Topological applications of long ω_1-approximation sequences

David Milovich
Texas A&M International University

2015 Winter School in Abstract Analysis
Hejnice, Czech Republic

A long time ago, some authors used "curve" to denote an isometric copy of a graph of a function $\mathbb{R} \to \mathbb{R}$. (Continuity is not required.)

If such a curve is a measurable subset of $\mathbb{R}^2$, then it is null.

However, Sierpiński showed (1933) that, assuming CH, the plane is a countable union of graphs of functions and their converses:

- Let $\lhd$ order $\mathbb{R}$ with type ω_1.
- Let f_x map ω onto $\{y : y \unlhd x\}$.
- Let $g_n(x) = f_x(n)$.
- $\bigcup_{n<\omega}(g_n \cup g_n^{-1}) = \bigcup_{n<\omega}\bigcup_{x\in\mathbb{R}}\{(x, g_n(x)), (g_n(x), x)\} = \mathbb{R}^2$

Thus, CH implies that the plane is a countable union of curves.

Sierpiński asked (1951) if CH is needed to cover the plane by countably many curves.

Roy O. Davies answered "no" (1963) with an ingenious ZFC covering. (Never underestimate the axiom of choice!)

To cover the plane by countable many curves, it is enough to partition the plane into countably many partial curves.

Fix an ω-sequence pairwise non-parallel lines $(L_n : n < \omega)$. (For us, identical lines are considered parallel.)

Davies constructed a partition $\bigsqcup_{n<\omega} C_n = \mathbb{R}^2$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L||L_n$.

(Davies remarked that an argument of Sierpiński implicitly shows that, given a covering of $\mathbb{R}^2$ by countably many curves, there is a covering of $\mathbb{R}^2$ by countably many pairwise isometric curves.)

To a set theorist, the tastiest ingredient of Davies' proof is his following implicit lemma.

Lemma (Davies' Lemma). *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*
- $\bigcup \operatorname{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- $\overline{\mathfrak{M}}$ *has the* **Davies property***: for all $\alpha \leq \eta$,*

$\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is a* **finite** *union of substructures of $\mathfrak{A}$.*

Davies' partition of the plane applies his lemma to a partial Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$ where $\mathscr{P}$ is the set $\mathbb{R}^2$ of points in the plane and $\mathscr{L}$ is the set of lines in the plane.

We will simply let $\mathfrak{A}$ be a complete Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$. Therefore, all substructures are elementary substructures.

Let $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ be as in Davies' Lemma.

Suppose that $\alpha < \eta$ and we have constructed a partition $\bigsqcup_{n<\omega} C_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha}$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that we can extend $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C_n''' = \mathscr{P} \cap \mathfrak{M}_{<\alpha+1}$ such that $|L \cap C_n'''| \leq 1$ for all n and all lines $L || L_n$.

Let $\nu \leq \omega$ and let $\bar{p} = (p_k)_{k<\nu}$ biject from ν to $\mathscr{P} \cap \mathfrak{M}_\alpha \setminus \mathfrak{M}_{<\alpha}$.

Suppose that $k < \nu$ and we have extended $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C'_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ such that $|L \cap C'_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that that we can extend $\overline{C}'$ to a partition $\bigsqcup_{n<\omega} C''_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k+1\}$ such that $|L \cap C''_n| \leq 1$ for all n and all lines $L || L_n$.

Let $d < \omega$ and $\overline{\mathfrak{N}} = (\mathfrak{N}_i)_{i<d}$ be such that $\mathfrak{M}_{<\alpha} = \bigcup \mathrm{ran}(\overline{\mathfrak{N}})$ and each $\mathfrak{N}_i$ is a substructure of $\mathfrak{A}$.

For each $n < \omega$, let K_n be the line through p_k that is parallel to L_n.

It suffices to show that there exists $n < \omega$ such that K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$.

For each $j < k$, there is at most one $n < \omega$ such that $p_j \in K_n$.

For each $i < d$, there is at most one $n < \omega$ such that K_n intersects $\mathscr{P} \cap \mathfrak{N}_i$. Why? If $m < n < \omega$, $x \in K_m \cap \mathfrak{N}_i$, and $y \in K_n \cap \mathfrak{N}_i$, then $K_m, K_n \in \mathfrak{N}_i$; then $p_k \in \mathfrak{N}_i$ because $K_m \cap K_n = \{p_k\}$. But $p \notin \mathfrak{N}_i$.

Thus, K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ for almost all n. $\Box$

Davies' Lemma apparently was not used in print again until 2002 by Jackson and Mauldin, and then by Milovich starting in 2008.

Jackson and Mauldin constructed (in ZFC) a Steinhaus set, that is, a subset of $\mathbb{R}^2$ that intersects every isometric copy of $\mathbb{Z}^2$ at exactly one point.

Without Davies' Lemma, Jackson and Mauldin's proof would have needed CH.

We do not know if higher-dimensional analogs of Steinhaus sets exist.

References

R. O. Davies, *Covering the plane with denumerably many curves*, J. London Math. Soc. **38** (1963), 433–438.

S. Jackson and R. D. Mauldin, *On a lattice problem of H. Steinhaus*, J. Amer. Math. Soc. **15** (2002), no. 4, 817–856.

D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, Topology Appl. **156** (2008), 443–464.

D. Milovich, *The* (λ, κ)*-Freese-Nation property for Boolean algebras and compacta*, Order **29** (2012), 361–379.

D. Milovich, *On the strong Freese-Nation property* (2014), arXiv:1412.7443.

How did Davies prove his lemma? Recall:

Lemma (Davies' Lemma)**.** *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructures of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- $\overline{\mathfrak{M}}$ *has the* **Davies property***: for all $\alpha \leq \eta$,*

  $\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is a* **finite** *union of substructures of $\mathfrak{A}$.*

Proof: The Davies tree. Recursively construct as follows a sequence $(\mathfrak{B}_t : t \in T)$ with T a subtree of $\mathrm{O}rd^{<\omega}$.

