Hybrid mice, scales, and the core model induction

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Outline

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The core model induction

- The core model induction is a general method, pioneered by Woodin, further developed by Steel, Schindler, and others, used for mining strength of a given theory. In a typical core model induction, one constructs models of determinacy (inductively) that extend one another and this is achieved by constructing mouse operators that capture the relevant sets of reals.

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- This relationship is given by a theorem of Neeman: if N is a (sufficiently iterable) mouse with iteration strategy Σ and $N \vDash \delta$ is Woodin. Suppose N *captures* $A \subseteq \mathbb{R}$, i.e. there is a $Col(\omega, \delta)$-term relation τ_A in N such that whenever $i : N \to M$ is according to Σ, letting $g \subseteq Col(\omega, i(\delta))$ be M-generic, then $A \cap M[g] = (i(\tau_A)_g)$. Then A is determined.

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- For instance, to show all projective sets are determined (PD), it suffices to show the operator $x \mapsto \mathcal{M}_n^\sharp(x)$ is total (on HC) for all n.
- In general, the "successor step" in the core model induction is: suppose $\mathcal{F}$ is a "nice" mouse operator, one wants to construct the "next operator", namely $\mathcal{M}_1^{\mathcal{F},\sharp}$.
- Examples of mouse operators are: $x \mapsto x^\sharp$, $x \mapsto \mathcal{M}_1^\sharp(x)$, a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a fine structural mouse (or a hod mouse) and Σ is $\mathcal{P}$'s iteration strategy with nice condensation property (roughly, trees according to Σ collapses to trees according Σ).

The core model induction (cont.)

To go further in general, we need to organize our induction according to the pattern of scales. The key case (the gap case) is when we have already constructed a point class Γ such that

- Γ is inductive-like, i.e. Γ has the scales property, is closed under real quantifications, and non-self-dual.
- For some operator $\mathcal{F}$, Γ-MC(F) holds, that is, for every $x, y \in \mathbb{R}$, $x \in OD^{\mathcal{F},\Gamma}(y)$ implies x is in a $\mathcal{F}$-mouse over y with iteration strategy in Γ.

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We next construct a pair $(\mathcal{P}, \Sigma)$ that captures a countable, cofinal subset of **Env**(Γ) and Σ has condensation. The existence of $(\mathcal{P}, \Sigma)$ is hypothesis-dependent. It is not always possible to get such a pair and/or a cofinal, countable subset of **Env**(Γ).

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We then show $x \mapsto \mathcal{M}_n^{\Sigma,\sharp}(x)$ is total. This has the consequence that it gives us determined, scaled pointclasses strictly extending Γ.

The core model induction (cont.)

- For instance, Γ is $\Sigma_1^{\mathcal{M}}$ for $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R})$ admissible and begins a Σ_1-gap in $\mathrm{Lp}(\mathbb{R})$. Two cases: the gap is not the last gap and is the last gap. The solutions are different in each case. The first requires the scales analysis inside $\mathrm{Lp}(\mathbb{R})$.

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- The solution to the second case is hypothesis-dependent.
- Recall Θ is the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α. Working under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$, we say that $(\theta_\alpha : \alpha \leq \Omega)$ is the *Solovay sequence* if: (a) θ_0 is the supremum of ordinals α such that there is an *OD* surjection from $\mathbb{R}$ onto α, (b) for $\alpha < \Omega$ (and $\theta_\alpha < \Theta$), $\theta_{\alpha+1}$ is the supremum of ordinals α such that for some $A \subseteq \mathbb{R}$ of Wadge rank θ_α, there is an OD_A surjection from $\mathbb{R}$ onto α, (c) for $\beta \leq \Omega$ limit, $\theta_\beta = \sup_{\alpha<\beta}\theta_\alpha$, and (d) $\theta_\Omega = \Theta$.

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- We list some important determinacy theories in increasing consistency strength: (1) AD^+, (2) $\mathrm{AD}^+ + \Theta > \theta_0$, (3) $\mathrm{AD}_{\mathbb{R}}$, (4) $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$, (5) $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.

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- "Maximal models" of "$\mathsf{AD}^+ + \Theta = \theta_0$ or $\theta_{\alpha+1}$" have the form $\mathrm{Lp}^{\mathcal{F}}(\mathbb{R})$ for some $\mathcal{F}$.

