PROVIDENT SET THEORY

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[MB] A. R. D. Mathias and N. J. Bowler, Rudimentary recursion, gentle functions and provident sets,

[M4] A. R. D. Mathias, Set forcing over models of Zermelo or Mac Lane,
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Rudimentary functions
\[ R_0(x, y) = \{x, y\} \]
\[ R_1(x, y) = x \setminus y \]
\[ R_2(x) = \bigcup x \]
\[ R_3(x) = \text{Dom}(x) \]
\[ R_4(x, y) = x \times y \]
\[ R_5(x) = x \cap \{(a, b) \mid a, b \in b\} \]
\[ R_6(x) = \{(b, a, c) \mid a, b, c \mid (a, b, c) \in x\} \]
\[ R_7(x) = \{(b, c, a) \mid a, b, c \mid (a, b, c) \in x\} \]
\[ A_{14}(x, y) = x^{\{y\}} \left[= \text{Dom}((x \cap ([\bigcup \bigcup x] \times \{y\}))^{-1})\right] \]
\[ R_8(x, y) = \{x^{\{w\}} \mid w \in y\} \]
**Definition** Let $\mathcal{B}$ be the closure of $R_0 \ldots R_7$ under composition.

**Proposition** For each $\Delta_0$ class $A$ the separator $x \mapsto x \cap A$ is in $\mathcal{B}$.

**Definition** Let $\mathcal{R}$ be the closure of $R_0 \ldots R_8$ under composition.

**Proposition** If $F$ is in $\mathcal{R}$ so is $x \mapsto F^{-1}(x) =_{df} \{F(y) | y \in x\}$.

In words: the collection $\mathcal{R}$ of **rudimentary functions** is closed under formation of images.
$$T(u) \triangleq u \cup \{u\} \cup [u]^1 \cup [u]^2$$

$$\cup \{x \setminus y \mid x, y \in u\}$$

$$\cup \{\bigcup x \mid x \in u\}$$

$$\cup \{\text{Dom}(x) \mid x \in u\}$$

$$\cup \{u \cap (x \times y) \mid x, y \in u\}$$

$$\cup \{x \cap \{(a, b) \mid a, b \in b\} \mid x \in u\}$$

$$\cup \{u \cap \{(b, a, c) \mid a, b, c \in x\} \mid x \in u\}$$

$$\cup \{u \cap \{(b, c, a) \mid a, b, c \in x\} \mid x \in u\}$$

$$\cup \{x\{w\} \mid x, w \in u, w \in u\}$$

$$\cup \{u \cap \{x\{w\} \mid w \in y\} \mid x, y \in u\}.$$
**Proposition**  \( \mathcal{T} \) is rudimentary, \( u \subseteq \mathcal{T}(u) \) and \( u \in \mathcal{T}(u) \). Further, if \( u \) is transitive, then \( \mathcal{T}(u) \) is a set of subsets of \( u \), and hence \( \mathcal{T}(u) \) is transitive.

**Remark**  It will not in general be true that \( u \subseteq v \implies \mathcal{T}(u) \subseteq \mathcal{T}(v) \), the problem being that \( u \in \mathcal{T}(u) \), but if \( v \) is countably infinite, so is \( \mathcal{T}(v) \) which therefore cannot contain all the subsets of \( v \). Fortunately, \( u \subseteq \mathcal{T}(u) \subseteq \mathcal{T}^2(u) \ldots \)

**Proposition**  For any transitive \( u \), \( \bigcup_{n \in \omega} \mathcal{T}^n(u) \) is the rudimentary closure of \( u \cup \{ u \} \) and models \( \mathcal{T}\text{Co} \).
**Proposition** If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$, denoted by $c_F$, such that for all transitive $u$, if each argument in $\vec{x}$ is in $u$, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.

**Proof:** The stated property holds of the nine generating functions and is preserved under composition. ⊥

**Corollary** (Gandy; Jensen) If $F$ is rudimentary, then there is a finite $\ell$ such that the rank of the value is at most the maximum of the ranks of the arguments, plus $\ell$.

**Proof:** the function $\mathbb{T}$ increases rank by exactly 1. ⊥
Rudimentary recursion
Many set-theoretic functions are defined by recursions of the form

\[ F(x) = G(F \upharpoonright x) \]

For example, the \( \Sigma_1 \) recursion theorem of Kripke-Platek set theory KP proves that if \( G \) is a total \( \Sigma_1 \) function, so is the function \( F \).

If the defining function \( G \) is rudimentary in the sense of Jensen, we shall speak of \( F \) as given by a \textit{rudimentary recursion}, or, more briefly, that \( F \) is \textit{rud rec}.