- $\mathfrak{B}_{()} = \mathfrak{A}$.

- If $\mathfrak{B}_t$ is countable, declare t to be a leaf of T.

- If $|\mathfrak{B}_t| = \kappa > \aleph_0$, declare $t^\frown(\alpha) \in T$ for all $\alpha < \kappa$ and choose an increasing sequence $(\mathfrak{B}_{t^\frown(\alpha)})_{\alpha<\kappa}$ of substructures of $\mathfrak{B}_t$ with union $\mathfrak{B}_t$ such that $|\mathfrak{B}_{t^\frown(\alpha)}| < |\mathfrak{B}_t|$ for all α.

T is well-founded. Therefore, the set L of leaves of T is well ordered by its lexicographic order $<_{\mathrm{lex}}$.

Moreover, $\bigcup_{t\in L} \mathfrak{B}_t = \mathfrak{A}$.

Finally, if $t \in L$, then $\bigcup_{s<_{\mathrm{lex}}t} \mathfrak{B}_s = \bigcup_{i<\mathrm{dom}(t)} \bigcup_{\alpha<t_i} \mathfrak{B}_{(t\restriction i)^\frown(\alpha)}$. □

Note that if $|\mathfrak{A}| = \aleph_n < \aleph_\omega$, then the Davies tree has height $n + 1$. Therefore:

Lemma. *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure* **of size** $\aleph_n < \aleph_\omega$. **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- *for all $\alpha \leq \eta$, $\mathfrak{M}_{<\alpha}$ is a union* **at most n** *substructures of $\mathfrak{A}$.*

For each cardinal κ, let $H(\kappa)$ denote the set of all sets x with transitive closure $\bigcup_{n<\omega}\bigcup^n x$ of cardinality less than κ.

For each regular uncountable cardinal θ, $(H(\theta), \in)$ is a model of ZFC except possibly for the power set axiom.

We will always implicitly choose θ large enough to include all the sets and power sets we need for the problem at hand.

The notation $N \prec H(\theta)$ means that N is an elementary $\{\in\}$-substructure of $H(\theta)$.

A *long ω_1-approximation sequence* is a transfinite sequence $\overline{M} = (M_\alpha)_{\alpha<\eta}$ of countable elementary substructures of $(H(\theta), \in)$ that is **retrospective**:

for each $\alpha < \eta$, the sequence $(M_\beta)_{\beta<\alpha}$ is an element of M_α.

Warning: If α is uncountable, then $(M_\beta)_{\beta<\alpha}$, $\{M_\beta : \beta < \alpha\}$, and $M_{<\alpha} = \bigcup_{\beta<\alpha} M_\beta$ are not subsets of M_α.

If $\overline{M}$ is a long ω_1-approximation sequence, $A \in M_0$, and $0 < \alpha < \mathrm{dom}(\overline{M})$, then M_0 and α are definable from $(M_\beta)_{\beta<\alpha}$, and hence elements of M_α.

Recall that if $X \in N \prec H(\theta)$ and $|X| \leq \aleph_0$, then $X \subset N$.

Therefore, $M_0 \subset M_\alpha$ for all $\alpha \in \mathrm{dom}(\overline{M})$.

Also, $M_\beta \subset M_\alpha$ for all $\beta \leq \alpha \in \omega_1 \cap \mathrm{dom}(\overline{M})$.

More generally, for all $\alpha, \beta \in \mathrm{dom}(\overline{M})$, we have

$$M_\beta \subsetneq M_\alpha \Leftrightarrow M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha.$$

Recall that if $\mathfrak{A}$ is a first order structure for a countable language $\mathfrak{L}$ and $\mathfrak{A} \in N \prec H(\theta)$, then $\mathfrak{A} \cap N \prec_{\mathfrak{L}} \mathfrak{A}$.

Therefore, assuming $\mathfrak{A} \in M_0$, we have $\mathfrak{A} \cap M_\alpha \prec_{\mathfrak{L}} \mathfrak{A}$ for all $\alpha \in \mathrm{dom}(\overline{M})$.

Moreover, if every $M_{<\alpha}$ is a finite union of elementary substructures of $H(\theta)$ (and we will show that it is), then every $\mathfrak{A} \cap M_{<\alpha}$ is a finite union of $\mathfrak{L}$-elementary substructures of $\mathfrak{A}$.

Choose a surjection $f\colon |\mathfrak{A}| \to \mathfrak{A}$ in M_0. Assuming $|\mathfrak{A}| \leq \mathrm{dom}(\overline{M})$, we have $f(\alpha) \in M_\alpha$ for all $\alpha < |\mathfrak{A}|$. Therefore, $\bigcup_{\alpha<|\mathfrak{A}|}(\mathfrak{A} \cap M_\alpha) = \mathfrak{A}$.

Long ω_1-approximation sequences are canonical sequences of countable structures that are sufficiently rich to encode Davies trees of which they are leaves.

A Davies tree is built top-down, starting from a large structure. Long ω_1-approximation sequences are more flexibly built up from countable structures, which simplifies the construction of large structures "from scratch."

Long ω_1-approximation sequences provide a uniformly definable version of the Davies property and additional coherence properties.

The *cardinal normal form* of an ordinal α is the polynomial

$$\omega_{\beta_0} \cdot \gamma_0 + \omega_{\beta_1} \cdot \gamma_1 + \cdots + \omega_{\beta_{m-1}} \cdot \gamma_{m-1} + \gamma_m$$

that equals α and satisfies

- $\beta_0 > \cdots > \beta_{m-1} \geq 1$,
- $1 \leq \gamma_i < \omega_{\beta_i}^{+}$ for all $i < m$, and
- $\gamma_m < \omega_1$.

