the last critical step of $u$. That is, if $u = u_1 \cdots u_k$ with $u_i \in W_k$ as in Definition 3.6, let $h = |u_2 \cdots u_k|$. We can pick $q_1 \in Q_p$ such that $\phi(x_{pq_1}) > z_p$. Then $(v', v'') = (t \setminus z_p, \phi(x_{pq_1}) \setminus z_p)$ must be again a rung of type $\tau_1$ for high enough $t$, and we made sure that this condition satisfies the top-comb condition as illustrated in Figure 4.

13$\phi$ is one-to-one so there is at most one $q$ such that $\phi(x_{pq}) = z_p$. 

Figure 7. The nodes $x_{pq_0}$ in a sequence with $(*)$.

Figure 8. The nodes $\phi(x_{pq_0})$ as a set of type $\tau_1$ above the branch $B$.

Figure 9. Sets of type $\tau_0$ over a $\tau_1$-set.

Claim A: The branch $B$ does not depend on the choice of the sequences $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$ with property $(*)$ above. Proof of Claim A: Choose different sequences $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$, and consider $X'$ and $B'$ the analogues of the set $X$ and the branch $B$ obtained from this new sequences of integers. Observe that $X$ and $X'$ can be alternated to produce a set of the form

$$Y = \{x_{p_1, q_1}, x_{p_2, q_2}, x_{p_3, q_3}, x_{p_4, q_4}, \cdots\}$$

and the sequence $k_1 < k_2 < \cdots$ can be chosen to grow fast enough so that property $(*)$ is satisfied, and $Y$ is again a set of type $\tau_1$. Then $\phi(Y)$ is a set of type $\tau_1$ again of the form presented in Figure 8 with underlying branch $B$. But $\phi(Y)$ contains both an infinite subsequence contained in $\phi(X)$ and an infinite subsequence contained in $\phi(X')$. This implies that the equality of the underlying branches $B = B' = B'$, and finishes the proof of Claim A.
Proof of the domination theorem

That finished the proof of the case when \( n = 2 \) and \( k = 1 \). For the general case, consider a normal embedding \( \psi : 2^{<\omega} \to n^{<\omega} \) given by \( \psi(t_0, \ldots, t_p) = (k - 1 + t_0, \ldots, k - 1 + t_p) \). Then we can apply the case when \( n = 2 \) and \( k = 1 \) to \( \phi' = \phi \circ \psi, r_0' = r_{k-1} \) and \( r_1' = r_k \).

\[ \square \]

**Corollary 4.5.** If \( \phi : n^{<\omega} \to m^{<\omega} \) is a normal embedding, \( \tau \gg \tau' \) and \( \hat{\phi} \tau \neq \hat{\phi} \tau' \), then \( \hat{\phi} \tau \gg \hat{\phi} \tau' \).

**Corollary 4.6.** Let \( \phi : n^{<\omega} \to m^{<\omega} \) be a normal embedding, \( \tau \) a top-comb type with \( \max(\tau^1) = k \), and suppose that \( \phi \) is not constant equal to \( \hat{\phi} \tau \) on the set of types of maximum at most \( k \). Then \( \hat{\phi} \tau \) is a top-comb type.

**Corollary 4.7.** Let \( M \) be the minimal \( n \)-gap of Corollary 2.4, and let \( \{ S_i : i < n \} \) be pairwise disjoint nonempty families of types in \( m^{<\omega} \). The following are equivalent:

1. \( M \leq \{ \Gamma S_i : i < n \} \).
2. We can pick \( \tau_i \in S_i \) such that \( \tau_0 \ll \tau_1 \ll \cdots \ll \tau_{n-1} \).

5. Subdomination

When we remove from domination the condition of being a top-comb, we obtain the notion of subdomination.

**Definition 5.1.** We say that a type \( \tau \) subdominates another type \( \sigma \), and we will write \( \tau \gg_\sigma \sigma \), if \( \tau = (\tau', \tau^1) \) is a comb type which is not top-comb, and \( \max(\tau^1) \geq \max(\sigma) \).

Lemma 4.3 says that when a type dominates \( \tau \) the range of a normal embedding \( \phi \), then it is possible to define a new normal embedding \( \psi \) whose range equals the range of \( \phi \) plus the type \( \tau \). In this section, we shall see that if \( \tau \) only subdominates the range of \( \phi \), then we can find a normal embedding \( \psi \) whose range contains the range of \( \phi \) plus the type \( \tau \), plus maybe at most five more types, which are formally described in Definition 5.2 and illustrated in Figures 11 and 12.