In favourable cases we may also use this terminology when \( F \) is intended to be a function defined on \( On \) rather than on \( V \), or defined by recursion on other well-founded relations related to the epsilon relation.
Some rudimentary recursions

**Example** \( \varrho(x) = \bigcup \{ \varrho(y) + 1 \mid y \in x \} \)

**Example** \( \text{tcl}(x) = x \cup \bigcup \{ \text{tcl}(y) \mid y \in x \} \)

**Example** Let \( S(x) \) be the set of finite subsets of \( x \). *Restricted to ordinals*, this has a rudimentarily recursive definition:

\[
S(0) = \{ \emptyset \}; \quad S(\zeta + 1) = S(\zeta) \cup \{ x \cup \{ \zeta \} \mid x \in S(\zeta) \}; \quad S(\lambda) = \bigcup_{\nu < \lambda} S(\nu).
\]

*Restricted to the hereditarily finite sets*, it hasn’t.
The constructible universe

**Definition** \[ T(x) = \bigcup_{y \in x} T(T(y)) \]

**Remark** \( T(x) \) always equals \( T_\varnothing(x) \), where

\[ T_0 = \varnothing; \quad T_{\nu+1} = T(T_\nu); \quad T_\lambda = \bigcup_{\nu < \lambda} T_\nu \]

which can be said in one breath as

\[ T_\zeta = \bigcup_{\nu < \zeta} T(T_\nu). \]

Then \( L = \bigcup_{\nu \in ON} T_\nu \), and \( J_\nu = T_{\omega \nu} \); but \( \nu \mapsto \omega \nu \) is not rud rec.
Rudimentary recursion from parameters

Let \( p \) be a set. We call a unary function \( F \) \( p\)-rud rec if there is a binary rudimentary \( G \) such that for all \( x \),

\[
F(x) = G(p, F \upharpoonright x).
\]

**Example** Ordinal addition is given by the recursion

\[
A(\alpha, 0) = \alpha; \ A(\alpha, \beta + 1) = A(\alpha, \beta) + 1; \ A(\alpha, \lambda) = \bigcup_{\nu < \lambda} A(\alpha, \nu)
\]

For each \( \alpha \) that is an \( \alpha \)-rud recursion on the second variable \( \beta \).

**Remark** If \( F \) is rud rec (in a parameter), so is \( x \mapsto F \upharpoonright x \) (in the same parameter).
Gentle functions
A *gentle* function is one of the form $H \circ F$ where $H$ is rudimentary and $F$ is rud rec. The importance of the notion lies in the results of Nathan Bowler, that while the collection of rud rec functions is *not* closed under composition, the slightly larger collection of gentle functions *is*.

An alternative definition is this: $F_1$ is *gentle* iff there are $0 < \ell < \omega$ and $\ell$-ary rudimentary functions $G_1, \ldots, G_\ell$ such that for each $i \in [1, \ell]$ and $x \in V$, $F_i(x) = G_i(F_1 \upharpoonright x, \ldots, F_\ell \upharpoonright x)$.

Thus $F_1$ is a projection of the rud rec function

$$x \mapsto (F_1(x), F_2(x), \ldots, F_\ell(x))_\ell.$$
**Definition** (Scott, McCarty)

\[ \langle x, y \rangle_2^{SM} = \{ \langle 0, t \rangle_2^{SM} \mid t \in x \} \cup \{ \langle 1, u \rangle_2^{SM} \mid u \in y \} \]

Consider these six definitions:

\[
\begin{align*}
\tau(y) &= \{ \emptyset \} \cup \{ \tau(u) \mid u \in y \}; & \phi(y) &= \{ \phi(u) \mid u \in y \land \emptyset \in u \}; \\
\sigma(x) &= \{ \sigma(t) \cup \{ \emptyset \} \mid t \in x \}; & \psi(y) &= \{ \psi(u \setminus \{ \emptyset \}) \mid u \in y \};
\end{align*}
\]

\(\text{left}^{SM}(a) =_{df} \psi``(a \cap \{ d \mid d \emptyset \notin d \}); \quad \text{right}^{SM}(a) =_{df} \phi``(a \cap \{ c \mid c \emptyset \in c \}).\)

**Remark** \(\tau, \sigma\) and \(\phi\) are pure rud rec; \(\psi\), \(\text{left}^{SM}\) and \(\text{right}^{SM}\) are gentle; \(\langle \cdot, \cdot \rangle_2^{SM}\) is a composite of gentle functions.

**Proposition** (Scott, McCarty) \(\langle x, y \rangle_2^{SM} =_{df} \sigma``x \cup \tau``y\)

**Lemma** Let \(a = \langle x, y \rangle_2^{SM}\): then \(\text{left}^{SM}(a) = x\) and \(\text{right}^{SM}(a) = y\).
**PROPOSITION** (Bowler) The function $H : x \mapsto \begin{cases} \omega & \text{if } \varrho(x) = \omega \\ 0 & \text{otherwise} \end{cases}$ is a composite of a rud function with a rud rec function, but is not rud rec.