An example cardinal normal form:

$$\omega_{\omega+1} \cdot 4 + \omega_\omega + \omega_7 \cdot \left(\omega_7^{\omega_3^{\omega_7}} + \omega_6 \cdot \omega\right) + \omega_1 \cdot \omega_1 + (\omega^\omega + \omega \cdot 2 + 3)$$

The mapping sending each ordinal α to the code $\left(\overline{\beta}, \overline{\gamma}\right)$ for its unique cardinal normal form is uniformly definable without parameters according to the following computation.

- For every $\zeta \geq \omega_1$, let $\lfloor \zeta \rfloor$ be the greatest $|\zeta| \cdot \delta \leq \zeta$.
- For every $\zeta < \omega_1$, let $\lfloor \zeta \rfloor = \zeta$.
- For every ordinal ζ, let $\partial \zeta$ be the unique ε such that $\lfloor \zeta \rfloor + \varepsilon = \zeta$.
- For every ordinal ζ, let $\alpha_0 = \alpha$ and $\alpha_{i+1} = \partial \alpha_i$ for each $i < \omega$.
- For each $i < \omega$, let $\partial_i \alpha = \lfloor \alpha_i \rfloor$.
- Let m be least such that $\alpha_m < \omega_1$.
- For each $i < m$, let β_i satisfy $\omega_{\beta_i} = |\partial_i \alpha|$.
- For each $i < m$, let γ_i satisfy $\omega_{\beta_i} \cdot \gamma_i = \partial_i \alpha$.
- Let $\gamma_m = \partial_m \alpha$.

Given a cardinal normal form $\alpha = \sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$:

We have $\partial_i \alpha = \omega_{\beta_i} \cdot \gamma_i$ for each $i < m$ and $\partial_m \alpha = \gamma_m$.

Let $\lfloor \alpha \rfloor_i = \sum_{j<i} \partial_j \alpha$ for each $i \leq m$.

Let $\daleth(\alpha) = m + 1$ if $\gamma_m > 0$ and $\daleth(\alpha) = m$ if $\gamma_m = 0$.

Let $I_i(\alpha) = [\lfloor \alpha \rfloor_i , \lfloor \alpha \rfloor_{i+1})$ for all $i < \daleth(\alpha)$.

Fundamental Lemma. *If $(M_\alpha)_{\alpha<\eta}$ is a long ω_1-approximation sequence and $i < \daleth(\eta)$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed (with respect to $\subset$). Hence, $\bigcup\{M_\alpha : \alpha \in I_i(\eta)\} \prec H(\theta)$.*

The lemma applies to every initial segment of $\overline{M}$. Therefore, $\overline{M}$ has (the analog of) the Davies property.

Proof. Proceed by induction on η.

• If $\eta \leq \omega_1$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is a chain.

• If $\daleth(\eta) \geq 2$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed by our induction hypothesis.

Why? First, $I_i(\eta) = [\lfloor\eta\rfloor_i, \lfloor\eta\rfloor_i + \partial_i\eta)$ and $I_0(\partial_i\eta) = \partial_i\eta < \eta$.

Second, $\lfloor\alpha\rfloor_i = \lfloor\eta\rfloor_i$ for all $\alpha \in I_i(\eta)$, so each M_α can compute a decomposition $\alpha = \lfloor\eta\rfloor_i + \beta$ from the cardinal normal of α, so $\left(M_{\lfloor\eta\rfloor_i+\beta}\right)_{\beta<\partial_i\eta}$ is retrospective.

• If $\eta = \kappa\cdot\gamma$ where κ is a an uncountable cardinal, γ is a limit ordinal, and $\gamma < \kappa^+$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is directed because by our induction hypothesis $\{M_\alpha : \alpha < \kappa \cdot \beta\}$ is directed for all $\beta < \gamma$.

• The only remaining case is that $\eta = \kappa \cdot (\beta + 1)$ where κ is a an uncountable cardinal and $1 \leq \beta < \kappa^+$.

$M_{\kappa \cdot \beta}$ can compute κ and β from $\kappa \cdot \beta$ and then compute η. Therefore, $M_{\kappa \cdot \beta}$ knows that $|\eta| = \kappa$. Choose a surjection $f \colon \kappa \to \eta$ in $M_{\kappa \cdot \beta}$.

For each $\alpha < \kappa$, $M_{\kappa \cdot \beta + \alpha}$ knows the cardinal normal form $\kappa \cdot \beta + \alpha$. Hence, $f \in M_{\kappa \cdot \beta} \subset M_{\kappa \cdot \beta + \alpha}$ and $\alpha \in M_{\kappa \cdot \beta + \alpha}$; hence, $M_{f(\alpha)} \subset M_{\kappa \cdot \beta + \alpha}$.

Thus, $\{M_\alpha : \kappa \cdot \beta \leq \alpha < \eta\}$ is cofinal in $\{M_\alpha : \alpha < \eta\}$.

$\{M_\alpha : \kappa \cdot \beta \leq \alpha < \eta\}$ is directed by our induction hypothesis applied to $(M_{\kappa \cdot \beta + \alpha})_{\alpha < \kappa}$. $\square$

The Fundamental Lemma implies that every $M_{<\alpha}$ is the union of $\daleth(\alpha)$-many elementary substructures of $H(\theta)$.

By definition, $|I_{\daleth(\alpha)-2}(\alpha)| \geq \aleph_1$ and

$$|\alpha| = |I_0(\alpha)| > |I_1(\alpha)| > \cdots > |I_{\daleth(\alpha)-1}(\alpha)|.$$

Hence, if $1 \leq n < \omega$ and $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$.

Therefore, for all $n \in [1, \omega)$ and all $\alpha < \omega_n$, $M_{<\alpha}$ is the union at most n elementary substructures of $H(\theta)$.

$n = 1$ is the trivial case where $\alpha < \omega_1$ and $M_{<\alpha} \prec H(\theta)$ because $\{M_\beta : \beta < \alpha\}$ is a chain.

