Universality, Self-iterability, and Definability

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Apr 26, 2014

Introduction

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These questions turn out to be related. Roughly speaking, sufficient iterability implies ordinal definability.
For mice that are sufficiently simple (e.g. $\leq \mathcal{M}_1$), we can answer positively the above questions to some extent. For example, in $\mathcal{M}_1$, letting δ be the Woodin cardinal of $\mathcal{M}_1$ and $\gamma < \delta$, $\mathcal{M}_1|\gamma$ is fully iterable in $\mathcal{M}_1$. Furthermore, for any cardinal $\gamma < \delta$ in $\mathcal{M}_1$, $\mathcal{M}_1$ can ordinal define a well-ordering of its own $\mathcal{P}(\gamma)$. More is true, $\mathcal{M}_1 \vDash V = \mathrm{HOD}$.

Some results concerning self-iterability

Theorem

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We say that a pre mouse $\mathcal{M}$ is **nontame** if there is some γ such that $\mathcal{M}|\gamma$ is active with top extender F and there is some $\mathrm{crt}(F) \leq \delta \leq \gamma$ such that $\mathcal{M}|\gamma \vDash$ "δ is Woodin". Otherwise, $\mathcal{M}$ is **tame**.

Some results concerning self-iterability (cont.)

Theorem (Schindler, Steel)

Suppose $\mathcal{M}$ is a fully iterable tame mouse. Suppose γ is an $\mathcal{M}$-cardinal or $\gamma = o(\mathcal{M})$. Suppose $\mathcal{M}|\gamma \vDash$ "there are only boundedly many Woodin cardinals". Then there is a strong cutpoint $\xi < \gamma$ such that $\mathcal{M} \vDash \mathcal{M}|\gamma$ is $(\omega, \omega_1, \gamma)$-iterable.

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While the amount of self-iterability above is sufficient for proving principles like $\diamond^*_{\kappa,\lambda}$ hold in $\mathcal{M}$ for $\omega_1^{\mathcal{M}} < \kappa < \lambda$, it is not sufficient to prove definability. In particular, it is still open whether every iterable tame mouse $\mathcal{M}$ can ordinal define a well-ordering of its own $\mathcal{P}(\omega)$.

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While the amount of self-iterability above is sufficient for proving principles like $\diamond^*_{\kappa,\lambda}$ hold in $\mathcal{M}$ for $\omega_1^{\mathcal{M}} < \kappa < \lambda$, it is not sufficient to prove definability. In particular, it is still open whether every iterable tame mouse $\mathcal{M}$ can ordinal define a well-ordering of its own $\mathcal{P}(\omega)$. Steel has shown that $\mathcal{M}_\omega^\sharp$ can ordinal define a well-ordering of its own $\mathcal{P}(\omega)$ using genericity iterations and derived models.

Ordinal definability

Theorem (Woodin)

Suppose $\mathcal{N}$ is premouse such that $\mathcal{N} \models$ "there is a Woodin cardinal". Let $\delta^{\mathcal{N}}$ be the least Woodin cardinal of $\mathcal{N}$. Suppose further that for every $\alpha < \delta^{\mathcal{N}}$, in $\mathcal{N}$, $\mathcal{N}|\alpha$ is $(\omega, \delta^{\mathcal{N}})$ iterable via an OD iteration strategy. For any $\gamma < \delta^{\mathcal{N}}$, in $\mathcal{N}$ there is an OD well-ordering of $\mathcal{P}(\gamma)$. In particular, in $\mathcal{N}$ there is an OD well-ordering of the reals.

Ordinal definability (cont.)

Let $\gamma < \delta^{\mathcal{N}}$ be a regular uncountable cardinal in $\mathcal{N}$. We show that $\mathcal{P}(\gamma)^{\mathcal{N}} \subset \mathrm{HOD}^{\mathcal{N}}$. Working in $\mathcal{N}$ for the rest of the proof, let

$$\mathcal{F} = \{\mathcal{M} \mid \mathcal{N}|\gamma^+ \lhd \mathcal{M} \wedge \rho_\omega(\mathcal{M}) = \gamma^+ \wedge \mathcal{M} \text{ is } (\omega, \delta^{\mathcal{N}}) \text{ iterable via an } \mathit{OD} \text{ strategy}\}.$$