**Proof:** $H$ is the composite $\delta_\omega \circ \varrho$, where $\delta_\omega : x \mapsto \begin{cases} \omega & \text{if } x = \omega \\ 0 & \text{otherwise} \end{cases}$

For any unary rud $G$, suitable $\ell(= c_G)$ and transitive $x$,

$$\bigcup^{\ell+1} G(x) \subseteq \bigcup^{\ell+1} G''(x \cup \{x\}) \subseteq \bigcup^{\ell+1} T^\ell(x \cup \{x\}) = \bigcup (x \cup \{x\}) = x.$$

Suppose that $H$ were rud rec, given by $G_0$ say. Let $\ell = c_G$ where $G$ is the rud function $G : y \mapsto G_0(\{0\} \times y)$. Let $Z$ be the transitive set, of rank $\omega$, of Zermelo integers: $Z = \{ s^n(\varnothing) \mid n \in \omega \}$, where $s : x \mapsto \{x\}$. Then

$$\omega = \bigcup^{\ell+1} \omega = \bigcup^{\ell+1} H(Z) = \bigcup^{\ell+1} (G_0(H \upharpoonright Z)) = \bigcup^{\ell+1} (G_0(\{0\} \times Z)) = \bigcup^{\ell+1} (G(Z)) \subseteq Z — a falsehood !$$
Provident sets
**Definition** A set $A$ is $p$-*provident*, where $p$ is a set, if it is non-empty, transitive, closed under pairing and for all $p$-rud rec $F$ and all $x$ in $A$, $F(x) \in A$.

**Remark** If $A$ is $p$-provident, $p \in A$.

**Example** The Jensen fragment $J_{\nu}$ is $\emptyset$-provident for all $\nu \geq 1$.

**Definition** $A$ is *provident* if it is $p$-provident for every $p \in A$.

**Example** Each $J_{\omega \nu}$ is provident.

**Remark** For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x, y\}$ is $x$-rud rec, being given by the recursion $F(y) = \{x, \text{Dom } F \upharpoonright y\}$.
Bounding rudimentary functions in a finite progress

**Definition** A $\xi$-progress is a sequence $\langle P_\nu \mid \nu < \xi \rangle$ of transitive sets such that for each $\nu$ with $\nu + 1 < \xi$, $T(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda < \xi$, $\bigcup_{\nu < \lambda} P_\nu \subseteq P_\lambda$.

The progress is *strict* if for each $\nu$ with $\nu + 1 < \xi$, $P_{\nu+1} \subseteq P(\nu)$; *continuous* if for each limit $\lambda < \xi$, $P_\lambda = \bigcup_{\nu < \lambda} P_\nu$; and *solid* if it is strict and continuous and $P_0 = \emptyset$.

**Proposition** If the progress is strict and continuous then for each $\nu \leq \xi$, $\varrho(P_\nu) = \varrho(P_0) + \nu$.

**Theorem** Let $R$ be a rudimentary function of $n$ variables. There is a $c_R \in \omega$ such that for every $c_R$-progress $P_0, P_1, \ldots, P_{c_R}$,

$$R^{\langle P^n_0 \rangle} \subseteq P_{c_R}.$$
The canonical progress towards a given transitive set

Let $c$ be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since $c$ is transitive, $c_{\zeta+1}$ will be a set of subsets of $c_\zeta$; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$; we shall use this as a direct recursive definition below.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using $c$ as a parameter we define a sequence of pairs $((c_\nu, P_c^\nu))_\nu$ by a rud recursion on $\nu$. Each $P_c^\nu$ will be of rank $\nu$; we shall use the function $T$, but we shall also “feed” stages of $c$ into the process.
**Definition**

\[ c_0 = \emptyset \quad c_{\nu + 1} = c \cap \{x \mid x \subseteq c_\nu\} \quad c_\lambda = \bigcup_{\nu < \lambda} c_\nu \]

\[ P_0^c = \emptyset \quad P_{\nu + 1}^c = \mathbb{T}(P_\nu^c) \cup \{c_\nu\} \cup c_{\nu + 1} \quad P_\lambda^c = \bigcup_{\nu < \lambda} P_\nu^c \]

**Lemma** Each \( P_\nu^c \) is transitive; \( P_\nu^c \subseteq P_{\nu + 1}^c \); \( P_\nu^c \in P_{\nu + 1}^c \); and so for \( \nu < \zeta \), \( P_\nu^c \subseteq P_\zeta^c \) and \( P_\nu^c \in P_\zeta^c \). \( c_\nu = c \cap P_\nu^c \); \( \varrho(P_\nu^c) = \nu \).

**Remark** \( P_\nu^c \) may be defined by a single rud recursion on ordinals:

\[ P_0^c = \emptyset; \quad P_{\nu + 1}^c = \mathbb{T}(P_\nu^c) \cup \{c \cap P_\nu^c\} \cup (c \cap \{x \mid x \subseteq P_\nu^c\}); \quad P_\lambda^c = \bigcup_{\nu < \lambda} P_\nu^c. \]

**Remark** Each \( P_\lambda^c \) is rud closed, for \( \lambda \) a limit ordinal; \( P_\omega^c = V_\omega \).
**Proposition** Let $A$ be a provident set, and write $\theta(A)$ for the least ordinal not in $A$.