Given a long ω_1-approximation sequence $(M_\alpha)_{\alpha<\eta}$, let:

- $M_{<\alpha} = \bigcup\{M_\beta : \beta < \alpha\}$ for each $\alpha \leq \eta$;

- $N^i_\alpha = \bigcup\{M_\alpha : \alpha \in I_i(\eta)\}$ for each $\alpha \leq \eta$ and $i < \daleth(\alpha)$;

- $P^i_\alpha = N^i_\alpha \cap M_\alpha$ for each $\alpha < \eta$ and $i < \daleth(\alpha)$.

By the Fundamental Lemma, $M_{<\alpha} = \bigcup_{i<\daleth(\alpha)} N^i_\alpha$ and $N^i_\alpha \prec H(\theta)$.

Some easily proved coherence properties:

Starting from $\overline{M} \restriction \alpha$, M_α can compute α, then $I_i(\alpha)$, and then N^i_α. Hence, $N^i_\alpha \in M_\alpha$ and, for every $n < \omega$, M_α knows that $N^i_\alpha \prec_{\Sigma_n} H(\theta)$. Hence, $P^i_\alpha \prec M_\alpha$.

If $j < i < \daleth(\alpha)$, then $\lfloor\lfloor\alpha\rfloor_i\rfloor_j = \lfloor\alpha\rfloor_j$, so $N^j_\alpha \in M_{\lfloor\alpha\rfloor_i} \subset P^i_\alpha \subset N^i_\alpha$.

Additional coherence properties of $(M_\alpha)_{\alpha<\eta}$:

• Each $\{M_\alpha : \alpha \in I_i(\eta)\}$ is a $\vee$-semilattice (with respect to $\subset$).

• For every nonempty $I \subset \eta$, there exists $J \subset \min(I)+1$ such that $\bigcup_{\beta\in J} M_\beta$ is a directed union equal to $\bigcap_{\alpha\in I} M_\alpha$.

• For every nonempty $s \subset \beth(\eta)$,

$$\bigcap_{i\in s}\{M_\alpha : \alpha < \eta \text{ and } \exists\beta \in I_i(\eta)\ \ M_\alpha \subset M_\beta\}$$

is directed.

• If $D \subset \eta$ and $\{M_\alpha : \alpha \in D\}$ is directed (and nonempty), then there exists $i < \beth(\eta)$ such that for every $\alpha \in D$ there exists $\beta \in I_i(\eta)$ such that $M_\alpha \subset M_\beta$.

Suppose $\mathfrak{A}$ is an uncountable first order structure for a countable language $\mathfrak{L}$, $(M_\alpha)_{|\mathfrak{A}|}$ is a long ω_1-approximation sequence, and $\mathfrak{A} \in M_0$. We can recover a Davies tree from $\overline{M}$ as follows.

Let S denote the set of all $\alpha \leq |\mathfrak{A}|$ whose cardinal normal forms $\sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$ are such that $\gamma_{\daleth(\alpha)}$ is a successor ordinal.

Let $\mathcal{C}_\alpha = \mathfrak{A} \cap N_\alpha^{\daleth(\alpha)-1}$ for all $\alpha \in S$. (So $\mathcal{C}_{\beta+1} = M_\beta$ for all $\beta < |\mathfrak{A}|$.)

For each $\alpha \in S \cap |\mathfrak{A}|$, let

$$\alpha' = \begin{cases} \lfloor \alpha \rfloor_{\daleth(\alpha)-1} + |\partial_{\daleth(\alpha)-2}\alpha| & : \daleth(\alpha) \geq 2; \\ |\mathfrak{A}| & : \daleth(\alpha) = 1. \end{cases}$$

Let $\mathcal{T} = \{\mathcal{C}_\alpha : \alpha \in S\}$ and order $\mathcal{T}$ by declaring $\mathcal{C}_{\alpha'}$ to be the parent of $\mathcal{C}_\alpha$ for all $\alpha \in S \cap |\mathfrak{A}|$.

$\mathcal{T}$ is a tree with root $\mathfrak{A}$; nodes are leaves iff they are countable; the children of each non-leaf node $\mathcal{C}_\alpha$ are well-ordered by $\subset$, have cardinality less than $|\mathcal{C}_\alpha|$, and have union $\mathcal{C}_\alpha$.

Given a regular uncountable cardinal λ, define a *long λ-approximation sequence* to be a retrospective sequence $(M_\alpha)_{\alpha<\eta}$ of elementary substructures of $H(\theta)$ such that $|M_\alpha| < \lambda$ and $\lambda \cap M_\alpha \in \lambda$ for all α.

Requiring $\lambda \cap M_\alpha \in \lambda$ is equivalent to requiring that if $X \in M_\alpha$ and $|X| < \lambda$, then $X \subset M_\alpha$.

To prove the Fundamental Lemma for long λ-approximation sequences, simply replace ω_1 with λ in the proof of the lemma and in the definition of cardinal normal form.

Topological applications of long ω_1-approximation sequences

David Milovich
Texas A&M International University

2015 Winter School in Abstract Analysis
Hejnice, Czech Republic

A long time ago, some authors used “curve” to denote an isometric copy of a graph of a function $\mathbb{R} \to \mathbb{R}$. (Continuity is not required.)

If such a curve is a measurable subset of $\mathbb{R}^2$, then it is null.

However, Sierpiński showed (1933) that, assuming CH, the plane is a countable union of graphs of functions and their converses:

- Let $\lhd$ order $\mathbb{R}$ with type ω_1.
- Let f_x map ω onto $\{y : y \unlhd x\}$.
- Let $g_n(x) = f_x(n)$.
- $\bigcup_{n<\omega}(g_n \cup g_n^{-1}) = \bigcup_{n<\omega}\bigcup_{x\in\mathbb{R}}\{(x, g_n(x)), (g_n(x), x)\} = \mathbb{R}^2$

Thus, CH implies that the plane is a countable union of curves.