Lots of definitions

Definition

Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{a}$, $\dot{\mathfrak{P}}$. A **$\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$) (for $\mathcal{L}_0$)** is a structure $\mathcal{M}$ for $\mathcal{L}_0$ such that $a^{\mathcal{M}} = a$, $(\mathfrak{P}^{\mathcal{M}} = \mathfrak{P})$, and there is $\lambda \in [1, \mathrm{Ord})$ such that $\lfloor \mathcal{M} \rfloor = \mathcal{J}_\lambda^{S^{\mathcal{M}}}(\hat{a})$. Let $l(\mathcal{M})$ denote λ, the **length** of $\mathcal{M}$, and let $\hat{a}^{\mathcal{M}}$ denote $\hat{a}$.

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Definition

A **$\mathcal{J}$-model over a (with parameter $\mathfrak{P}$)** is an acceptable $\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$), of the form

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P})$$

where $\dot{E}^{\mathcal{M}} = E$, etc, and letting $\lambda = l(\mathcal{M})$, the following hold.

1. $\mathcal{M}$ is amenable.
2. $S = \langle S_\xi \mid \xi \in [1, \lambda) \rangle$ is a sequence of $\mathcal{J}$-models over a (with parameter $\mathfrak{P}$).
3. For each $\xi \in [1, \lambda)$, $\dot{S}^{S_\xi} = S \restriction \xi$.
4. Suppose $E \neq \emptyset$. Then $B = \emptyset$ and E amenably codes an extender F over $\mathcal{M}$ which is $\hat{a} \times \gamma$-complete for all $\gamma < \mathrm{crit}(F)$ and the axioms from FSIT hold for $(\mathcal{M}, F)$.

Lots of definitions (cont.)

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An **operator** $\mathcal{F}$ **with domain** D is a function with domain D, such that for some cone C, possibly self-wellordered, D is the set of pairs (i, X) such that either:

1. $i = 0$ and $X \in C$, or
2. $i = 1$ and X is a $\mathcal{J}$-model over $X_1 \in C$,

and for each $(i, X) \in D$, $\mathcal{F}_i(X) = \mathcal{F}(i, X)$ is a $\mathcal{J}$-model over X such that for each $\mathcal{P} \trianglelefteq \mathcal{F}_i(X)$, $\mathcal{P}$ is fully sound. (Note that $\mathcal{P}$ is a $\mathcal{J}$-model over X, so soundness is in this sense.)

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Let $\mathcal{F}, D$ be as above. We say $\mathcal{F}$ is **forgetful** iff $\mathcal{F}_0(X) = \mathcal{F}_1(X)$ whenever $(0, X), (1, X) \in D$, and whenever X is a $\mathcal{J}$-model over X_1, and X_1 is a $\mathcal{J}$-model over $X_2 \in C$, we have $\mathcal{F}_1(X) = \mathcal{F}_1(X{\downarrow}X_2)$. Otherwise we say $\mathcal{F}$ is **historical**.

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We say $\mathcal{F}$ is **basic** iff for all $(i, X) \in D$ and $\mathcal{P} \trianglelefteq \mathcal{F}_i(X)$, we have $E^{\mathcal{P}} = \emptyset$. We say $\mathcal{F}$ is **projecting** iff for all $(i, X) \in D$, we have $\rho_\omega^{\mathcal{F}_i(X)} = X$.

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Here are some illustrations. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. The operator $\mathcal{F}(X) = X^{\#}$ is forgetful and projecting, and not basic.

Lots of definitions (cont.)

Given a $\mathcal{J}$-model $\mathcal{M}_1$ over b and a $\mathcal{J}$-model $\mathcal{M}_2$ over $\mathcal{M}_1$, we write $\mathcal{M}_2{\downarrow}b$ for the $\mathcal{J}$-model $\mathcal{M}$ over b, such that $\mathcal{M}$ is "$\mathcal{M}_1{}^\frown\mathcal{M}_2$", if this is well-defined. That is, $\mathcal{M}_2{\downarrow}b$ is the unique $\mathcal{J}$-model $\mathcal{M}$ such that $\lfloor\mathcal{M}\rfloor = \lfloor\mathcal{M}_2\rfloor$, $a^{\mathcal{M}} = b$, $E^{\mathcal{M}} = E^{\mathcal{M}_2}$, $B^{\mathcal{M}} = B^{\mathcal{M}_2}$, and $\mathcal{P} \lhd \mathcal{M}$ iff $\mathcal{P} \unlhd \mathcal{M}_1$ or there is $\mathcal{Q} \lhd \mathcal{M}_2$ such that $\mathcal{P} = \mathcal{Q}{\downarrow}b$, when such an $\mathcal{M}$ exists.

