Weak square and the failure of SCH

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Joint work with Dima Sinapova

1. The singular cardinals hypothesis

2. Trees, weak squares and scales

3. A question of Woodin
**Theorem (Easton)**

Given a model $V$ of GCH and a function $F$ in $V$ with inputs the $V$ regular cardinals and outputs $V$ cardinals satisfying:

- $\kappa \leq \lambda \rightarrow F(\kappa) \leq F(\lambda)$,
- $F(\kappa) > \kappa$ and
- $\mathrm{cf}(F(\kappa)) > \kappa$.

There is a cardinal preserving generic extension $V[G]$ such that

$$V[G] \vDash 2^\kappa = F(\kappa)$$
What about the continuum function for singular cardinals?

**Definition**

Let $\mu$ be a singular cardinal. The *singular cardinals hypothesis* (SCH) at $\mu$ is the assertion "$\mu$ strong limit implies $2^\mu = \mu^+$."

Note that every singular cardinal in Easton’s model satisfies the singular cardinals hypothesis.
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The singular cardinals hypothesis

Theorem (Gitik)

The assertion "There is a singular cardinal $\mu$ at which SCH fails" is equiconsistent with "There is a measurable cardinal $\kappa$ with $\omega(\kappa) = \kappa^{++}$."

Theorem (Shelah)

If $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{\omega^4}$.  

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The height of an element $t$ is the order-type of the collection of the predecessors of $t$ under $<_T$. That is, the unique ordinal $\alpha$ such that $(\alpha, \in) \simeq (\{x \in T \mid x <_T t\},<_T)$. 

The $\alpha$th level of the tree is the collection of nodes of height $\alpha$.

The height of a tree $T$ is the least ordinal $\beta$ such that there are no nodes of height $\beta$. 

A set $b$ is a cofinal branch through $T$ if $b \subseteq T$ and $(b,<_T)$ is a linear order whose order-type is the height of the tree.
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The tree property and weak square

Definition
A regular cardinal $\kappa$ has the tree property if every tree of height $\kappa$ with levels of size less than $\kappa$ has a cofinal branch. A counter-example to the tree property at $\kappa$ is called a $\kappa$-Aronszajn tree.

Theorem (Tarski and Keisler)
A cardinal $\kappa$ is weakly compact if and only if it is inaccessible and has the tree property.

Theorem (Jensen)
There is a special $\mu^+$-Aronszajn tree if and only if the principle weak square at $\mu$ $(\Box^* \mu)$ holds.
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- There are also weakenings of weak square known as the *Approachability property* and the assertion “All scales are good”.
- These weakenings are decreasing in strength.
Question

Is it consistent that the tree property holds at $\aleph_{\omega+1}$ and the singular cardinals hypothesis fails at $\aleph_\omega$?

Remark

The question is still open if we weaken it to ask for the failure of $\square^*_{\aleph_\omega}$.

Theorem (Joint with Sinapova)

Starting from a model $V$ with a supercompact cardinal, there is a generic extension in which SCH fails at $\aleph_\omega$ and $\square_{\aleph_\omega}$, $\aleph_n$ fails for all $n < \omega$.

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A typical scheme to get the failure of SCH.

- Start with a large cardinal $\kappa$.
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**Theorem (Gitik and Sharon)**

*Given a model $V$ with a supercompact cardinal $\kappa$, there is a generic extension in which $\kappa$ is singular strong limit of cofinality $\omega$, $\kappa^+ = (\kappa + \omega + 1)^V$, $2^\kappa = \kappa^{++}$ and $\square^*_\kappa$ fails.*