$A$ is rud closed;

$A$ contains the rank $\varrho(x)$ of each member $x$ of $A$;

$A$ contains the transitive closure of each of its members;

$\theta(A)$ is indecomposable;

$\theta(A) = \varrho(A)$;

$A = \bigcup\{P_{\theta(A)}^d \mid d \cup d \subseteq d \in A\}$.

**Proposition** Let $\theta$ be an indecomposable ordinal, and let $(Q_\nu)_{\nu \leq \theta}$ be a $\theta$-progress with $Q_\theta = \bigcup_{\nu < \theta} Q_\nu$. Then $Q_\theta$ is provident.
**Theorem** If $\theta$ is an indecomposable ordinal and $C$ is a set of transitive sets such that any two members of $C$ are members of a third, then $B = \text{df} \bigcup_{c \in C} P_\theta^c$ is provident. More generally, the union of a directed system of provident sets is provident.

**Proof:** Given a parameter $p$ in $B$ and an argument $x$ in $B$, choose $c \in C$ with both $p$ and $x$ in $P_\theta^c$. We know that $P_\theta^c$ is provident, and so if $F$ is $p$-rud rec, $F(x)$ is in $P_\theta^c$ and therefore in $B$. $\dashv$
Provident levels of the Jensen and Gödel hierarchies

**Proposition**  If $u$ is transitive and $\emptyset$-provident then so is $\text{rud}(u)$.

*Proof:* We take $P_n = T^n(u)$, and $P_{\omega} = \bigcup_n P_n$. $\langle P_\nu \mid \nu \leq \omega \rangle$ is then a strict continuous $\omega$-progress, so we may apply a previous proposition with $p = \emptyset$. ⊣

**Corollary**  Each non-empty $J_\nu$ is $\emptyset$-provident,

**Theorem**  $J_\nu$ is provident iff $\omega \nu$ is indecomposable. More generally, if $c$ is a transitive set, $J_\nu(c)$ will be provident iff $\omega \nu$ is indecomposable and strictly greater than the rank of $c$. 
**Remark** We need $\omega \nu$ to exceed the rank of $c$, as provident sets contain the ranks of their members.

**Remark** So although for a given $p$ in $L$ we must go to the first indecomposable ordinal above the moment of construction of $p$ to find a $J_\nu$ which is $p$-provident, every subsequent $J_\xi$ will also be $p$-provident.

**Proposition** $J_\omega$ is provident. The next one will be $J_{\omega^2}$.

**Proposition** Each $L_\lambda$ is $\emptyset$-provident for limit $\lambda$.

**Proposition** $L_\lambda$ is provident iff $\lambda$ is indecomposable.
Provident closures
Theorem Suppose that $M$ is a non-empty set. Let $\theta$ be the least indecomposable ordinal not less than $\varrho(M)$. Set

$$\text{Prov}(M) = \text{df} \bigcup \{ P_\theta^{\text{tcl}(s)} \mid s \in S(M) \}.$$ 

Then $\text{Prov}(M)$ is provident and includes $M$, and if $P$ is any other such, $\text{Prov}(M) \subseteq P$.

Here $S(M)$ denotes, as before, the set of finite subsets of $M$.

Definition We call $\text{Prov}(M)$ the \textit{provident closure} of $M$.

The theories PROV and PROVI

There is a finitely axiomatisable set theory (which we call PROV) of which the transitive models are the provident sets.
Its axioms are extensionality and the thirteen axioms

\[
\emptyset \in V \quad \bigcup x \in V \quad \quad a \cap \{ (x, y) \mid x \in y \} \in V
\]
\[
\{x, y\} \in V \quad \text{Dom}(x) \in V \quad \quad \{ (y, x, z) \mid (x, y, z) \in b \} \in V
\]
\[
x \setminus y \in V \quad x \times y \in V \quad \quad \{ (y, z, x) \mid (x, y, z) \in c \} \in V
\]
\[
\{x\{w\} \mid w \in y\} \in V
\]

each set is in the domain of an attempt at the rank function; (whence both TCo and set foundation)

any two ordinals are in the domain of an attempt at ordinal addition;

for each transitive \( c \) each ordinal is in the domain of an attempt at the sequence \( \langle P^c \nu \mid \nu \in ON \rangle \).

We write PROVI for PROV + \( \omega \in V \).
François Dorais has established the following “reversals”:

Let $HC$ denote the statement that every set is countable.

Then $PROVI + HC$ is bi-interpretable with $ACA_0^+$;

and $PROVI + HC +$ Mostowski Collapse is bi-interpretable with $ATR_0$.

