A Theory of Stationary Trees, and the Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees

Ari Meir Brodsky

Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)

Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

Finite Ramsey Theorem For all positive integers m, r, and k, there is some positive integer n such that

$$n \to (m)^r_k \,.$$

Infinite Ramsey Theorem For all positive integers r and k, we have

$$\omega \to (\omega)^r_k \,.$$

Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

  Finite Ramsey Theorem For all positive integers m, r, and k, there is some positive integer n such that

  $$n \to (m)^r_k \,.$$

  Infinite Ramsey Theorem For all positive integers r and k, we have

  $$\omega \to (\omega)^r_k \,.$$

- Erdős and Rado (1956): the first systematic study of partition relations for linear orders (especially cardinals and ordinals) of all cardinalities

Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

  Finite Ramsey Theorem For all positive integers m, r, and k, there is some positive integer n such that
  $$n \to (m)^r_k \,.$$

  Infinite Ramsey Theorem For all positive integers r and k, we have
  $$\omega \to (\omega)^r_k \,.$$

- Erdős and Rado (1956): the first systematic study of partition relations for linear orders (especially cardinals and ordinals) of all cardinalities
- Todorcevic (early 1980s): systematically extending the partition calculus to trees and partial orders

Arrow Notation for Partial Orders

Suppose $\langle P, <_P \rangle$ is any partial order.

- If α is any ordinal, $[P]^\alpha$ denotes the set of all linearly ordered chains in P of order-type α.
- If μ is any cardinal and α is any ordinal, the statement

  $$P \to (\alpha)^2_\mu$$

  means: For any colouring (partition function) $c : [P]^2 \to \mu$, there is a chain $X \in [P]^\alpha$ that is c-homogeneous, that is, $c''[X]^2 = \{\chi\}$ for some colour $\chi < \mu$.

Trees

Definition

A partial order $\langle T, <_T \rangle$ is a tree if for every $t \in T$,

$$t{\downarrow} = \{s \in T : s <_T t\}$$

is a well-ordered set.

Nonspecial Trees

Definition (Todorcevic, 1981)

A tree T is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

Nonspecial Trees

Definition (Todorcevic, 1981)

A tree T is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

Fact

For any tree T,

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0}.$$

Nonspecial Trees

Definition (Todorcevic, 1981)

A tree T is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

Fact

For any tree T,

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0} .$$

Examples

- ω_1

Nonspecial Trees

Definition (Todorcevic, 1981)

A tree T is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

Fact

For any tree T,

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0}.$$

Examples

- ω_1
- (Kurepa, 1956) $w\mathbb{Q}$ (and its variant $\sigma\mathbb{Q}$), the collection of all well-ordered subsets of $\mathbb{Q}$, ordered by end-extension

Taller Nonspecial Trees

Definition (Todorcevic, 1985)

For any infinite cardinal κ, a tree T is non-κ-special if it cannot be written as a union of $\leq \kappa$ many antichains:

$$T \to (2)^1_\kappa\,.$$

Taller Nonspecial Trees

Definition (Todorcevic, 1985)

For any infinite cardinal κ, a tree T is non-κ-special if it cannot be written as a union of $\leq \kappa$ many antichains:

$$T \rightarrow (2)^1_\kappa \,.$$

Fact

For any tree T,

$$T \text{ is non-}\kappa\text{-special} \iff T \rightarrow (\kappa)^1_\kappa.$$

From Trees to Partial Orders

Theorem (Todorcevic 1985)

Let r be any positive integer, let κ and θ be cardinals, and for each $\gamma < \theta$ let α_γ be an ordinal. If every non-κ-special tree T satisfies

$$T \to (\alpha_\gamma)^r_{\gamma<\theta}\,, \qquad (**)$$

then every partial order P satisfying $P \to (\kappa)^1_\kappa$ also satisfies the above partition relation $(**)$.