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Note that in $\mathcal{N}$, card$(\mathcal{F}) \leq \gamma^{++}$ and $\mathcal{P}(\gamma) = \mathcal{P}(\gamma)^{\mathcal{M}}$ for all $\mathcal{M} \in \mathcal{F}$. For each $\mathcal{M} \in \mathcal{F}$, let $\Sigma_{\mathcal{M}}$ be the least $OD^{\mathcal{N}}$ $(\omega, \delta^{\mathcal{N}})$ strategy for $\mathcal{M}$ in $\mathcal{N}$. Now we perform the simultaneous comparison of $\mathcal{C} = \{(\mathcal{M}, \Sigma_{\mathcal{M}}) \mid \mathcal{M} \in \mathcal{F}\}$, that is at successor steps, we iterate away a pair of disagreeing extenders (E, F) with least index and at limit steps, we use the strategies to choose branches. Since $\gamma^{++} < \delta^{\mathcal{N}}$ and $\delta^{\mathcal{N}}$ is a limit cardinal, the process terminates successully. For each such $\mathcal{M}$, let $\mathcal{T}_{\mathcal{M}}$ be the iteration tree according to $\Sigma_{\mathcal{M}}$ and $\mathcal{M}_\infty^{\mathcal{T}_{\mathcal{M}}}$ be the end model of $\mathcal{T}_{\mathcal{M}}$.

Ordinal definability (cont.)

Let $\mathcal{M}_{\gamma,\infty}$ be a model of the form $\mathcal{M}_\infty^{\mathcal{T}_\mathcal{M}}$ with least ordinal height. So there are an $\mathcal{M} \in \mathcal{F}$ and an iteration embedding $i : \mathcal{M} \to \mathcal{M}_{\gamma,\infty}$ according to $\Sigma_\mathcal{M}$.

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It's not hard to see that if $A \in \mathcal{M}_{\gamma,\infty}$ then A is OD in $V_{\delta\mathcal{N}}^{\mathcal{N}}$. This is because $\mathcal{M} \mapsto \mathcal{T}_\mathcal{M}$ and $\mathcal{M}_{\gamma,\infty}$ are uniformly definable over $V_{\delta\mathcal{N}}^{\mathcal{N}}$ from γ.

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Let $T = \{\alpha < \gamma \mid \mathrm{cf}(\alpha) = \omega\}$ and let

$$\vec{S} = \{S_\alpha \mid \alpha < \gamma\}$$

be a partition of T into disjoint stationary sets. So $T, \vec{S} \in \mathcal{M}$. Let

$$\vec{T} =_{\mathrm{def}} \langle T_\alpha \mid \alpha < i(\gamma)\rangle = i(\vec{S})$$

and

$$\gamma^* = \sup\{i(\alpha) \mid \alpha < \gamma\}.$$

Ordinal definability (cont.)

Note that $\vec{T} \in \mathcal{M}_\infty$ and in fact T_α is OD for every $\alpha < i(\gamma)$. Finally, let

$$Z = \{\alpha < \gamma^* \mid T_\alpha \text{ is stationary in } \gamma^*\}.$$

Clearly Z is ordinal definable.

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Note that $\vec{T} \in \mathcal{M}_\infty$ and in fact T_α is *OD* for every $\alpha < i(\gamma)$. Finally, let

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The key point is that: $Z = i[\gamma]$. This implies that $i \upharpoonright \gamma$ is ordinal definable.

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The key point is that: $Z = i[\gamma]$. This implies that $i \restriction \gamma$ is ordinal definable.
Suppose $\alpha \in i[\gamma]$. So $\alpha = i(\beta)$ for some $\beta < \gamma$. We need to see that $T_\alpha = i(S_\beta)$ is stationary in γ^*. Let $C \subseteq \gamma^*$ be a club. It's easy to see that $i^{-1}[C]$ contains an ω-club, hence $i^{-1}[C] \cap S_\beta \neq \emptyset$ (otherwise, the set D of limit points of $i^{-1}[C]$ is a club and is disjoint from S_β because elements of S_β have cofinality ω. This contradicts the fact that S_β is stationary.). This shows $i[\gamma] \subseteq Z$.

Ordinal definability (cont.)

Suppose $\alpha \in Z \backslash i[\gamma]$. So T_α is stationary in γ^* but

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Now it's easy to see that $\cup_{\beta<\gamma} i(S_\beta)$ contains an ω-club in γ^*; this is because i is continuous at every element of T. By the same argument as in the previous paragraph and the fact that T_α is stationary in γ^*, we have

$$T_\alpha \cap \cup_{\beta<\gamma} i(S_\beta) \neq \emptyset.$$

Contradiction. So in fact $Z \subseteq i[\gamma]$.