**Definition 5.2.** Given a comb type \( \tau \) which is not top-comb, we associate to it other comb types:

1. \( \check{\tau} \) is exactly equal to \( \tau \) except that the last element of \( \tau^1 \) is moved to the penultimate position in the order \( \triangleleft \) in order to make \( \check{\tau} \) a comb type.

For example, if \( \tau = [2^{16}_7] \), then \( \check{\tau} = [2^{16}_7] \).
We shall sketch the proof of the results announced at the beginning:

**Theorem 1**

If $\Gamma_0, \ldots, \Gamma_{n-1}$ is an analytic $n$-gap, then $\exists M \subseteq N$:

- $\Gamma_0|_M, \Gamma_1|_M$ form a 2-gap.
- $\Gamma_k|_M = \emptyset$ for all but at most 6 many of the remaining $k$

**Theorem 2**

If $\Gamma_0, \ldots, \Gamma_{n-1}$ is an analytic $n$-gap, then $\exists M \subseteq N$ and $i < j < n$:

- $\Gamma_i|_M, \Gamma_j|_M$ form a 2-gap.
- $\Gamma_k|_M = \emptyset$ for all other $k$
Step 1: We apply our general theorem to the gap \( \{ \Gamma_0, \Gamma_1 \} \)
Sketch of some proofs

Step 1: We apply our general theorem to the gap \( \{\Gamma_0, \Gamma_1\} \)
Step 2: Apply the Ramsey theorem

\[\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[0_1] & \rightarrow \Gamma_b \text{ or } X \\
[1_0] & \rightarrow \Gamma_c \text{ or } X \\
[01_1] & \rightarrow \Gamma_d \text{ or } X \\
[1_01] & \rightarrow \Gamma_e \text{ or } X \\
[0_11] & \rightarrow \Gamma_f \text{ or } X
\end{align*}\]
Step 2: Apply the Ramsey theorem and we have Theorem 1!
Now we go for Theorem 2.
Observe that $[01]$ dominates $[0]$,
Observe that $[0_1]$ dominates $[0]$, so we have $u : 2^{<\omega} \rightarrow 2^{<\omega}$
Observe that $[0_1]$ dominates $[0]$, So if $b \neq 0$ we are done.
Observe that $[01]$ dominates $[0]$, So if $b \neq 0$ we are done.
The same argument works for these other types.

\[
\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[0_1] & \rightarrow \Gamma_0 \text{ or } X \\
[1_0] & \rightarrow \Gamma_c \text{ or } X \\
[01_1] & \rightarrow \Gamma_d \text{ or } X \\
[1_01] & \rightarrow \Gamma_e \text{ or } X \\
[0_11] & \rightarrow \Gamma_f \text{ or } X
\end{align*}
\]
The same argument works for these other types.

\[
\begin{align*}
\Gamma_0 & \quad [0] \\
\Gamma_1 & \quad [1] \\
\Gamma_a \text{ or } X & \quad [01] \\
\Gamma_0 \text{ or } X & \quad [0]_1 \\
\Gamma_0 \text{ or } X & \quad [1]_0 \\
\Gamma_0 \text{ or } X & \quad [0]_1_1 \\
\Gamma_e \text{ or } X & \quad [1]_0_1 \\
\Gamma_0 \text{ or } X & \quad [0]_1_1 
\end{align*}
\]
Sketch of some proofs

But these types also dominate \[1\].

\[
\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[01] & \rightarrow \Gamma_0 \text{ or } X \\
[10] & \rightarrow \Gamma_0 \text{ or } X \\
[01] & \rightarrow \Gamma_0 \text{ or } X \\
[10] & \rightarrow \Gamma_e \text{ or } X \\
[01] & \rightarrow \Gamma_0 \text{ or } X
\end{align*}
\]
But these types also dominate [1]. So if they go to \( \Gamma_0 \), we are done.

\[
\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[0_1] & \rightarrow \Gamma_0 \text{ or } X \\
[1_0] & \rightarrow \Gamma_0 \text{ or } X \\
[01_1] & \rightarrow \Gamma_0 \text{ or } X \\
[1_01] & \rightarrow \Gamma_e \text{ or } X \\
[01_1] & \rightarrow \Gamma_0 \text{ or } X
\end{align*}
\]
But these types also dominate [1]. So if they go to $\Gamma_0$, we are done.

\[\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[01] & \rightarrow \Gamma_0 \text{ or } X \\
[10] & \rightarrow X \\
[01] & \rightarrow X \\
[10] & \rightarrow X \\
[101] & \rightarrow \Gamma_e \text{ or } X \\
[01] & \rightarrow X
\end{align*}\]
So far, we isolated at most four families.