Balanced Partition Relations for Pairs

Theorem (Erdős-Rado, 1956)

Let κ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,

$$\left(2^{<\kappa}\right)^{+} \to (\kappa+1)^{2}_{\mu}.$$

Balanced Partition Relations for Pairs

Theorem (Erdős-Rado, 1956)

Let κ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,

$$\left(2^{<\kappa}\right)^+ \to (\kappa+1)^2_\mu .$$

Theorem (Todorcevic, 1985)

Let κ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \to (\kappa+1)^2_\mu .$$

Balanced Partition Relations for Pairs

Theorem (Baumgartner-Hajnal-Todorcevic, 1991)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \xi)^2_k .$$

Balanced Partition Relations for Pairs

Theorem (Baumgartner-Hajnal-Todorcevic, 1991)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \xi)^2_k \,.$$

Theorem (Brodsky — Main Theorem)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree } \to (\kappa + \xi)^2_k \,.$$

Optimality of the Result

Theorem (Brodsky)

Let κ be any infinite regular cardinal. If T is any tree, then the following are equivalent:

1. $T \to (2)^1_{2^{<\kappa}}$
2. $T \to (2^{<\kappa})^1_{2^{<\kappa}}$
3. $T \to (\kappa + 1, \kappa)^2$
4. *For any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have*
$$T \to (\kappa + \xi)^2_k \, .$$

Optimality of the Result

Theorem (Main Theorem)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\text{non-}\left(2^{<\kappa}\right)\text{-special tree} \rightarrow (\kappa + \xi)^2_k \,.$$

Theorem (Rebholz, Donder)

If $V = L$, and if κ is any regular uncountable cardinal that is not weakly compact, then

$$\kappa^+ \not\rightarrow (\kappa + \tau)^2_2 \,,$$

where τ is the first cardinal such that $2^\tau \geq \kappa$.

That is, our Main Theorem is the best possible balanced generalization to trees of the Erdős-Rado Theorem for finitely many colours to ordinal goals.

Optimality of the Result

Theorem (Main Theorem)

Let κ *be any infinite regular cardinal, let* ξ *be any ordinal such that* $2^{|\xi|} < \kappa$*, and let* k *be any natural number. Then*

$$\text{non-}\left(2^{<\kappa}\right)\text{-special tree} \rightarrow (\kappa + \xi)^2_k \, .$$

Theorem

If $V = L$*, and if* κ *is any successor cardinal, then*

$$\kappa^+ \not\rightarrow (\kappa + 2)^2_{\kappa^-} \, .$$

Examples for Particular Cardinals

Example

Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers k and n,

$$\text{nonspecial tree } \rightarrow (\omega + n)^2_k\,.$$

Examples for Particular Cardinals

Example

Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers k and n,

$$\text{nonspecial tree } \rightarrow (\omega + n)^2_k \,.$$

But this is subsumed by:

Theorem (Todorcevic, 1985)

For all $\alpha < \omega_1$ and $k < \omega$ we have

$$\textit{nonspecial tree } \rightarrow (\alpha)^2_k \,.$$

Examples for Particular Cardinals

Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have, for any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \to (\omega_1 + n)^2_k \,.$$

Examples for Particular Cardinals

Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have, for any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \to (\omega_1 + n)^2_k\,.$$

Since $\mathfrak{p}$ (the pseudo-intersection number) is regular and $2^{<\mathfrak{p}} = \mathfrak{c}$, we have:

Example

For any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \to (\mathfrak{p} + n)^2_k\,.$$

Ideals on Nonspecial Trees

Ideal of special subtrees corresponds to ideal of bounded subsets of an ordinal.

Ideals on Nonspecial Trees

Ideal of special subtrees corresponds to ideal of bounded subsets of an ordinal.

Question

How do we generalize club, stationary, and nonstationary sets to trees?

Diagonal Unions

Definition

Let T be a tree. For a collection of subsets of T indexed by nodes of T, i.e.

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t{\uparrow}) .$$

Diagonal Unions

Definition

Let T be a tree. For a collection of subsets of T indexed by nodes of T, i.e.

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\uparrow) .$$

Definition

Let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal. We define

$$\bigtriangledown \mathcal{I} = \left\{ \bigtriangledown_{t \in T} A_t : \langle A_t \rangle_{t \in T} \subseteq \mathcal{I} \right\} .$$

Diagonal Unions

Lemma

Let T be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. Then

$$\bigtriangledown \mathcal{I} = \{X \subseteq T : \exists \text{ regressive } f : X \to T \text{ constant only on sets from } \mathcal{I}\}.$$

Diagonal Unions

Lemma

Let T be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. Then

$$\nabla \mathcal{I} = \{X \subseteq T : \exists \text{ regressive } f : X \to T \text{ constant only on sets from } \mathcal{I}\}.$$

Lemma (Idempotence Lemma)

Let $\lambda = \mathrm{ht}(T)$, and suppose λ is any cardinal. If $\mathcal{I}$ is a λ-complete ideal on T, then $\nabla \nabla \mathcal{I} = \nabla \mathcal{I}$, that is, $\nabla \mathcal{I}$ is normal.