Sierpiński asked (1951) if CH is needed to cover the plane by countably many curves.

Roy O. Davies answered "no" (1963) with an ingenious ZFC covering. (Never underestimate the axiom of choice!)

To cover the plane by countable many curves, it is enough to partition the plane into countably many partial curves.

Fix an ω-sequence pairwise non-parallel lines $(L_n : n < \omega)$. (For us, identical lines are considered parallel.)

Davies constructed a partition $\bigsqcup_{n<\omega} C_n = \mathbb{R}^2$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L||L_n$.

(Davies remarked that an argument of Sierpiński implicitly shows that, given a covering of $\mathbb{R}^2$ by countably many curves, there is a covering of $\mathbb{R}^2$ by countably many pairwise isometric curves.)

To a set theorist, the tastiest ingredient of Davies' proof is his following implicit lemma.

Lemma (Davies' Lemma)**.** *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*

- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*

- $\overline{\mathfrak{M}}$ *has the* **Davies property***: for all $\alpha \leq \eta$,*

  $\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is a* **finite** *union of substructures of $\mathfrak{A}$.*

Davies' partition of the plane applies his lemma to a partial Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$ where $\mathscr{P}$ is the set $\mathbb{R}^2$ of points in the plane and $\mathscr{L}$ is the set of lines in the plane.

We will simply let $\mathfrak{A}$ be a complete Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$. Therefore, all substructures are elementary substructures.

Let $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ be as in Davies' Lemma.

Suppose that $\alpha < \eta$ and we have constructed a partition $\bigsqcup_{n<\omega} C_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha}$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that we can extend $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C_n''' = \mathscr{P} \cap \mathfrak{M}_{<\alpha+1}$ such that $|L \cap C_n'''| \leq 1$ for all n and all lines $L || L_n$.

Let $\nu \leq \omega$ and let $\bar{p} = (p_k)_{k<\nu}$ biject from ν to $\mathscr{P} \cap \mathfrak{M}_\alpha \setminus \mathfrak{M}_{<\alpha}$.

Suppose that $k < \nu$ and we have extended $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C'_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ such that $|L \cap C'_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that that we can extend $\overline{C}'$ to a partition $\bigsqcup_{n<\omega} C''_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k+1\}$ such that $|L \cap C''_n| \leq 1$ for all n and all lines $L || L_n$.

Let $d < \omega$ and $\overline{\mathfrak{N}} = (\mathfrak{N}_i)_{i<d}$ be such that $\mathfrak{M}_{<\alpha} = \bigcup \mathrm{ran}(\overline{\mathfrak{N}})$ and each $\mathfrak{N}_i$ is a substructure of $\mathfrak{A}$.

For each $n < \omega$, let K_n be the line through p_k that is parallel to L_n.

It suffices to show that there exists $n < \omega$ such that K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$.

For each $j < k$, there is at most one $n < \omega$ such that $p_j \in K_n$.

For each $i < d$, there is at most one $n < \omega$ such that K_n intersects $\mathscr{P} \cap \mathfrak{N}_i$. Why? If $m < n < \omega$, $x \in K_m \cap \mathfrak{N}_i$, and $y \in K_n \cap \mathfrak{N}_i$, then $K_m, K_n \in \mathfrak{N}_i$; then $p_k \in \mathfrak{N}_i$ because $K_m \cap K_n = \{p_k\}$. But $p \notin \mathfrak{N}_i$.

Thus, K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ for almost all n. $\square$

Davies' Lemma apparently was not used in print again until 2002 by Jackson and Mauldin, and then by Milovich starting in 2008.

Jackson and Mauldin constructed (in ZFC) a Steinhaus set, that is, a subset of $\mathbb{R}^2$ that intersects every isometric copy of $\mathbb{Z}^2$ at exactly one point.

Without Davies' Lemma, Jackson and Mauldin's proof would have needed CH.

We do not know if higher-dimensional analogs of Steinhaus sets exist.

References

R. O. Davies, *Covering the plane with denumerably many curves*, J. London Math. Soc. **38** (1963), 433–438.

S. Jackson and R. D. Mauldin, *On a lattice problem of H. Steinhaus*, J. Amer. Math. Soc. **15** (2002), no. 4, 817–856.

D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, Topology Appl. **156** (2008), 443–464.

D. Milovich, *The* (λ,κ)*-Freese-Nation property for Boolean algebras and compacta*, Order **29** (2012), 361–379.

D. Milovich, *On the strong Freese-Nation property* (2014), arXiv:1412.7443.

How did Davies prove his lemma? Recall:

Lemma (Davies' Lemma). *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructures of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- $\overline{\mathfrak{M}}$ *has the* **Davies property**: *for all $\alpha \leq \eta$,*

  $\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is a* **finite** *union of substructures of $\mathfrak{A}$.*

Proof: The Davies tree. Recursively construct as follows a sequence $(\mathfrak{B}_t : t \in T)$ with T a subtree of $\mathrm{O}rd^{<\omega}$.

- $\mathfrak{B}_{()} = \mathfrak{A}$.

- If $\mathfrak{B}_t$ is countable, declare t to be a leaf of T.

- If $|\mathfrak{B}_t| = \kappa > \aleph_0$, declare $t^\frown(\alpha) \in T$ for all $\alpha < \kappa$ and choose an increasing sequence $(\mathfrak{B}_{t^\frown(\alpha)})_{\alpha<\kappa}$ of substructures of $\mathfrak{B}_t$ with union $\mathfrak{B}_t$ such that $|\mathfrak{B}_{t^\frown(\alpha)}| < |\mathfrak{B}_t|$ for all α.

T is well-founded. Therefore, the set L of leaves of T is well ordered by its lexicographic order $<_{\mathrm{lex}}$.

Moreover, $\bigcup_{t\in L} \mathfrak{B}_t = \mathfrak{A}$.