\[
\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[0_1] & \rightarrow \Gamma_0 \text{ or } X \\
[1_0] & \rightarrow X \\
[01_1] & \rightarrow X \\
[1_{01}] & \rightarrow \Gamma_e \text{ or } X \\
[0_{11}] & \rightarrow X
\end{align*}
\]
Now look at the types $[101]$ and $[0]$. 

\[
\begin{align*}
[0] & \rightarrow \Gamma_0 \\
[1] & \rightarrow \Gamma_1 \\
[01] & \rightarrow \Gamma_a \text{ or } X \\
[01] & \rightarrow \Gamma_0 \text{ or } X \\
[10] & \rightarrow X \\
[01] & \rightarrow X \\
[101] & \rightarrow \Gamma_e \text{ or } X \\
[01] & \rightarrow X
\end{align*}
\]
Sketch of some proofs

\[
\max[1_{01}] = 1 \geq 0 = \max[0].
\]
After some painful computation...

Sketch of some proofs

\[ \begin{align*}
0 & \quad \rightarrow \quad 0 \quad \rightarrow \quad \Gamma_0 \\
1 & \quad \rightarrow \quad 1 \quad \rightarrow \quad \Gamma_1 \\
01 & \quad \rightarrow \quad 01 \quad \rightarrow \quad \Gamma_a \text{ or } X \\
0_1 & \quad \rightarrow \quad 0_1 \quad \rightarrow \quad \Gamma_0 \text{ or } X \\
1_0 & \quad \rightarrow \quad 1_0 \quad \rightarrow \quad X \\
01_1 & \quad \rightarrow \quad 01_1 \quad \rightarrow \quad X \\
1_01 & \quad \rightarrow \quad 1_01 \quad \rightarrow \quad \Gamma_e \text{ or } X \\
0_11 & \quad \rightarrow \quad 0_11 \quad \rightarrow \quad X 
\end{align*} \]
After some painful computation... So if $e \neq 0$ we are done

Sketch of some proofs

$\Gamma_0$ or $X$

$\Gamma_1$ or $X$

$\Gamma_a$ or $X$

$\Gamma_e$ or $X$
Now..

\[
\begin{align*}
[0] & \longrightarrow \Gamma_0 \\
[1] & \longrightarrow \Gamma_1 \\
[01] & \longrightarrow \Gamma_a \text{ or } X \\
[0\ 1] & \longrightarrow \Gamma_0 \text{ or } X \\
[1\ 0] & \longrightarrow X \\
[01\ 1] & \longrightarrow X \\
[1\ 01] & \longrightarrow \Gamma_0 \text{ or } X \\
[0\ 1\ 1] & \longrightarrow X
\end{align*}
\]
Sketch of some proofs

Looking similarly at [01], we have...

\[
\begin{align*}
[0] & \rightarrow [0] \rightarrow \Gamma_0 \\
[1] & \rightarrow [1] \rightarrow \Gamma_1 \\
[01] & \rightarrow [01] \rightarrow \Gamma_a \text{ or } X \\
[0_1] & \rightarrow [0_1] \rightarrow \Gamma_0 \text{ or } X \\
[1_0] & \rightarrow [1_0] \rightarrow X \\
[0_{11}] & \rightarrow [0_{11}] \rightarrow X \\
[1_{01}] & \rightarrow [1_{01}] \rightarrow \Gamma_0 \text{ or } X \\
[0_{11}] & \rightarrow [0_{11}] \rightarrow X
\end{align*}
\]
So if \( a \neq 0 \) we are done,
So if \( a \neq 0 \) we are done, and otherwise as well.
Theorem 2

If $\Gamma_0, \Gamma_1, \Gamma_2$ is an analytic 3-gap, then at least two of the following three hold:

1. $\exists M \subset N$: $\{\Gamma_0|_M, \Gamma_1|_M\}$ form a 2-gap but $\Gamma_2|_M = \emptyset$.
2. $\exists M \subset N$: $\{\Gamma_0|_M, \Gamma_2|_M\}$ form a 2-gap but $\Gamma_1|_M = \emptyset$.
3. $\exists M \subset N$: $\{\Gamma_1|_M, \Gamma_2|_M\}$ form a 2-gap but $\Gamma_0|_M = \emptyset$.

Proof: Just check it for each of the 933 minimal analytic 3-gaps.
Theorem 2

If $\Gamma_0, \Gamma_1, \Gamma_2$ is an analytic 3-gap, then at least two of the following three hold:

1. $\exists M \subset N : \{\Gamma_0|_M, \Gamma_1|_M\}$ form a 2-gap but $\Gamma_2|_M = \emptyset$. 
Another sample result

Theorem 2

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- $\exists M \subset N : \{\Gamma_0|_M, \Gamma_1|_M\}$ form a 2-gap but $\Gamma_2|_M = \emptyset$.
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