Mathias forcing and combinatorial covering properties of filters

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Oberwolfach Workshop in Set Theory,
January 13, 2014
A subset $\mathcal{F}$ of $[\omega]^\omega$ is called a filter if $\mathcal{F}$ contains all cofinite sets, is closed under finite intersections of its elements, and under taking supersets.

$M_\mathcal{F}$ consists of pairs $\langle s, F \rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F \rangle$ is stronger than $\langle t, U \rangle$ if $F \subset U$, $s$ is an end-extension of $t$, and $s \setminus t \subset U$.

$M_\mathcal{F}$ is usually called Mathias forcing associated with $\mathcal{F}$.

$M_\mathcal{F}$ is a natural forcing adding a pseudointersection of $\mathcal{F}$: if $G$ is a $M_\mathcal{F}$-generic, then $X = \bigcup \{s : \exists F \in \mathcal{F}(\langle s, F \rangle \in G)\}$ is almost contained in any $F \in \mathcal{F}$.

Applications: killing mad families, making the ground model reals not splitting, etc.
$\mathbb{M}_F$ and dominating reals

Let $x, y \in \omega^\omega$. The notation $x \leq^* y$ means $x(n) \leq y(n)$ for all but finitely many $n$.

$b$ (resp. $d$) is the minimal size of an $\leq^*$-unbounded (resp. dominating) $A \subset \omega^\omega$.

A poset $\mathbb{P}$ is said to add a dominating real if in $V^\mathbb{P}$ there exists $x \in \omega^\omega$ such that $y \leq^* x$ for all ground model $y \in \omega^\omega$.

Example: Laver forcing, Hechler forcing. Miller and Cohen forcing do not add dominating reals.

**Theorem (Canjar 1988)**

$d = c$ implies the existence of an ultrafilter $F$ such that $\mathbb{M}_F$ does not add dominating reals. □

**Definition (Guzman-Hrusak-Martinez)**

A filter $F$ on $\omega$ is called Canjar if $\mathbb{M}_F$ does not add dominating reals.

Let $B$ be an unbounded subset of $\omega^\omega$. A filter $F$ on $\omega$ is called $B$-Canjar if $\mathbb{M}_F$ adds no reals dominating all elements of $B$. □

There is a combinatorial characterization of Canjar filters by Hrusak and Minami in terms of the filter $F^{<\omega}$ on $[\omega]^{<\omega}$ generated by $\{[F]^{<\omega} : F \in F\}$. 


Theorem (Brendle 1998)

1) Every $\sigma$-compact filter is Canjar.
2) $(b = c)$. Let $A$ be a mad family. Then for any unbounded $B = \{b_\alpha : \alpha < b\} \subset \omega^\omega$ such that $b_\alpha \leq^* b_\beta$ for all $\alpha < \beta$, there exists a $B$-Canjar $\mathcal{F} \supset \mathcal{F}_A$.

If an ultrafilter $\mathcal{F}$ is Canjar, then it is a $P$-filter and there is no monotone surjection $\varphi : \omega \to \omega$ such that $\varphi(\mathcal{F})$ is rapid. The converse is consistently not true by a recent result of Blass, Hrusak and Verner. Its proof relies on the following characterization

Theorem (Guzman-Hrusak-Martinez 2013; Blass-Hrusak-Verner 2011 for ultrafilters)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong $P^+$-filter.

Recall that a filter $\mathcal{F}$ is a coherent strong $P^+$-filter if for every sequence $\langle C_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in C_n$ for all $n$ and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1}) \subset \text{for } n < m$, then $\bigcup_{n \in \omega} (X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$. Strong $P^+$-filters are defined by removing the coherence requirement.
A topological space $X$ has the Menger covering property (or simply is Menger), if for every sequence $\langle U_n : n \in \omega \rangle$ of open covers of $X$ there exists a sequence $\langle V_n : n \in \omega \rangle$ such that $V_n \in [U_n]^\omega$ and $\bigcup V_n : n \in \omega \}$ is a cover of $X$.

If, moreover, we can choose $V_n$ in such a way that for any $x \in X$ we have $x \in \bigcup V_n$ for all but finitely many $n \in \omega$, then $X$ is called Hurewicz.

\[ \square \]

Example: every $\sigma$-compact space is Hurewicz. More generally: a union of fewer than $b$ (resp. $\mathfrak{d}$) compacts is Hurewicz (resp. Menger).

$\omega^\omega$ is not Menger as witnessed by $\langle U_n : n \in \omega \rangle$, $U_n = \{ \{x : x(n) = k\} : k \in \omega \}$. 
Main results

Theorem (Chodounský-Repovš-Z. 2013)
\( \mathbb{M}_F \) is Canjar iff \( F \) has the Menger covering property as a subspace of \( \mathcal{P}(\omega) \).