Finally, if $t \in L$, then $\bigcup_{s<_{\mathrm{lex}}t} \mathfrak{B}_s = \bigcup_{i<\mathrm{dom}(t)} \bigcup_{\alpha<t_i} \mathfrak{B}_{(t\restriction i)^\frown(\alpha)}$. □

Note that if $|\mathfrak{A}| = \aleph_n < \aleph_\omega$, then the Davies tree has height $n+1$. Therefore:

Lemma. *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure* **of size** $\aleph_n < \aleph_\omega$. **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- *for all $\alpha \le \eta$, $\mathfrak{M}_{<\alpha}$ is a union* **at most n** *substructures of $\mathfrak{A}$.*

For each cardinal κ, let $H(\kappa)$ denote the set of all sets x with transitive closure $\bigcup_{n<\omega} \bigcup^n x$ of cardinality less than κ.

For each regular uncountable cardinal θ, $(H(\theta), \in)$ is a model of ZFC except possibly for the power set axiom.

We will always implicitly choose θ large enough to include all the sets and power sets we need for the problem at hand.

The notation $N \prec H(\theta)$ means that N is an elementary $\{\in\}$-substructure of $H(\theta)$.

A *long ω_1-approximation sequence* is a transfinite sequence $\overline{M} = (M_\alpha)_{\alpha<\eta}$ of countable elementary substructures of $(H(\theta), \in)$ that is **retrospective**:

for each $\alpha < \eta$, the sequence $(M_\beta)_{\beta<\alpha}$ is an element of M_α.

Warning: If α is uncountable, then $(M_\beta)_{\beta<\alpha}$, $\{M_\beta : \beta < \alpha\}$, and $M_{<\alpha} = \bigcup_{\beta<\alpha} M_\beta$ are not subsets of M_α.

If $\overline{M}$ is a long ω_1-approximation sequence, $A \in M_0$, and $0 < \alpha <$ dom($\overline{M}$), then M_0 and α are definable from $(M_\beta)_{\beta<\alpha}$, and hence elements of M_α.

Recall that if $X \in N \prec H(\theta)$ and $|X| \leq \aleph_0$, then $X \subset N$.

Therefore, $M_0 \subset M_\alpha$ for all $\alpha \in$ dom($\overline{M}$).

Also, $M_\beta \subset M_\alpha$ for all $\beta \leq \alpha \in \omega_1 \cap$ dom($\overline{M}$).

More generally, for all $\alpha, \beta \in$ dom($\overline{M}$), we have

$$M_\beta \subsetneq M_\alpha \Leftrightarrow M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha.$$

Recall that if $\mathfrak{A}$ is a first order structure for a countable language $\mathfrak{L}$ and $\mathfrak{A} \in N \prec H(\theta)$, then $\mathfrak{A} \cap N \prec_{\mathfrak{L}} \mathfrak{A}$.

Therefore, assuming $\mathfrak{A} \in M_0$, we have $\mathfrak{A} \cap M_\alpha \prec_{\mathfrak{L}} \mathfrak{A}$ for all $\alpha \in \mathrm{dom}(\overline{M})$.

Moreover, if every $M_{<\alpha}$ is a finite union of elementary substructures of $H(\theta)$ (and we will show that it is), then every $\mathfrak{A} \cap M_{<\alpha}$ is a finite union of $\mathfrak{L}$-elementary substructures of $\mathfrak{A}$.

Choose a surjection $f\colon |\mathfrak{A}| \to \mathfrak{A}$ in M_0. Assuming $|\mathfrak{A}| \leq \mathrm{dom}(\overline{M})$, we have $f(\alpha) \in M_\alpha$ for all $\alpha < |\mathfrak{A}|$. Therefore, $\bigcup_{\alpha<|\mathfrak{A}|}(\mathfrak{A} \cap M_\alpha) = \mathfrak{A}$.

Long ω_1-approximation sequences are canonical sequences of countable structures that are sufficiently rich to encode Davies trees of which they are leaves.

A Davies tree is built top-down, starting from a large structure. Long ω_1-approximation sequences are more flexibly built up from countable structures, which simplifies the construction of large structures "from scratch."

Long ω_1-approximation sequences provide a uniformly definable version of the Davies property and additional coherence properties.

The *cardinal normal form* of an ordinal α is the polynomial

$$\omega_{\beta_0} \cdot \gamma_0 + \omega_{\beta_1} \cdot \gamma_1 + \cdots + \omega_{\beta_{m-1}} \cdot \gamma_{m-1} + \gamma_m$$

that equals α and satisfies

- $\beta_0 > \cdots > \beta_{m-1} \geq 1$,
- $1 \leq \gamma_i < \omega_{\beta_i}^+$ for all $i < m$, and
- $\gamma_m < \omega_1$.

An example cardinal normal form:

$$\omega_{\omega+1} \cdot 4 + \omega_\omega + \omega_7 \cdot \left(\omega_7^{\omega_3^{\omega_7}} + \omega_6 \cdot \omega\right) + \omega_1 \cdot \omega_1 + (\omega^\omega + \omega \cdot 2 + 3)$$

The mapping sending each ordinal α to the code $\left(\overline{\beta}, \overline{\gamma}\right)$ for its unique cardinal normal form is uniformly definable without parameters according to the following computation.

- For every $\zeta \geq \omega_1$, let $\lfloor\zeta\rfloor$ be the greatest $|\zeta| \cdot \delta \leq \zeta$.
- For every $\zeta < \omega_1$, let $\lfloor\zeta\rfloor = \zeta$.
- For every ordinal ζ, let $\partial\zeta$ be the unique ε such that $\lfloor\zeta\rfloor + \varepsilon = \zeta$.
- For every ordinal ζ, let $\alpha_0 = \alpha$ and $\alpha_{i+1} = \partial\alpha_i$ for each $i < \omega$.
- For each $i < \omega$, let $\partial_i\alpha = \lfloor\alpha_i\rfloor$.
- Let m be least such that $\alpha_m < \omega_1$.
- For each $i < m$, let β_i satisfy $\omega_{\beta_i} = |\partial_i\alpha|$.
- For each $i < m$, let γ_i satisfy $\omega_{\beta_i} \cdot \gamma_i = \partial_i\alpha$.
- Let $\gamma_m = \partial_m\alpha$.