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Definition (Potential $\mathcal{F}$-premouse)

Let $\mathcal{F}$ be an operator with domain D. Let $b \in D$. A **potential $\mathcal{F}$-premouse over** b is a $\mathcal{J}$-model $\mathcal{M}$ over b such that there is an ordinal $\iota > 0$ and an increasing, closed sequence $\langle\zeta_\alpha\rangle_{\alpha\leq\iota}$ of ordinals such that for each $\alpha \leq \iota$, we have:

1. $0 = \zeta_0 \leq \zeta_\alpha \leq \zeta_\iota = l(\mathcal{M})$ (so $\mathcal{M}|\zeta_0 = b$ and $\mathcal{M}|\zeta_\iota = \mathcal{M}$).
2. If $1 < \iota$ then $\mathcal{M}|\zeta_1 = \mathcal{F}_0(b)$.
3. If $1 = \iota$ then $\mathcal{M} \trianglelefteq \mathcal{F}_0(b)$.
4. If $1 < \alpha + 1 < \iota$ then $\mathcal{M}|\zeta_{\alpha+1} = \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.
5. If $1 < \alpha + 1 = \iota$, then $\mathcal{M} \trianglelefteq \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.
6. If α is a limit then $\mathcal{M}|\zeta_\alpha$ is B-passive.

We say that $\mathcal{M}$ is **($\mathcal{F}$-)whole** iff ι is a limit or else, $\iota = \alpha + 1$ and $\mathcal{M} = \mathcal{F}_1(\mathcal{M}|\zeta_\alpha){\downarrow}b$.
A **(potential) $\mathcal{F}$-premouse** is a (potential) $\mathcal{F}$-premouse over b, for some b.

Lots of definitions (cont.)

Definition

1. Let $\mathcal{F}$ be an operator with domain D and let $b \in D$. Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over b. A **potential continuing $\mathcal{F}$-premouse over $\mathcal{N}$** is a $\mathcal{J}$-model $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M}{\downarrow}b$ is a potential $\mathcal{F}$-premouse over b. (Therefore $\mathcal{N}$ is a whole strong cutpoint of $\mathcal{M}$.)

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2. $\mathrm{Lp}^{\mathcal{F}}(a)$ denotes the stack of all countably iterable $\mathcal{F}$-premice $\mathcal{M}$ over a such that $\mathcal{M}$ is fully sound and projects to a. Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse. Then $\mathrm{Lp}^{\mathcal{F}}_{+}(\mathcal{N})$ denotes the stack of all countably iterable continuing $\mathcal{F}$-premice $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M}$ is fully sound and projects to $\mathcal{N}$.

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Remark: In defining $\mathcal{F}$-iterability, iteration trees on $\mathcal{F}$-mice are only allowed to use non-$\mathcal{F}$-extenders. Our assumptions then ensure that $\mathrm{Lp}^{\mathcal{F}}(a)$ is indeed a stack: the usual comparison argument just needs iterability, the ISC and fine structure. (Because all models which appear during the comparison are $\mathcal{F}$-premice, all extenders used are non-$\mathcal{F}$.)

Lots of definitions (cont.)

Definition (Mouse operator)

Let Y be a basic, projecting, uniformly Σ_1 operator. A **(continuing)** Y**-mouse operator** $\mathcal{F}$ is an operator with domain D such that for each $(0, X) \in D$, $\mathcal{F}_0(X) \trianglelefteq \mathrm{Lp}^Y(X)$, and for each $(1, X) \in D$, X is a sound whole Y-premouse and $\mathcal{F}_1(X) \trianglelefteq \mathrm{Lp}_+^Y(X)$.

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Let $\mathcal{F}$ be an operator and let C be some class of E-active $\mathcal{F}$-premice. Let b be transitive. A **(**C**-certified)** $L^{\mathcal{F}}[\mathbb{E}, b]$**-construction** is a sequence $\langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ with the following properties. We omit the phrase "over b".

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We have $\mathcal{N}_0 = b$ and $\mathcal{N}_1 = \mathcal{F}(0, b)$.
Let $\alpha \in (0, \lambda]$. Then $\mathcal{N}_\alpha$ is an $\mathcal{F}$-premouse. Now suppose that $\alpha < \lambda$. Then either:

1. $\mathcal{N}_\alpha$ is passive and is a limit of whole proper segments and $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, G)$ for some extender G (with $\mathcal{N}_{\alpha+1} \in C$); or
2. $\mathcal{N}_\alpha$ is ω-$\mathcal{F}$-solid. Let $\mathcal{M}_\alpha = \mathfrak{C}_\omega(\mathcal{N}_\alpha)$. Let $\mathcal{M}$ be the largest whole initial segment of $\mathcal{M}_\alpha$. (Therefore $\mathcal{M}_\alpha \triangleleft F_1(\mathcal{M})$, because $\mathcal{M}_\alpha$ is an $\mathcal{F}$-premouse.) Then $\mathcal{N}_{\alpha+1}$ is the least $\mathcal{M}_\alpha \triangleleft \mathcal{N} \trianglelefteq F_1(\mathcal{M}){\downarrow}b$ such that either $\mathcal{N} = F_1(\mathcal{M}){\downarrow}b$ or for some $k+1 < \omega$, $(\mathcal{N}, k+1)$), $\rho_{k+1}(\mathcal{N}) < \rho_\omega(\mathcal{M}_\alpha)$.