Theorem (Chodounský-Repovš-Z. 2013)
Let \( F \) be a filter. Then \( \mathbb{M}_F \) is almost \( \omega^\omega \)-bounding iff \( F \) is \( B \)-Canjar for all unbounded \( B \subset \omega^\omega \) iff \( F \) is Hurewicz.
Some corollaries

Corollary

Let $\mathcal{F}$ be an analytic filter on $\omega$. Then $\mathbb{M}_\mathcal{F}$ does not add a dominating real iff $\mathcal{F}$ is $\sigma$-compact.

Answers a question of Hrusak and Minami. For Borel filters has been independently proved by Guzman, Hrusak, and Martinez.

Corollary (Hrušák-Martínez 2012)

There exists a mad family $A$ on $\omega$ such that $\mathbb{M}_\mathcal{F}(A)$ adds a dominating real ($= \mathcal{F}(A)$ is not Canjar).

Answers a question of Brendle.

Corollary

($\mathfrak{d} = \mathfrak{c}$.) There exists a mad family $A$ on $\omega$ such that $\mathbb{M}_\mathcal{F}(A)$ does not add a dominating real ($= \mathcal{F}$ is Canjar).

Under $\mathfrak{d} = \mathfrak{c} = \mathfrak{u}$ it was proved by Guzman, Hrusak, and Martinez.

Corollary

A filter $\mathcal{F}$ is Canjar iff it is a strong $P^+$-filter.
Theorem (Guzman-Hrusak-Martinez 2013)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong $P^+\text{-filter}$.

Recall that a filter $\mathcal{F}$ is a coherent strong $P^+\text{-filter}$ if for every sequence $\langle C_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in C_n$ for all $n$ and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1}) \subset$ for $n < m$, then $\bigcup_{n \in \omega} (X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.

Strong $P^+\text{-filters}$ are defined by removing the coherence requirement.
A Tychonov space $X$ is called a $\gamma$-space if $C_p(X)$, the space of continuous functions from $X$ to $\mathbb{R}$ with the topology inherited from $\mathbb{R}^X$, has the Fréchet-Urysohn property.

For $a \in [\omega]^{<\omega}$ and $n \in \omega$, $a(n)$ denotes the $n$-th element in the increasing enumeration of $a$. For $a, b \in [\omega]^{<\omega}$, $a \leq^* b$ means that $a(n) \leq b(n)$ for all but finitely many $n$.

A $b$-scale is an unbounded set $S = \{s_\alpha : \alpha < b\}$ in $([\omega]^{<\omega}, \leq^*)$ such that $s_\alpha \leq^* s_\beta$ for $\alpha < \beta$. It is easy to see that $b$-scales exist in ZFC. For each $b$-scale $S$, $S \cup [\omega]^{<\omega}$ is $b$-concentrated on $[\omega]^{<\omega}$ in the sense that $|S \setminus U| < b$ for any open $U \supset [\omega]^{<\omega}$.

For brevity, the union of a $b$-scale with $[\omega]^{<\omega}$, viewed as a subset of the Cantor space $\mathcal{P}(\omega)$, will be called $b$-scale set.
Applications to $\gamma$-spaces and scales, continued

Easy: every second countable space of size $< p$ is a $\gamma$-space.

**Theorem (Galvin-Miller 1984)**

*Under $p = c$ there exists a $b$-scale set which is a $\gamma$-space.*

Their $b$-scale was a tower, where $S = \{ s_\alpha : \alpha < \kappa \} \subset [\omega]^{\omega}$ is called a tower if $s_\alpha \subseteq^* s_\beta$ for all $\beta < \alpha$.

**Theorem (Orenshtein-Tsaban 2011)**

*If $p = b$ then any $b$-scale set is a $\gamma$-space provided that it is a tower.*

On the other hand, we have

**Theorem (Reповš-Tsaban-Z. 2008)**

*Under $b = c$ there exists a $b$-scale set which fails to be a $\gamma$-space.*

**Corollary**

*If $p = c$, then some $b$-scale sets are $\gamma$-spaces, and some are not.*
γ-spaces have strong measure zero.

**Corollary (Laver 1976)**

> It is consistent with ℵ = ℵ₀ that no ℵ-scale set is a γ-space.

The following result completes the picture of the relations between ℵ-scale sets and γ-spaces, and answers a question of Tsaban.

**Theorem (Chodounský-Repovš-Z. 2013)**

> It is consistent with ZFC that every ℵ-scale set is a γ-space.
Questions

Question

Let $\mathcal{A} \subseteq [\omega]^\omega$ be a mad family. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{F}(\mathcal{A})$?

Question

(CH) Let $\mathcal{U}$ be a meager filter generated by a tower. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{U}$?
Thank you for your attention.