Ordinal definability (cont.)

So for any $A \subseteq \gamma$, for any $\alpha < \gamma$,

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So for any $A \subseteq \gamma$, for any $\alpha < \gamma$,

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So we define A in terms of $i \restriction \gamma$ and $i(A)$. But since $i \restriction \gamma$ is OD, and $i(A) \in \mathcal{M}_\infty$ is OD, hence $A \in \mathrm{HOD}$. This shows in $\mathcal{N}$, $\mathcal{P}(\gamma) \subset \mathrm{HOD}$.

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Remark. By minimizing parameters in the above proof, we can also show that the extender sequence $E^{\mathcal{N}}|\delta^{\mathcal{N}}$ is ordinal definable over $V^{\mathcal{N}}_{\delta^{\mathcal{N}}}$.

Self-iterability

Let $\mathcal{M}$ be a mouse such that $o(\mathcal{M}) = \Omega$ is a regular uncountable cardinal. Let Σ be $\mathcal{M}$'s iteration strategy. We say that $(\mathcal{M}, \Sigma)$ is **universal** if whenever $\mathcal{M}'$ is a non-dropping Σ-iterate of $\mathcal{M}$ via a $<$-Ω iteration and whenever $\mathcal{P}$ is a mouse of size $\leq \Omega$, then $\mathcal{M}'$ does not lose the comparison between $\mathcal{M}'$ and $\mathcal{P}$.

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Theorem (T., Zeman)

Suppose $(\mathcal{M}, \Sigma)$ and Ω are as above. Suppose $\mathcal{M}$ is tame. Suppose $\delta^{\mathcal{M}}$ is the least Woodin cardinal of $\mathcal{M}$. Suppose $\gamma < \delta^{\mathcal{M}}$. Then $\mathcal{M} \vDash$ "$\mathcal{M}|\gamma$ is $(\omega, \delta^{\mathcal{M}})$-iterable via the $\mathcal{Q}$-structure guided strategy."

Self-iterability (cont.)

Theorem (Schlutzenberg-Steel)

Let Ω be a regular cardinal or $\Omega = On$. Assume further:

- $(R, \delta) \in H_\Omega$ is an $(\Omega + 1)$-iterable coarse premouse, and $x \in V_\delta^R$.
- $(\mathcal{N}_\xi^R | \xi \leq \zeta)$, where $\zeta \leq \delta$, is a reasonable $L[E]_\mu$-construction over x in R.
- $\mathcal{M}$ is a normally $(\Omega + 1)$-iterable premouse over x satisfying all fine structural consequences of countable $(0, \omega_1, \omega_1 + 1)$-iterability.

Then there is an ω-maximmal normal iteration tree $\mathcal{T}$ on $\mathcal{M}$ and an iteration tree $\mathcal{U}$ on R both of successor length $< \Omega$ such that, letting $R' = \mathcal{M}_\infty^\mathcal{U}$ and $(\mathcal{N}_\alpha^{R'} | \alpha \leq \zeta') = i^\mathcal{U}(\mathcal{N}_\xi^R | \xi \leq \zeta)$, one of the following holds:

(a) $\mathfrak{C}_\omega(\mathcal{N}_{\zeta'}^{R'})) \lhd \mathcal{M}_\infty^\mathcal{T}$.

(b) There is no drop on the main branch of $\mathcal{T}$ in model or degree, and $\mathcal{M}_\infty^\mathcal{T} \trianglelefteq \mathfrak{C}_\omega(\mathcal{N}_\alpha^{R'})$ for some $\alpha \leq \zeta'$.

Self-iterability (cont.)

Let $\mu < \delta^{\mathcal{M}}$ and $\mu > \gamma$, where $\gamma < \delta^{\mathcal{M}}$ is a successor cardinal of an inaccessible cardinal in $\mathcal{M}$. Apply Schlutzenberg-Steel theorem with $R = \mathcal{M}$, $\zeta = \delta = \delta^{\mathcal{M}}$, $\mathcal{M} = \mathcal{M}|\gamma =_{def} \mathcal{N}$, $x = \emptyset$. Let $\mathcal{T}$ be on $\mathcal{N}$ and $\mathcal{U}$ be on $\mathcal{M}$, $R', \mathcal{M}^{\mathcal{T}}_{\infty}$ be as in the theorem. We claim that conclusion (a) **cannot occur**.

