Transcending $\omega_1$-sequences of reals

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Observe that if $\langle a_\xi : \xi < \omega_1 \rangle$ is a $\diamond$-sequence, then $g(\delta) = F(a_\delta)$ defines a witness $g$ in the above theorem.
Devlin’s coding

Consider the following statement (U):

\[ \text{for every ladder system } \langle C_\alpha : \alpha \in \text{lim}(\omega_1) \rangle \text{ and every } g : \omega_1 \to 2, \]
\[ \text{there is an } f : \omega_1 \to 2 \text{ such that for all limit ordinals } \delta, \]
\[ f \upharpoonright C_\delta \equiv^* g(\delta). \]

It is not difficult to show that for any \( \vec{C} \) and any \( g \), the collection

of countable approximations to an \( f \) satisfying the conclusion of \((U)\) is proper and does not add new reals.

On the other hand, by Devlin-Shelah, these forcings cannot be iterated to obtain a model of \((U) + CH\).

Shelah isolated the problematic feature of this forcing which is responsible for the addition of new reals at limit stages of the iteration. He moreover formulated a general condition on forcings—(essentially) complete properness— which avoids this pathology.
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Shelah’s iteration theorem

Theorem (Shelah)

Let $\langle P_\alpha; \dot{Q}_\alpha : \alpha < \theta \rangle$ be a countable support iteration of forcings which are:

1. completely proper;
2. $\alpha$-proper for every $\alpha < \omega_1$.

Then $P_\theta$ does not add new reals.

Many forcings can be shown to satisfy these two hypotheses. For instance there is a forcing to specialize an Aronszajn tree which satisfies these conditions.
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The role of $< \omega_1$-properness?

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Additionally, a seemingly unrelated hypothesis — that the forcing remains proper in all proper forcing extensions with the same reals — can be substituted for “$< \omega_1$-properness” in Shelah’s iteration theorem.
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Shelah has constructed an example which shows that $< \omega_1$-properness can not be eliminated entirely from the iteration theorem by constructing a counterexample iteration in $L$. However:

- The counterexample does not correspond to a consequence of CH.
- The construction does not work in the presence of a measurable cardinal.
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- The counterexample does not correspond to a consequence of CH.
- The construction does not work in the presence of a measurable cardinal.
The main result

Theorem

(M.) Assume CH. There is a tree $T$ of countable closed subsets of $\omega_1$, ordered by end extensions such that:

- if $s, t \in T$ and $\lim(s) \cap \lim(t)$ are cofinal in some limit ordinal $\delta$, then $s \cap \delta = t \cap \delta$;
- every level in $T$ is predense when $T$ is regarded as a forcing;
- $T$ is completely proper as a forcing and remains so in any outer model with the same reals in which $T$ has no uncountable branch.

The first condition implies that $T^2 \setminus \Delta = \{(s, t) \in T^2 : ht(s) = ht(t) \land s \neq t\}$ is a countable union of antichains. In particular, $T$ can have at most one uncountable branch; if $T$ has an uncountable branch, then forcing with $T$ collapses $\omega_1$.
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What the proof really gives

Suppose \( r : \omega_1 \to \mathbb{R} \) is an injection. To any such \( r \) and club \( E \subseteq \omega_1 \), there is a \( \Sigma_0 \)-definable \( T^r_E \) (in the parameters \( r \) and \( E \)) which satisfies all properties specified above, except that \( T^r_E \) may fail to be completely proper if \( T^r_E \) contain an uncountable branch. Furthermore, there is an \( E \in L[r] \) such that \( T^r_E \) does not contain an uncountable branch which is in \( L[r] \). Define \( E_0 \) to be the \( L[r] \)-least such club. Define \( E_\xi \) recursively so that if \( T^r_{E_\xi} \) has an uncountable branch, then \( E_{\xi + 1} \) is the union of that branch and \( E_\eta = \bigcap_{\xi < \eta} E_\xi \) if \( \eta \) is a limit ordinal.
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Let $\gamma(r)$ be the greatest ordinal $\gamma$ such that $E_\gamma$ is defined. Then one of the following is true:

• $\gamma(r) < \omega_2$, $T_r E_{\gamma(r)}$ is completely proper, and yet has no uncountable branch.

• $\gamma(r) < \omega_2$ and any diagram witnessing that $T_r E_{\gamma(r)}$ is not completely proper fails to be in $L[r]$.

• $\gamma(r) = \omega_2$ and $\langle E_\xi \cap \delta : \xi \in \omega_2 \rangle$ is not in $L[r]$ whenever $\delta \in E_{\gamma(r)}$.

Moreover, if $r$ enumerates $R$, only the first item can occur.
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