Stationary Subtrees

Definition

Let $B \subseteq T$, where T is a tree of height κ^+.
B is a nonstationary subtree of T if we can write

$$B = \bigtriangledown_{t \in T} A_t,$$

where each A_t is a κ-special subtree of T.
Otherwise, B is a stationary subtree of T.

$$\begin{aligned} NS_\kappa^T &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \bigtriangledown \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

Stationary Subtrees

Definition

Let $B \subseteq T$, where T is a tree of height κ^+.
B is a nonstationary subtree of T if we can write

$$B = \bigtriangledown_{t \in T} A_t,$$

where each A_t is a κ-special subtree of T.
Otherwise, B is a stationary subtree of T.

$$\begin{aligned} NS^T_\kappa &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \bigtriangledown \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

Theorem

For any tree T of height κ^+, the ideal NS^T_κ is a normal ideal on T.

Nonreflecting Ideals Determined by Elementary Submodels

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. Then the collection $\mathcal{P}(T) \cap N$ is a field of sets (set algebra) over the set T.

Nonreflecting Ideals Determined by Elementary Submodels

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. Then the collection $\mathcal{P}(T) \cap N$ is a field of sets (set algebra) over the set T.

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection

$$\{B \subseteq T : B \in N \text{ and } t \in B\}$$

is an ultrafilter in the set algebra $\mathcal{P}(T) \cap N$, and the collection

$$\{B \subseteq T : B \in N \text{ and } t \notin B\}$$

is the corresponding maximal (proper) ideal in the same set algebra.

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Define a collapsing function

$$\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t\downarrow)$$

by setting, for $B \subseteq T$ with $B \in N$,

$$\pi_{N,t}(B) = B \cap t\downarrow.$$

We then define the collection

$$\begin{aligned} \mathcal{A}_{N,t} = \operatorname{range}(\pi_{N,t}) &= \{B \cap t\downarrow : B \in \mathcal{P}(T) \cap N\} \\ &= \{B \cap t\downarrow : B \in N\} \subseteq \mathcal{P}(t\downarrow). \end{aligned}$$

Nonreflecting Ideals Determined by Elementary Submodels

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$.
Then the collection $\mathcal{A}_{N,t}$ is a set algebra over the set $t{\downarrow}$, and the collapsing function $\pi_{N,t}$ defines a surjective homomorphism of set algebras

$$\pi_{N,t} : \langle \mathcal{P}(T) \cap N, \cup, \cap, \setminus, \emptyset, T \rangle \to \langle \mathcal{A}_{N,t}, \cup, \cap, \setminus, \emptyset, t{\downarrow} \rangle .$$

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define the collections

$$\begin{aligned}
\mathcal{G}_{N,t} &= \{\pi_{N,t}(A) : A \in \mathcal{P}(T) \cap N \text{ and } t \notin A\} \\
&= \{A \cap t{\downarrow} : A \in N \text{ and } t \notin A\} \subseteq \mathcal{A}_{N,t}, \text{ and} \\
\mathcal{G}^*_{N,t} &= \{\pi_{N,t}(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \in B\} \\
&= \{B \cap t{\downarrow} : B \in N \text{ and } t \in B\} \subseteq \mathcal{A}_{N,t}.
\end{aligned}$$

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define the collections

$$\begin{aligned}
\mathcal{G}_{N,t} &= \{\pi_{N,t}(A) : A \in \mathcal{P}(T) \cap N \text{ and } t \notin A\} \\
&= \{A \cap t\!\downarrow : A \in N \text{ and } t \notin A\} \subseteq \mathcal{A}_{N,t}, \text{ and} \\
\mathcal{G}^*_{N,t} &= \{\pi_{N,t}(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \in B\} \\
&= \{B \cap t\!\downarrow : B \in N \text{ and } t \in B\} \subseteq \mathcal{A}_{N,t}.
\end{aligned}$$

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $\mathcal{G}_{N,t}$ is a (not necessarily proper) ideal in the set algebra $\mathcal{A}_{N,t}$, and $\mathcal{G}^*_{N,t}$ is the dual filter corresponding to $\mathcal{G}_{N,t}$.