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If α is a limit then $\mathcal{N}_\alpha$ is the lim inf of the $\mathcal{N}_\beta$ for $\beta < \alpha$.

Lots of definitions (cont.)

Theorem (Schlutzenberg, T.)

Let Y be a basic, projecting, uniformly Σ_1 operator which **condenses finely**. *Let $\mathcal{F}$ be a projecting, uniformly Σ_1, whole, continuing Y-mouse operator. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ be an $L^{\mathcal{F}}[\mathbb{E}, b]$-construction ($C$-certified for some C). Suppose $k < \omega$ and that for a club of countable elementary $\pi : \mathcal{M} \to \mathfrak{C}_k(\mathcal{N}_\lambda)$, there is a Y-putative, $(k, \omega_1, \omega_1 + 1)$-iteration strategy Σ for $\mathcal{M}$, such that every Y-putative tree $\mathcal{T}$ via Σ is $(\pi, \mathbb{C})$-realizable. Then $\mathcal{N}_\lambda$ are ω-$\mathcal{F}$-solid.*

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Part of condenses finely is the following: Let Y be an operator. Suppose for all $i \in \{0, 1\}$ and $(i, \bar{X}), (i, X) \in \mathrm{dom}(Y)$, and all $\mathcal{J}$-models $\mathcal{M}^+$ over $\bar{X}$, if $\pi : \mathcal{M}^+ \to Y_i(X)$ is fully elementary, then

- if $i = 0$ then $\mathcal{M}^+ \trianglelefteq Y_0(\bar{X})$; and
- if $i = 1$ and X is a sound whole Y-premouse, then $\mathcal{M}^+ \trianglelefteq Y_1(\bar{X})$.

One also has to consider π being core maps, weak k-embeddings etc.

Strategy mice

Definition ($\mathfrak{B}(a, \mathcal{T}, b)$, $b^{\mathcal{N}}$)

Let $a, \mathcal{P}$ be transitive, with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. Let $\lambda > 0$ and let $\mathcal{T}$ be an iteration tree on $\mathcal{P}$, of length $\omega\lambda$, with $\mathcal{T}\restriction\beta \in a$ for all $\beta \leq \omega\lambda$. Let $b \subseteq \omega\lambda$. We define $\mathcal{N} = \mathfrak{B}(a, \mathcal{T}, b)$ recursively on $\mathrm{lh}(\mathcal{T})$, as the $\mathcal{J}$-model $\mathcal{N}$ over a, with parameter $\mathcal{P}$, such that:

1. $l(\mathcal{N}) = \lambda$,
2. for each $\gamma \in (0, \lambda)$, $\mathcal{N}|\gamma = \mathfrak{B}(a, \mathcal{T}\restriction\omega\gamma, [0, \omega\gamma]_{\mathcal{T}})$,
3. $B^{\mathcal{N}}$ is the set of ordinals $\mathrm{o}(a) + \gamma$ such that $\gamma \in b$,
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Let φ be an $\mathcal{L}_0$-formula. Let $\mathcal{P}$ be transitive. Let $\mathcal{M}$ be a $\mathcal{J}$-model (over some a), with parameter $\mathcal{P}$. Let $\mathcal{T} \in \mathcal{M}$. We say that φ **selects** $\mathcal{T}$ **for** $\mathcal{M}$, and write $\mathcal{T} = \mathcal{T}_{\varphi}^{\mathcal{M}}$, iff

1. $\mathcal{T}$ is the unique $x \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(x)$,
2. $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ of limit length,
3. for every $\mathcal{N} \triangleleft \mathcal{M}$, we have $\mathcal{N} \not\models \varphi(\mathcal{T})$, and
4. for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, there is $\mathcal{N} \triangleleft \mathcal{M}$ such that $\mathcal{N} \models \varphi(\mathcal{T}\restriction\lambda)$.

Strategy mice (cont.)