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Suppose $\mathfrak{C}_{\omega}(\mathcal{N}^{R'}_{\delta'})) = \mathcal{N}^{R'}_{\delta'} \lhd \mathcal{M}^{\mathcal{T}}_{\infty}$. In general, $\delta' \geq \delta(\mathcal{T})$. We assume equality (the other case is similar). Let $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\infty}$ be the $\mathcal{Q}$-structure for δ'.

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Case 2: $\rho_\omega(\mathcal{Q}) < \delta'$.

Self-iterability (cont.)

Case 1: $R'|\delta'$ is generic over $\mathcal{Q}$ for the (δ'-generator) extender algebra at δ'. We can re-organize $\mathcal{Q}[R'|\delta']$ into a premouse $\mathcal{Q}'$ over $R'|\delta'$; $\mathcal{Q}'$ is δ'-sound and projects to $\leq \delta'$. Now compare $\mathcal{Q}'$ against R' (the comparison is above δ'). Universality of $(\mathcal{M}, \Sigma)$ implies R' wins the comparison and in fact, $\mathcal{Q}' \lhd R'$. But then $A \in R'$ and kills the Woodinness of δ' in R'. Contradiction.

Self-iterability (cont.)

Case 2: Again, $\mathcal{Q}' \lhd \mathcal{R}'$. We claim that $\mathcal{Q}$ and hence $\mathcal{Q}'$ definably singularizes δ'. This will give us that in R', δ' is singular. But δ' is Woodin in R'. Contradiction.

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Suppose not. For simplicity, suppose $\rho_1(\mathcal{Q}) < \delta'$. Find cofinally in δ' many $\delta^* > \rho_1(\mathcal{Q})$ such that: letting $H(\delta^*) = H_1^{\mathcal{Q}}(\delta' \cup p_{\mathcal{Q}})$, we have $\delta' = \delta \cap H(\delta^*)$ and $\delta \in H(\delta^*)$.

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Let $\mathcal{Q}^*$ be the transitive collapse of $H(\delta^*)$. Since $\mathcal{Q}^*, \mathcal{Q}$ have the same ρ_1 and are not sound, they coiterate to the same model $\mathcal{P}$. Note that $\mathcal{Q}^*$ agrees with $\mathcal{Q}$ up to δ^* and $\mathcal{Q}^*$ has no extender overlapping δ^* by tameness, the $\mathcal{Q}^*$-side is above δ^*. The $\mathcal{Q}$-side is also above δ^* by tameness of $\mathcal{Q}$ and the fact that δ^* is Woodin in all models on the $\mathcal{Q}^*$-side. This implies $\mathcal{P}(\delta^*)^{\mathcal{Q}} = \mathcal{P}(\delta^*)^{\mathcal{Q}^*}$. So δ^* is Woodin in $\mathcal{Q}$.

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The argument above gives that δ' is a Woodin limit of Woodins in $\mathcal{Q}$. Contradicts tameness of $\mathcal{Q}$.

Self-iterability (cont.)

So case (b) occurs. In R', $\mathcal{N}$ is (ω, δ')-iterable by the strategy Λ defined as follows: suppose $\mathcal{S} \in R'$ is an iteration tree on $\mathcal{N}$ according to Λ of limit length $< \delta'$. Then $\Lambda(\mathcal{S})$ is the unique cofinal well-founded branch b such that

$$\Vdash^{R'}_{Col(\omega, lh(\mathcal{S}))} \mathcal{Q}(b, \mathcal{S}) \text{ is } (\omega, \omega_1 + 1)\text{-iterable.}$$

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By elementarily and the fact that the map $i^{\mathcal{U}}$ is above $o(\mathcal{N})$, the same holds in $\mathcal{M}$ about $\mathcal{N}$.

Going beyond

Self-iterability in windows between two consecutive Woodin cardinals is sufficient to guarantees ordinal definability up to the first measurable limit of Woodins (inside a tame, universal mouse). This uses the observation that the proof of Woodin's theorem can be made uniform so that it allows us to define the extender-sequence up to the Woodin inside the mouse.

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If in addition, $\mathcal{M}$ has the largest Woodin cardinal δ, then as far as ordinal definability goes, we can get up to δ. This requires some additional argument. But the basic method is still the Schlutzenberg-Steel theorem.

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If in addition, $\mathcal{M}$ has the largest Woodin cardinal δ, then as far as ordinal definability goes, we can get up to δ. This requires some additional argument. But the basic method is still the Schlutzenberg-Steel theorem. It's not clear whether one can prove the T.-Zeman theorem without assuming tameness.

Thank you!