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define

$$I_{N,t} = \{X \subseteq t\downarrow : X \subseteq Y \text{ for some } Y \in \mathcal{G}_{N,t}\}.$$

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq Y \text{ for some } Y \in \mathcal{G}_{N,t}\}.$$

Lemma

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq A \textit{ for some } A \in N \textit{ with } t \notin A\}$$

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq Y \text{ for some } Y \in \mathcal{G}_{N,t}\}.$$

Lemma

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq A \textit{ for some } A \in N \textit{ with } t \notin A\}$$

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $I_{N,t}$ is a (not necessarily proper) ideal on $t{\downarrow}$, that is, an ideal in the whole power set $\mathcal{P}(t{\downarrow})$.

Eligibility Condition

Remark

In the special case where T is a cardinal λ and $t = \sup(N \cap \lambda)$, we have $N \cap T \subseteq t{\downarrow}$, so that elementarity of N implies that $\pi_{N,t}$ is one-to-one, and the ideal $I_{N,t}$ is proper.
Not necessarily true for trees in general!

Eligibility Condition

Remark

In the special case where T is a cardinal λ and $t = \sup(N \cap \lambda)$, we have $N \cap T \subseteq t{\downarrow}$, so that elementarity of N implies that $\pi_{N,t}$ is one-to-one, and the ideal $I_{N,t}$ is proper.
Not necessarily true for trees in general!

To ensure that our ideals are proper, we impose an eligibility condition:

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. We say that a node $t \in T$ is N-eligible if

$$\forall B \in N \, [t{\downarrow} \subseteq B \implies t \in B] \, .$$

Eligibility Condition

Lemma
Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the following are all equivalent:

1. t *is* N*-eligible;*
2. $t{\downarrow} \notin \mathcal{G}_{N,t}$, *that is,* $\mathcal{G}_{N,t}$ *is a proper ideal in* $\mathcal{A}_{N,t}$*;*
3. $t{\downarrow} \notin I_{N,t}$, *that is,* $I_{N,t}$ *is a proper ideal on* $t{\downarrow}$*;*
4. *For all* $A, B \in N$ *with* $A \cap t{\downarrow} = B \cap t{\downarrow}$, *we have* $t \in A \iff t \in B$ *(even if* $\pi_{N,t}$ *is not injective);*
5. *For all* $B \in N$, *we have*

$$t \in B \iff B \cap t{\downarrow} \in \mathcal{G}^{+}_{N,t}.$$

Very Nice Collections of Elementary Submodels

Definition

Let λ be any regular uncountable cardinal, and let T be a tree of height λ. Suppose θ is a large enough regular cardinal, and let κ be an infinite cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a κ-very nice collection of elementary submodels of $H(\theta)$ indexed by T if:

1. For each $t \in T$, $N_t \prec H(\theta)$;
2. For each $t \in T$, $|N_t| < \lambda$;
3. For each $t \in T$, $t\downarrow \subseteq N_t$;
4. For $s, t \in T$ with $s <_T t$, $N_s \in N_t$.
5. For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.
6. The collection is continuous (with respect to its indexing), meaning that for all $t \in T$ with height a limit ordinal,

$$N_t = \bigcup_{s <_T t} N_s.$$

Very Nice Collections of Elementary Submodels

Lemma

Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and θ is a large enough regular cardinal. If κ is an infinite cardinal such that

$$(\forall \text{ cardinals } \nu < \lambda)\left[\nu^{<\kappa} < \lambda\right],$$

then there is a κ-very nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$.

κ-Completeness and Eligibility Condition

Lemma

Suppose ν is any infinite cardinal, T is a non-ν-special tree (necessarily of height ν^+), and θ is a large enough regular cardinal. Suppose κ is an infinite cardinal, and $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then the set

$$\left\{ t \in T : t \text{ is } N_t\text{-eligible and } [N_t]^{<\kappa} \subseteq N_t \right\}$$

is a stationary subtree of T.