Definition

Let $\mathcal{P}$ be transitive and Σ a partial iteration strategy for $\mathcal{P}$. Let φ be a formula of $\mathcal{L}_0$. Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ be the operator such that:

1. $\mathcal{F}_0(a) = \mathcal{J}_1(a;\mathcal{P})$, for all transitive a such that $\mathcal{P} \in \mathcal{J}_1(\hat{a})$;
2. Let $\mathcal{M}$ be a sound branch-whole Σ-premouse of type φ. Let $\lambda = l(\mathcal{M})$ $\mathcal{T}$ be the "next tree" selected by φ. If $\mathcal{T} = \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathcal{J}_1(\mathcal{M};\mathcal{P})$. If $\mathcal{T} \neq \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathfrak{B}(\mathcal{M},\mathcal{T},b)$ where $b = \Sigma(\mathcal{T})$.

We say that $\mathcal{F}$ is a **strategy operator corresponding to Σ**.

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Can the define a Σ-premouse of type φ by "feeding in branches according to Σ using $\mathcal{F}$ and use φ to select the next tree to process. Note that if $\mathcal{M}$ is as in case 2, then every initial segment of $\mathcal{F}_1(\mathcal{M})$ is sound, and $\mathcal{F}_1(\mathcal{M})$ is indeed a $\mathcal{J}$-model.

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If Σ is suitably condensing, then $\mathcal{F}_{\Sigma,\varphi}$ is uniformly Σ_1, basic, and condenses finely.

g-organized $\mathcal{F}$-mice

Definition

Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ for a suitably condensing Σ and suppose $\mathcal{M}_1^{\mathcal{F},\sharp}$ exists. We say that $\mathcal{F}$ **determines itself on generic extensions** iff there is an $\mathcal{L}_0$ formula Ψ such that for any non-dropping $\Sigma_{\mathfrak{M}}$-iterate $\mathcal{N}$ of $\mathfrak{M}$, via an iteration tree $\mathcal{T}$, any $\mathcal{N}$-cardinal δ, and any g which is set-generic over $\mathcal{N}$, then $\mathcal{N}[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$, and $\mathcal{G}_{\mathcal{F}}\restriction(\mathcal{N}|\gamma)[g]$ is defined over $(\mathcal{N}|\gamma)[g]$ by Ψ. We call such an operator $\mathcal{F}_{\Sigma,\varphi}$ **nice operator**.

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Fix $\Sigma, \mathcal{F}$ as before; let $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\sharp}$; let M be a transitive structure. The **tree $\mathcal{T}_M$ for making M generically generic**, is the iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of maximal length such that:

1. $\mathcal{T}$ is via Σ and is everywhere non-dropping.
2. $\mathcal{T} \restriction \mathrm{o}(M) + 1$ is the tree given by linearly iterating the first total measure of $\mathfrak{M}$ and its images.
3. Suppose $\mathrm{lh}(\mathcal{T}) \geq \mathrm{o}(M) + 2$ and let $\alpha + 1 \in (\mathrm{o}(M), \mathrm{lh}(\mathcal{T}))$. Let $\delta = \delta(\mathcal{M}_\alpha^{\mathcal{T}})$ and let $\mathbb{B} = \mathbb{B}(M_\alpha^{\mathcal{T}})$ be the extender algebra of $M_\alpha^{\mathcal{T}}$ at δ. Then $E_\alpha^{\mathcal{T}}$ is the extender E with least index in $M_\alpha^{\mathcal{T}}$ such that for some condition $p \in \mathrm{Col}(\omega, M)$, $p \Vdash$ "There is a $\mathbb{B}$-axiom induced by E which fails for $\dot{x}_{\dot{G}}$".

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g-organized $\mathcal{F}$-mice (cont.)

Sargsyan noticed that one can feed in $\mathcal{F}$ into a structure $\mathcal{N}$ indirectly, by just feeding in the branches for $\mathcal{T}_{\mathcal{M}}$, for various $\mathcal{M} \trianglelefteq \mathcal{N}$. We denote ${}^{\mathrm{g}}\mathcal{F}$ the corresponding operator.

g-organized $\mathcal{F}$-mice (cont.)

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Definition

Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$. A **potential g-organized $\mathcal{F}$-premouse over** b is a potential ${}^{\mathrm{g}}\mathcal{F}$-premouse over b, with parameter $\mathfrak{M}$.

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Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.
Then ${}^{\gamma}\mathcal{F}(b)$ denotes the least $\mathcal{N} \trianglelefteq {}^{\mathrm{g}}\mathcal{F}(b)$ such that either $\mathcal{N} = {}^{\mathrm{g}}\mathcal{F}(b)$ or $\mathcal{J}_1(\mathcal{N}) \vDash$ "Θ does not exist". (Therefore $\mathcal{J}_1^m(b) \trianglelefteq {}^{\gamma}\mathcal{F}(b)$.)