**Remark** Experience of the weak systems in [M3] suggests that if one wished to use $PROVI$ for syntactical reasoning, it would be desirable to enhance it by adding the axiom of infinity and the scheme of $\Pi_1$ foundation. The result will still be finitely axiomatisable in a subtle sense.
Set forcing over provident sets
**Example** Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member $y$ of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} = \text{df} \ (\mathbb{1}^\mathbb{P}, \hat{x}) | x \in y$$

This is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbb{1}^\mathbb{P}, F \upharpoonright a)$$

where $G$ is the rudimentary function $(\mathbb{1}^\mathbb{P}, a) \mapsto \{\mathbb{1}^\mathbb{P}\} \times \text{Im}(a)$. 
**Example** If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model $M$, and we follow Shoenfield in treating all members of $M$ as $\mathbb{P}$-names, the function $\text{val}_\mathcal{G}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$ \text{val}_\mathcal{G}(b) = \text{df} \ \{ \text{val}_\mathcal{G}(a) \mid \exists p \in \mathcal{G}(p, a) \in b \} $$

The generic extension $M[\mathcal{G}]$ will then be defined as

$$ \{ \text{val}_\mathcal{G}(a) \mid a \in M \}. $$

**Remark** Note that the definition of the forcing relation $\models$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has interesting properties.
Forcing in provident sets

**Definition** \( p \parallel_0 a \in b \iff (p, a) \in b. \)

\( \parallel_0 \) is our first approximation to the relation \( \vdash. \)

**Lemma** If \( p \parallel_0 a \in b \) then \( a \in \bigcup \bigcup b. \)

**Definition** In future we shall write \( \bigcup^2 x \) for \( \bigcup \bigcup x. \)

**Definition** \( p \parallel_1 a \in b \iff \exists q \in \bigcup^2 b [q \geq p \& (q, a) \in b]. \)

**Lemma** If \( p \parallel_1 a \in b \) and \( r \leq p \) then \( r \parallel_1 a \in b. \)

This last statement shows that \( \parallel_1 \) improves \( \parallel_0 \) and starts to resemble a forcing relation.
**Definition** \[ p \models b = c \iff \text{df} \]

\[
\forall \beta \in \bigcup^2 b \quad \forall r \leq p \\
\quad \left[ r \models_1 \beta \in b \Rightarrow \exists t \leq r \exists \gamma \in \bigcup^2 c \left( t \models \beta = \gamma \land t \models_1 \gamma \in c \right) \right] \quad \& \\
\forall \gamma \in \bigcup^2 c \quad \forall r \leq p \\
\quad \left[ r \models_1 \gamma \in c \Rightarrow \exists t \leq r \exists \beta \in \bigcup^2 b \left( t \models \gamma = \beta \land t \models_1 \beta \in b \right) \right]
\]

**Definition** Let \( \chi(p, b, c) \) be the characteristic function of the relation \( p \models^{P} b = c \), so that it takes the value 1 if \( p \models^{P} b = c \) and 0 otherwise.
The graph of $\chi_\equiv$ on transitive sets is given by a $\mathbb{P}$-rudimentary recursion.

**The Definability Lemma** “$f$ is a $\chi_\equiv$ attempt” is $\Delta_0(\mathbb{P}, f)$.

**The Propagation Lemma** Let $F(u) = \chi_\equiv\upharpoonright(\mathbb{P} \times u \times u)$. There is a rudimentary function $H_\equiv$ such that for any transitive $P$, if $P \subseteq P^+ \subseteq \mathcal{P}(P)$,

$$F(P^+) = H_\equiv(\mathbb{P}, F(P), P^+).$$
Propagation of $\chi_\equiv$

We have defined the progress $P^c_\nu$ for $c$ a transitive set. We could continue to work with progresses of the above kind, but a problem would then arise in the proof that a set-generic extension of a provident set is provident.

Hence it is better to work with other progresses, which might be called construction from $e$ as a set and $\chi_\equiv$ as a predicate, with the definition of $\chi_\equiv$ evolving during the construction.

**Definition** Let $e$ be a transitive set of which $\mathbb{P}$ is a member; let $\eta = \varrho(\mathbb{P})$. We define by a $p$-rudimentary recursion a sequence $((e_\nu, P^e_\nu; \equiv, \chi^e_\nu)_\nu)$ of triples, thus obtaining a new progress $(P^e_\nu; \equiv)_\nu$. 
For every $\nu$, $e_\nu$ will be defined as before; for $\nu \leq \eta$ we set $P^{e;=}_{\nu} = P^e_\nu$; for $\nu < \eta$, we set $\chi^e_\nu = \emptyset$ but at $\eta$, we set $\chi^e_\eta = \chi_\nu \uparrow P^e_\eta$, which will be a set by the last Corollary. Thereafter we set

$$
e_{\nu+1} = e \cap \{x \mid x \subseteq e_\nu\}$$

$$P^{e;=}_{\nu+1} = T(P^{e;=}_{\nu}) \cup \{e_\nu\} \cup e_{\nu+1} \cup \{\chi^e_\nu \cap P^{e;=}_{\nu}\}$$

$$\chi^e_{\nu+1} = H_{=}^{=}((P, \chi^e_\nu, P^{e;=}_{\nu+1}))$$

**Proposition** Let $e$ be transitive, with $P \in e$, and let $\theta$ be indecomposable and strictly greater than $\mathfrak{g}(P)$. Then $P^{e;=}_{\theta} = P^e_{\theta}$. 
This reconstruction of $P_\theta^e$ shortens the delay for most $\chi^e_\nu$:

**Proposition** For any ordinal $\nu \geq \eta$, any limit ordinal $\lambda > \eta$ and $k \in \omega$,

\[
\begin{align*}
\chi^e_\nu &= \chi_\nu \upharpoonright P^e_\nu;=; \\
\chi^e_\nu &\subseteq P^e_{\nu+6}; \\
\chi^e_\lambda &\subseteq P^e_\lambda;=; \\
\chi_\nu \upharpoonright P^e_\nu;= &\in P^e_{\nu+12}.
\end{align*}
\]

**Remark** Bowler has contributed another elegant simplification here by proving the general result that *any function which is gentle in a gentle predicate is gentle.*
Propagation of $\chi_\epsilon$

We may now define $p \models a \notin b$.

**Definition** $p \models a \notin b \iff \forall s \leq p \exists t \leq s \exists \beta \in \bigcup^2 b \left[ t \models \beta = a \land t \models \bar{\beta} = b \right]$.

**Remark** This is not a definition by recursion: indeed it is visibly rudimentary in $p \models b = c$.

**Definition** Let $\chi_\epsilon(p, a, b)$ be the characteristic function of the relation $p \models P_a \in b$.

**Proposition** There is a natural number $s_\epsilon$ such that for each ordinal $\nu \geq \eta$, $\chi_\epsilon \upharpoonright P_\nu^{e;=} \in P_\nu^{e;=} + s_\epsilon$. 
Construction of nominators for rudimentary functions

**Theorem** Let $R$ be a rudimentary function of some number of arguments. Then there is a function $R^P$, of the same number of arguments, which we shall call the nominator of $R$, with the property that if $A$ is a provident set and $P \in A$ a notion of forcing, then $A$ is closed under $R^P$ and, further, if $G$ is an $(A, P)$-generic, then (to take the case of a function of two variables) for all $x$ and $y$ in $A$, $\text{val}_G(R^P(x, y)) = R(\text{val}_G(x), \text{val}_G(y))$.

**Corollary** Let $A$ be provident, $P \in A$ and $G$ $(A, P)$-generic. Then $A[G]$ is rud closed.
Construction of rudimentarily recursive nominators for rank and transitive closure

Rank and transitive closure are pure rud rec; we show here that $\mathbb{P}$-rud rec nominators exist for them.

Let $S(\cdot)$ be the function $z \mapsto z \cup \{z\}$.

**Lemma** There is a rud function $S^\mathbb{P}(\cdot)$ such that $\text{val}_G(S^\mathbb{P}(x)) = S(\text{val}_G(x))$.

**Definition** $\varrho^\mathbb{P}(x) = \mathsf{df} \bigcup \mathbb{P} \{ (p, S^\mathbb{P}(\varrho^\mathbb{P}(y))) \mid (p, y) \in x \land p \in \mathbb{P} \}$

**Remark** $\varrho^\mathbb{P}$ is rud rec in the parameter $\mathbb{P}$.

**Lemma** Let $A$ be provident, and $\mathbb{P} \in A$. For all $x \in A$,

$$\text{val}_G(\varrho^\mathbb{P}(x)) = \varrho(\text{val}_G(x))$$
**Definition** \( tcl^P(x) =_{df} x \cup^P \bigcup^P \{(p, tcl^P(z)) \mid (p, z) \in x\} \).

**Remark** \( tcl^P \) is rud rec in the parameter \( P \).

**Lemma** Let \( A \) be provident, and \( P \in A \). For all \( x \in A \),

\[
\text{val}_G(tcl^P(x)) = tcl(\text{val}_G(x)).
\]

**Remark** More generally, it may be shown that gentle functions have gentle nominators, using the principle that provident sets are closed under gentle separators, which are functions of the form \( x \mapsto x \cap A \), where \( A \) is a class of which the characteristic function is gentle.
Construction of nominators for the stages of a progress.

Let $e$ be a transitive set in the ground model of which $\mathbb{P}$ is a member, and let $\theta$ be indecomposable, exceeding the rank of $e$. $P_\theta^e$ is provident. Let $\dot{d}$ be the nominator $e \cup \{ \dot{\mathcal{G}} \}^\mathbb{P}$, so that $\text{val}_\mathcal{G}(\dot{d})$ will be the transitive set $d = e \cup \{ \mathcal{G} \}$.

**Remark** $\dot{d}$ will be a member of $P_\theta^e(\mathbb{P}) + k$ for some (small) $k$, given the definition of $\dot{\mathcal{G}}$, our convention that $\mathbb{1} = 1$ and the fact that $\dot{\cdot}$ is $\mathbb{1}$-rud rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage $P_\nu^d$ of the progress towards $d$. 
A simplified progress

Now \( \varrho(\mathcal{G}) \leq \varrho(\mathcal{P}) < \varrho(\mathcal{P}) \), so that for \( \nu \geq \eta \), \( d_\nu = e_\nu \cup \{ \mathcal{G} \} \). It might be that \( \varrho(\mathcal{G}) < \varrho(\mathcal{P}) \); to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress \( (Q^d_\nu)_\nu \).