Given a cardinal normal form $\alpha = \sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$:

We have $\partial_i \alpha = \omega_{\beta_i} \cdot \gamma_i$ for each $i < m$ and $\partial_m \alpha = \gamma_m$.

Let $\lfloor \alpha \rfloor_i = \sum_{j<i} \partial_j \alpha$ for each $i \leq m$.

Let $\daleth(\alpha) = m + 1$ if $\gamma_m > 0$ and $\daleth(\alpha) = m$ if $\gamma_m = 0$.

Let $I_i(\alpha) = [\lfloor \alpha \rfloor_i , \lfloor \alpha \rfloor_{i+1})$ for all $i < \daleth(\alpha)$.

Fundamental Lemma. *If $(M_\alpha)_{\alpha<\eta}$ is a long ω_1-approximation sequence and $i < \daleth(\eta)$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed (with respect to $\subset$). Hence, $\bigcup\{M_\alpha : \alpha \in I_i(\eta)\} \prec H(\theta)$.*

The lemma applies to every initial segment of $\overline{M}$. Therefore, $\overline{M}$ has (the analog of) the Davies property.

Proof. Proceed by induction on η.

• If $\eta \leq \omega_1$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is a chain.

• If $\daleth(\eta) \geq 2$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed by our induction hypothesis.

Why? First, $I_i(\eta) = [\lfloor\eta\rfloor_i, \lfloor\eta\rfloor_i + \partial_i\eta)$ and $I_0(\partial_i\eta) = \partial_i\eta < \eta$.

Second, $\lfloor\alpha\rfloor_i = \lfloor\eta\rfloor_i$ for all $\alpha \in I_i(\eta)$, so each M_α can compute a decomposition $\alpha = \lfloor\eta\rfloor_i + \beta$ from the cardinal normal of α, so $\left(M_{\lfloor\eta\rfloor_i+\beta}\right)_{\beta<\partial_i\eta}$ is retrospective.

• If $\eta = \kappa \cdot \gamma$ where κ is a an uncountable cardinal, γ is a limit ordinal, and $\gamma < \kappa^+$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is directed because by our induction hypothesis $\{M_\alpha : \alpha < \kappa \cdot \beta\}$ is directed for all $\beta < \gamma$.

• The only remaining case is that $\eta = \kappa \cdot (\beta + 1)$ where κ is a an uncountable cardinal and $1 \le \beta < \kappa^+$.

$M_{\kappa \cdot \beta}$ can compute κ and β from $\kappa \cdot \beta$ and then compute η. Therefore, $M_{\kappa \cdot \beta}$ knows that $|\eta| = \kappa$. Choose a surjection $f \colon \kappa \to \eta$ in $M_{\kappa \cdot \beta}$.

For each $\alpha < \kappa$, $M_{\kappa \cdot \beta + \alpha}$ knows the cardinal normal form $\kappa \cdot \beta + \alpha$. Hence, $f \in M_{\kappa \cdot \beta} \subset M_{\kappa \cdot \beta + \alpha}$ and $\alpha \in M_{\kappa \cdot \beta + \alpha}$; hence, $M_{f(\alpha)} \subset M_{\kappa \cdot \beta + \alpha}$.

Thus, $\{M_\alpha : \kappa \cdot \beta \le \alpha < \eta\}$ is cofinal in $\{M_\alpha : \alpha < \eta\}$.

$\{M_\alpha : \kappa \cdot \beta \le \alpha < \eta\}$ is directed by our induction hypothesis applied to $(M_{\kappa \cdot \beta + \alpha})_{\alpha < \kappa}$. $\Box$

The Fundamental Lemma implies that every $M_{<\alpha}$ is the union of $\daleth(\alpha)$-many elementary substructures of $H(\theta)$.

By definition, $|I_{\daleth(\alpha)-2}(\alpha)| \geq \aleph_1$ and

$$|\alpha| = |I_0(\alpha)| > |I_1(\alpha)| > \cdots > |I_{\daleth(\alpha)-1}(\alpha)|.$$

Hence, if $1 \leq n < \omega$ and $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$.

Therefore, for all $n \in [1, \omega)$ and all $\alpha < \omega_n$, $M_{<\alpha}$ is the union at most n elementary substructures of $H(\theta)$.

$n = 1$ is the trivial case where $\alpha < \omega_1$ and $M_{<\alpha} \prec H(\theta)$ because $\{M_\beta : \beta < \alpha\}$ is a chain.

Given a long ω_1-approximation sequence $(M_\alpha)_{\alpha<\eta}$, let:

- $M_{<\alpha} = \bigcup\{M_\beta : \beta < \alpha\}$ for each $\alpha \le \eta$;
- $N^i_\alpha = \bigcup\{M_\alpha : \alpha \in I_i(\eta)\}$ for each $\alpha \le \eta$ and $i < \daleth(\alpha)$;
- $P^i_\alpha = N^i_\alpha \cap M_\alpha$ for each $\alpha < \eta$ and $i < \daleth(\alpha)$.

By the Fundamental Lemma, $M_{<\alpha} = \bigcup_{i<\daleth(\alpha)} N^i_\alpha$ and $N^i_\alpha \prec H(\theta)$.

Some easily proved coherence properties:

Starting from $\overline{M} \restriction \alpha$, M_α can compute α, then $I_i(\alpha)$, and then N^i_α. Hence, $N^i_\alpha \in M_\alpha$ and, for every $n < \omega$, M_α knows that $N^i_\alpha \prec_{\Sigma_n} H(\theta)$. Hence, $P^i_\alpha \prec M_\alpha$.