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We say that $\mathcal{M}$ is a **potential Θ-g-organized $\mathcal{F}$-premouse** (over X) for $X \subseteq \mathbb{R}^{\mathcal{M}}$ iff $\mathfrak{M} \in \mathrm{HC}^{\mathcal{M}}$ and $z \in \mathbb{R}^{\mathcal{M}}$, $\mathcal{M}$ is a potential ${}^{\gamma}\mathcal{F}$-premouse over $(\mathrm{HC}^{\mathcal{M}}, X, z)$ with parameter $\mathfrak{M}$ and $(\mathcal{M}|1) \vDash$"X is self-scaled via z". We write $X^{\mathcal{M}} = X$ and $z^{\mathcal{M}} = z$.

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Let $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R}, X)$ be the stack of (countably iterable) sound Θ-g-organized $\mathcal{F}$-premice $\mathcal{M}$ over X such that $\mathbb{R}^{\mathcal{M}} = \mathbb{R}$.

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The class of Θ-g-organized $\mathcal{F}$-premice has very nice condensation property (condenses finely, and even more).

g-organized $\mathcal{F}$-mice (cont.)

Lemma

Let $\mathcal{M}$ be an $\mathcal{F}$-closed (Θ)-g-organized $\mathcal{F}$-premouse over b. Then $\mathcal{M}$ is closed under $\mathcal{F}$. In fact, for any set generic extension $\mathcal{M}[g]$ of $\mathcal{M}$, with $g \in V$, $\mathcal{M}[g]$ is closed under $\mathcal{F}$ and $\mathcal{F}{\restriction}\mathcal{M}[g]$ is definable over $\mathcal{M}[g]$, via a formula in $\mathcal{L}_0^-$, uniformly in $\mathcal{M}, g$.

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Proof sketch.

We show that $\mathcal{M}$ is closed under $\mathcal{F}$; the generalization to generic extensions of $\mathcal{M}$ and the definability of $\mathcal{F}$ is similar.

g-organized $\mathcal{F}$-mice (cont.)

Lemma

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Proof sketch.

We show that $\mathcal{M}$ is closed under $\mathcal{F}$; the generalization to generic extensions of $\mathcal{M}$ and the definability of $\mathcal{F}$ is similar.
Let $z \in \lfloor\mathcal{M}\rfloor$; we want to see that $\mathcal{F}(z) \in \lfloor\mathcal{M}\rfloor$. Let $\kappa < l(\mathcal{M})$ be such that $z \in \mathcal{M}|\kappa$ and $\mathcal{M}|\kappa$ is ${}^{g}\mathcal{F}$-whole. Let $R = {}^{g}\mathcal{F}(\mathcal{M}|\kappa)$, so $R \trianglelefteq \mathcal{M}$. Let α_0 be the least $\alpha > \kappa$ such that $R|\alpha \vDash ZF^-$. Let P be the end model of $\mathcal{T}_{R|\alpha_0}$. Let $\mathbb{P} = \mathrm{Col}(\omega, R|\alpha_0)$. Let $\dot{x}$ be the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic real coding $R|\alpha_0$. Let $\dot{z}$ be the canonical $\mathbb{P}$-name for z. Now $R \vDash$"$\mathbb{P}$ forces that $\dot{x}$ is extender algebra generic over P". Let t be the theory of $\mathcal{F}(z)$, in parameters in $\hat{z}^{<\omega}$. Then for all $\vec{w} \in \hat{z}^{<\omega}$ and formulas φ, $\varphi(\vec{w}) \in t$ iff, letting $\dot{\vec{w}}$ be the canonical $\mathbb{P}$-name for $\vec{w}$, then in R, $\mathbb{P}$ forces that $P^{\Phi}[\dot{x}] \vDash$"There is y such that $\Psi(\dot{z}, y)$ and $\varphi(\dot{\vec{w}})$ is in the theory of y". □

Scales in $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R})$

Theorem (Schlutzenberg-T., Steel)

Let $\mathcal{M}$ be a countably iterable passive Θ-g-organized $\mathcal{F}$-premouse such that $\mathcal{M} \vDash AD$. Then $\mathcal{M} \vDash$ "$\Sigma_1^{\mathcal{M}}$ has the scale property".