**Definition**

for \( \nu < \eta \), \( Q^d_\nu = P^e_\nu \); 
\[
Q^d_\eta = P^e_\eta \cup \{ \mathcal{G} \};
\]

for \( \nu \geq \eta \), \( Q^d_{\nu+1} = \mathbb{T}(Q^e_\nu) \cup \{ d_\nu \} \cup d_{\nu+1} \); 
\[
Q^d_\lambda = \bigcup_{\nu<\lambda} Q^d_\nu \text{ if } \lambda = \bigcup \lambda > \eta.
\]

**Proposition** If \( \theta \) is indecomposable, then \( Q^d_\theta \) is provident and equals \( P^d_\theta \).
Generic extensions of provident sets

**Theorem** Let \( \theta \) be an indecomposable ordinal strictly greater than the rank of a transitive set \( e \) which contains the notion of forcing, \( \mathbb{P} \). Let \( \mathcal{G} \) be \((P^e_\theta, \mathbb{P})\)-generic. Then \((P^e_\theta)[\mathcal{G}] = P^{e \cup \{ \mathcal{G} \}}_\theta\) and hence is provident.

**Proof**: \((P^e_\theta)[\mathcal{G}]\) contains \( P^{e \cup \{ \mathcal{G} \}}_\theta \), as we have for each \( \nu < \theta \) built a name in \( P^e_\theta \) that evaluates under \( \mathcal{G} \) to \( Q^{e \cup \{ \mathcal{G} \}}_\nu \), and we know by the previous Proposition that \( Q^{e \cup \{ \mathcal{G} \}}_\theta \) equals \( P^{e \cup \{ \mathcal{G} \}}_\theta \). For the converse direction, we know that \( P^{e \cup \{ \mathcal{G} \}}_\theta \) is provident, and has \( \mathcal{G} \) as a member and hence can support the \( \mathcal{G} \)-rudimentary recursion defining \( \text{val}_\mathcal{G}(\cdot) \). Further \( P^{e \cup \{ \mathcal{G} \}}_\theta \) includes \((P^e_\nu)_\nu\), which is defined by an \( e \)-rudimentary recursion, and so includes \((P^e_\theta)[\mathcal{G}]\). \( \Box \)

**Remark** Thus, in this special case, a generic extension of a model of \text{PROVI} is a model of \text{PROVI}. The general case will now follow:
**Theorem** Let $A$ be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A^\mathbb{P}[\mathcal{G}]$ is provident.

**Proof**: Let $\theta =_{df} \text{On} \cap A$ and $T = \{c \mid c \in A \& c \text{ is transitive } \& \mathbb{P} \in c\}$. Then

$$A = \bigcup \{P_\theta^c \mid c \in T\}. $$

It follows, as each $P_\theta^c$ is provident and contains $\mathbb{P}$, that

$$A^\mathbb{P}[\mathcal{G}] = \bigcup_{c \in T} (P_\theta^c)^\mathbb{P}[\mathcal{G}] = \bigcup_{c \in T} P_\theta^{c \cup \{\mathcal{G}\}}$$

and each $P_\theta^{c \cup \{\mathcal{G}\}}$ is provident. But we have seen that the directed union of provident sets is provident. $\dashv$
Bowler has made the following elegant observation:

\[
\text{if } A \text{ is provident, } \mathbb{P} \in A \text{ and } \mathcal{G} \text{ is } (A, \mathbb{P})\text{-generic, then the generic extension } A^\mathbb{P}[\mathcal{G}] \text{ is the provident closure of } A \cup \{\mathcal{G}\}.\]
Let me suggest a way of discussion forcing extensions of transitive models of Morse-Kelley set theory. Suppose that $M$ is such a model, of height $\kappa + 1$, and let the notion of forcing, $\mathbb{P}$, be given by classes of $M$. Let $N$ be the provident closure of $M$, thus of height $\kappa \omega$. As $\mathbb{P}$ is a member of $N$, we may discuss instead set-generic extensions of $N$.

Chuaqui in his book discusses Morse-Kelley, but also a stronger theory which he calls $\text{BC}$, for Bernays class theory; not to be confused with $\text{NBG}$.

**The following speculative remarks have yet to be checked:**

If $M$ models $\text{BC}$ then $N$ adds no new sets of rank less than $\kappa$.

If $\kappa > \omega$ is indecomposable, $V_\kappa \in V$ and $V$ is the provident closure of the set $V_\kappa \cup (\kappa + 1)$, then $V_\kappa$ models Zermelo set theory.