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Definition

Let $\mathcal{M}$ be a $\mathcal{J}$-model such that $\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|1$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{B}\}$.
We write $\mathcal{N} \prec_1 \mathcal{M}$ iff $\mathcal{N} \trianglelefteq \mathcal{M}$ and whenever ψ is an $\mathcal{L}_0^-$-Σ_1 formula then for any $a_1, ..., a_n \in \mathbb{R}^{\mathcal{M}}$,

$$\mathcal{M} \vDash \psi[a_1, ..., a_n] \Rightarrow \mathcal{N} \vDash \psi[a_1, ..., a_n].$$

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Let $\alpha \leq \beta \leq l(\mathcal{M})$. We call the interval $[\alpha, \beta]$ a **Σ_1-gap** iff (i) $\mathcal{M}|\alpha \prec_1 \mathcal{M}|\beta$; (ii) for all $\alpha' \in [1, \alpha)$, $\mathcal{M}|\alpha' \not\prec_1 \mathcal{M}|\alpha$; (iii) for all $\beta' \in (\beta, l(\mathcal{M})]$, $\mathcal{M}|\beta \not\prec_1 \mathcal{M}|\beta'$; (iv) if $\beta = l(\mathcal{M})$ then $\mathcal{M}$ is fully sound and $\mathrm{HC}^{\mathcal{J}_1(\mathcal{M})} = \mathrm{HC}^{\mathcal{M}}$ and $\mathcal{M} \not\prec_1 \mathcal{J}_1^m(\mathcal{M}, \mathfrak{P}^{\mathcal{M}}){\downarrow}a^{\mathcal{M}}$.

Scales in $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R})$ (cont.)

Definition

Let $\mathcal{M}$ be an n-sound Θ-g-organized $\mathcal{F}$-premouse. Let $n > 0$ and $b \in \mathfrak{C}_0(\mathcal{M})$. The $\mathrm{r}\Sigma_n$ **type realized by** b **over** $\mathcal{M}$, denoted $\mathrm{r}\Sigma^{\mathcal{M}}_{n,b}$, is

$$\{\varphi(v) \in \mathcal{L}_0 \mid \varphi \text{ is either } \mathrm{r}\Sigma_n \text{ or } \mathrm{r}\Pi_n \text{ and } \mathfrak{C}_0(\mathcal{M}) \models \varphi[b]\}.$$

Let $[\alpha, \beta]$ be a Σ_1-gap of $\mathcal{M}$. We say the gap is **strong** iff $\mathcal{M}|\alpha$ is admissible and letting $n < \omega$ be the least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$, then every $\mathrm{r}\Sigma_n$-type realized over $\mathcal{M}|\beta$ is realized over $\mathcal{M}|\gamma$ for some $\gamma < \beta$. We say the gap is **weak** iff $\mathcal{M}|\alpha$ is admissible but not strong.

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Theorem (Kechris-Solovay)

Inside a Σ_1-gap there are no new scales. Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse which is countably 0-iterable. Suppose $[\alpha, \beta]$ is a Σ_1-gap of $\mathcal{M}$ and $\mathcal{M}|\alpha \vDash AD$. Then:

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$ with no uniformizing function $f \in \mathcal{M}|\beta$.*
2. *For $\alpha \leq \gamma < \beta$ and all $n \in [1, \omega)$, $\mathcal{M} \vDash$ "The pointclasses $\mathrm{r}\Sigma_n^{\mathcal{M}|\gamma}$ and $\mathrm{r}\Pi_n^{\mathcal{M}|\gamma}$ do not have the scale property."*

Scale in $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R})$ (cont.)

A relation witnesing item 2 of the previous theorem is $(\mathbb{R}^{\mathcal{M}})^2 \backslash \mathcal{C}^{\mathcal{M}|\alpha}$ where $\mathcal{C}^{\mathcal{M}|\alpha}(x,y)$ iff $x, y \in \mathbb{R}^{\mathcal{M}}$ and there is $\gamma < \alpha$ such that y is $\mathcal{L}_0$-definable over $\mathcal{M}|\gamma$ from parameters in $\mathrm{Ord} \cup \{x\}$. The same relation witnesses that there is no new scale definable over the end of a strong gap.

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Theorem (Martin)

Let $\mathcal{M}$ be a Θ-g-organized $\mathcal{F}$-premouse such that $\mathcal{M}$ is countably 0-iterable. Suppose $\mathcal{M} \vDash AD$. Let $[\alpha, \beta]$ be a strong Σ_1-gap of $\mathcal{M}$ such that $\beta < l(\mathcal{M})$. Then:

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$ which has no uniformization definable over $\mathcal{M}|\beta$.*
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The very next scales pointclass is $\Sigma_1^{\mathcal{M}|(\beta+1)}$ by the previous theorem and the fact that $\mathcal{M}|(\beta+1)$ is passive.

Scale in $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R})$ (cont.)

Theorem (Schlutzenberg-T., Steel)

Let $\mathcal{M}$ be a sound, Θ-g-organized $\mathcal{F}$-mouse satisfying AD. Let $[\alpha, \beta]$ be a weak gap of $\mathcal{M}$ (possibly $\beta = l(\mathcal{M})$). Let Γ be the pointclass $\Sigma_1^{\mathcal{M}|\alpha}$. Suppose that $\mathcal{F} = \mathcal{F}\restriction\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|\alpha$ and that $\mathcal{M} \vDash$ "${}^{\mathrm{g}}\mathcal{F}$-mouse capturing for Γ holds". Let $n < \omega$ be least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$. Then $\mathcal{M} \vDash$ "$\mathrm{r}\Sigma_n^{\mathcal{M}|\beta}$ has the scale property".

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Proof idea: use the mouse capturing condition to produce a suitable mouse $\mathcal{P}$ with a iteration strategy Λ with sufficient Dodd-Jensen property that moves the term relations for a countable set of reals that are cofinal in the Wadge rank of $\mathcal{P}(\mathbb{R}) \cap \mathcal{M}|\beta$ correctly. The scales then can be constructed from the direct limit system of Λ-iterates of $\mathcal{P}$.

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Remark: The mouse capturing hypothesis can be ensured during the core model induction.

Guessing models, PFA

Fix an uncountable cardinal θ. Let $R_\theta = H_\theta$ (or V_θ). Let $X \prec R_\theta$ and $\pi_X : M_X \to X$ be the uncollapse map with critical point μ_X. Let

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Definition

X is *γ-guessing* if whenever $z \in X$ and $b \subseteq z \cap X$, if for all $c \in \mathcal{P}_\gamma(X) \cap X$, $b \cap c \in X$, then there is some $d \in X$ such that $d \cap X = b$.

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Theorem

(i) (Viale, Weiss) PFA implies for all sufficiently large regular γ, there are stationary many $X \prec H_\gamma$, $|X| = \aleph_1$, and X is $\aleph_1$-guessing.

(ii) (T.) Suppose there is a supercompact cardinal. Then in a generic extension, for all sufficiently large regular γ, there are stationary many $X \prec H_\gamma$, $|X| = \aleph_2$, $X^\omega \subseteq X$, and X is $\aleph_2$-guessing.

Guessing models, PFA (cont.)

- Is the theory "$\forall n \geq 1 \mathrm{GM}_{\aleph_{n+1},\aleph_n}$" consistent?
  Itay Neeman has informed me that he can show "$\mathrm{GM}_{\aleph_2,\aleph_1} + \mathrm{GM}_{\aleph_3,\aleph_2}$" is consistent.

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- Does $\mathrm{GM}_{\aleph_3,\aleph_2}$ follow from a higher analog of PFA?

Guessing models, PFA (cont.)

Theorem (T., 2013-2014)

Suppose $2^{\omega_2} = \omega_3$. *Suppose*

1. *either PFA,*
2. *or for some large regular* γ, *there are stationary many* $X \prec H_\gamma$, $|X| = \aleph_2$, $X^\omega \subseteq X$, *and* X *is* $\aleph_2$*-guessing.*

Then in a (homogeneous) generic extension of V, *there are models of "*$AD_{\mathbb{R}} + \Theta$ *is regular".*

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The hypothesis takes care of the: $\mathcal{F} \rightarrow \mathcal{M}_1^{\mathcal{F},\sharp}$, "last gap", and limit steps in the core model induction. The scales analysis is used as follows: suppose we have a "nice" operator $\mathcal{F}$ that has been constructed in the core model induction, and we reach a level $\mathcal{M} \lhd \mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R}, \mathcal{F}\restriction\mathbb{R})$ such that $[\alpha, \beta]$ is, say, a weak gap in $\mathcal{M}$ ($I(\mathcal{M})$ may be β) and so this is not the "last gap" of $\mathrm{Lp}^{\gamma\mathcal{F}}(\mathbb{R}, \mathcal{F}\restriction\mathbb{R})$, then letting $\Gamma = \Sigma_1^{\mathcal{M}|\alpha}$, then $Env(\Gamma) = \mathcal{P}(\mathbb{R}) \cap \mathcal{M}|\beta$. The scales analysis (and a bit more work) gives that every set in $Env(\Gamma)$ has a scale whose norm is in $\mathcal{M}|\beta$. This is enough to construct a "nice" operator $\mathcal{G}$ "beyond" $\mathcal{M}|\beta$ from a countable collection of sets that are cofinal in $Env(\Gamma)$. Then we use the hypothesis to construct $\mathcal{M}_1^{\mathcal{G},\sharp}$ and go on with the induction.

Thank you!