If $\kappa > \omega$ is indecomposable, $V_{\kappa+1} \in V$ and $V$ is the provident closure of the set $V_\kappa \cup (\kappa + 1)$, then $V_{\kappa+1}$ models Morse–Kelley set theory.
L’s and T’s
Scott and McCarty showed that each infinite $V_\nu$ is closed under their pairing; it may be shown that every infinite $L_\nu$ is closed under SM-pairing and unpairing; $T_{\omega+1}$ is not so closed; of course each limit $T_\lambda$ is SM-closed, being $\emptyset$-provident.

That suggests the following question: can there be a set $c$ and a progress $(Q_\nu)_\nu$ defined by a $c$-rudimentary recursion such that for every limit ordinal $\lambda$, $Q_\lambda = L_\lambda$?

**Theorem** No such $c$ can be set-generic over $L$.

**Problem** If $0^\sharp$ exists, is there a set $d$ and a rud function $G$ such that for every $\nu$, $G(d, L_\nu) = L_{\nu+1}$?
Conjecture A
Let $\Phi(\cdot)$ be a $\Pi^1_1$ predicate of points in Baire space $\mathcal{N}$ (the set of functions from $\omega$ to $\omega$ with the usual topology). Kleene shows how to associate to each $\alpha \in \mathcal{N}$ a tree $T_\alpha$ of finite sequences of natural numbers closed under initial segment (which he called the tree of unsecured sequences) with the property that $\Phi(\alpha)$ holds iff $T_\alpha$ is well-founded (under the relation $s \prec t \iff s$ is a proper initial segment of $t$), which is to say that the set $[T_\alpha]$ of paths through $T_\alpha$ is empty. Mostowski proved that if $M$ is a transitive set in which every well-ordering is isomorphic to an ordinal, and in which enough set theory and arithmetic holds to construct such trees, then for $\alpha \in M$, $(\Phi(\alpha))^M \iff \Phi(\alpha)$, so that $\Pi^1_1$ predicates are absolute between $M$ and the universe.
Theorem If $M$ satisfies the weak set theory PROVI, which is much too weak to prove Mostowski’s collapsing theorem but supports set forcing, $\Pi^1_1$ predicates are absolute between $M$ and any set-forcing extension of $M$, (and also for those class-forcing extensions where every new set is added by some set sub-forcing.)

Proposition Let $M$ be provident with HF in $M$, and let $\mathbb{P}$ be a separative poset in $M$, $G (M, \mathbb{P})$-generic, and $N = M[G]$; let $T$ be a member of $M$ which is a tree of unsecured sequences of natural numbers (or more generally of ordinals less than some $\zeta \in M$). If in $N$, $[T]$ is non-empty, then it is non-empty in $M$. 

Proof: Let $p_0 \in \mathcal{G}$ be a condition and $\pi \in M$ a name such that in $M$

$$p_0 \Vdash \mathcal{P} \pi$$ is an infinite path through $\hat{T}$.

$$A =_{df} \{ s \in <^\omega \omega \mid \exists p \leq p_0 \, p \Vdash \mathcal{P} \hat{s} \in \pi \}$$ is in $M$ by gentle separation, and

$\emptyset \neq A \subseteq T \subseteq \text{HF}$.

**Lemma** \( \forall s \in A \, \exists t \in A \, (t < s) \)

Proof: Suppose that $p \leq p_0$ and $p \Vdash \mathcal{P} \hat{s} \in \pi$. Then

$$\forall q \leq p \, \exists r \leq q \, \exists t \, r \Vdash \mathcal{P} \hat{t}$$ is a member of $\pi$ that strictly extends $\hat{s}$

Then any such $t$ is in $A$ and $t < s$. \(\dashv\)
Now work in $M$ and use foundation to build a function $\omega \rightarrow A$ that gives an infinite path. Call an attempt a map $f : n \rightarrow A$, for some $n \in \omega$, such that for $m + 1 < n$, $f(m + 1)$ is the first (in some fixed well-ordering of $<\omega\omega$ that is definable over $\text{HF}$) member of $A$ strictly extending $f(m)$. Then being an attempt is $\Delta_0$ in the parameters $A$ and $\text{HF}$, which are both members of $M$.

All attempts, being hereditarily finite objects, are in $\text{HF}$. Consider the class

$$\omega \cap \{n \mid \text{no member of } \text{HF} \text{ is an attempt with domain } n\}.$$

That is a set by $\Delta_0$ separation, so by set foundation will, if non-empty have a least element, which will be of the form $k + 1$. But then the Lemma will rapidly yield a contradiction.
Let \( S =_{df} \text{HF} \cap \{f \mid f \text{ is an attempt}\} \). \( S \in M \) by \( \Delta_0 \) separation, and the set of all initial segments of the class \( \{f(n) \mid n \in \omega \& f \in S\} \)—which class will be a set in \( M \) as \( M \) is rud closed—will form an infinite path in \( M \) through \( T \), completing the proof of the Proposition.

**Corollary** If \( \alpha \in M \), \( (\Phi(\alpha))^M \iff (\Phi(\alpha))^N \).