If $j < i < \daleth(\alpha)$, then $\lfloor\lfloor\alpha\rfloor_i\rfloor_j = \lfloor\alpha\rfloor_j$, so $N^j_\alpha \in M_{\lfloor\alpha\rfloor_i} \subset P^i_\alpha \subset N^i_\alpha$.

Additional coherence properties of $(M_\alpha)_{\alpha<\eta}$:

• Each $\{M_\alpha : \alpha \in I_i(\eta)\}$ is a $\vee$-semilattice (with respect to $\subset$).

• For every nonempty $I \subset \eta$, there exists $J \subset \min(I) + 1$ such that $\bigcup_{\beta \in J} M_\beta$ is a directed union equal to $\bigcap_{\alpha \in I} M_\alpha$.

• For every nonempty $s \subset \daleth(\eta)$,

$$\bigcap_{i \in s} \{M_\alpha : \alpha < \eta \text{ and } \exists \beta \in I_i(\eta)\ \ M_\alpha \subset M_\beta\}$$

is directed.

• If $D \subset \eta$ and $\{M_\alpha : \alpha \in D\}$ is directed (and nonempty), then there exists $i < \daleth(\eta)$ such that for every $\alpha \in D$ there exists $\beta \in I_i(\eta)$ such that $M_\alpha \subset M_\beta$.

Suppose $\mathfrak{A}$ is an uncountable first order structure for a countable language $\mathfrak{L}$, $(M_\alpha)_{|\mathfrak{A}|}$ is a long ω_1-approximation sequence, and $\mathfrak{A} \in M_0$. We can recover a Davies tree from $\overline{M}$ as follows.

Let S denote the set of all $\alpha \leq |\mathfrak{A}|$ whose cardinal normal forms $\sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$ are such that $\gamma_{\daleth(\alpha)}$ is a successor ordinal.

Let $\mathcal{C}_\alpha = \mathfrak{A} \cap N_\alpha^{\daleth(\alpha)-1}$ for all $\alpha \in S$. (So $\mathcal{C}_{\beta+1} = M_\beta$ for all $\beta < |\mathfrak{A}|$.)

For each $\alpha \in S \cap |\mathfrak{A}|$, let

$$\alpha' = \begin{cases} \lfloor \alpha \rfloor_{\daleth(\alpha)-1} + |\partial_{\daleth(\alpha)-2}\alpha| & : \daleth(\alpha) \geq 2; \\ |\mathfrak{A}| & : \daleth(\alpha) = 1. \end{cases}$$

Let $\mathcal{T} = \{\mathcal{C}_\alpha : \alpha \in S\}$ and order $\mathcal{T}$ by declaring $\mathcal{C}_{\alpha'}$ to be the parent of $\mathcal{C}_\alpha$ for all $\alpha \in S \cap |\mathfrak{A}|$.

$\mathcal{T}$ is a tree with root $\mathfrak{A}$; nodes are leaves iff they are countable; the children of each non-leaf node $\mathcal{C}_\alpha$ are well-ordered by $\subset$, have cardinality less than $|\mathcal{C}_\alpha|$, and have union $\mathcal{C}_\alpha$.

Given a regular uncountable cardinal λ, define a *long λ-approximation sequence* to be a retrospective sequence $(M_\alpha)_{\alpha<\eta}$ of elementary substructures of $H(\theta)$ such that $|M_\alpha| < \lambda$ and $\lambda \cap M_\alpha \in \lambda$ for all α.

Requiring $\lambda \cap M_\alpha \in \lambda$ is equivalent to requiring that if $X \in M_\alpha$ and $|X| < \lambda$, then $X \subset M_\alpha$.

To prove the Fundamental Lemma for long λ-approximation sequences, simply replace ω_1 with λ in the proof of the lemma and in the definition of cardinal normal form.

Topological applications of long ω_1-approximation sequences II

David Milovich
Texas A&M International University

2015 Winter School in Abstract Analysis
Hejnice, Czech Republic

Let P be a poset. For $p \in P$, let $p\uparrow = \{q : q \geq p\}$.

Definition (Peregudov). *Define the Noetherian type* $\mathrm{Nt}(P)$ *of* P *to be the least infinite cardinal* κ *for which* $|p\uparrow| < \kappa$ *for all* $p \in P$.

Define the Noetherian type $\mathrm{Nt}(X)$ *of a topological space* X *to be the least* $\mathrm{Nt}(\mathcal{B})$ *where* $\mathcal{B}$ *is a base of* X *and* $\mathcal{B}$ *is ordered with respect to* $\subset$.

(Recall that a topological base is a family $\mathcal{B}$ of open sets such that for every $p \in U$ with U open, some $B \in \mathcal{B}$ satisfies $p \in B \subset U$.)

As a topological cardinal function, Nt is somewhat unusual. A few examples:

• If $\mathcal{B}$ is a base of X, then $\text{Nt}(X^{|\mathcal{B}|}) = \aleph_0$. Hence, there are compact spaces X, Y such that $\text{Nt}(X \times Y) < \max\{\text{Nt}(X), \text{Nt}(Y)\}$.

• There are Tychonoff spaces X, Y such that

$$\text{Nt}(X \times Y) < \min\{\text{Nt}(X), \text{Nt}(Y)\}.$$

We do not know if there is a compact example of this. However, GCH implies that $\text{Nt}(X^n) = \text{Nt}(X)$ for all compact homogeneous X.

• The countably supported box product topology on $2^{\aleph_\omega}$ has Noetherian type in $[\aleph_1, \aleph_4]$, with $\aleph_1$ and $\aleph_2$ consistent, and the consistency of $\aleph_3$ and $\aleph_4$ unknown.

References

M. Kojman, D. Milovich, and S. Spadaro, *Noetherian type in topological products*, *Israel Journal of Mathematics* **202** (2014), 195–225.

D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, *Topology and its Applications* **156** (2008), 443–464.

S. A. Peregudov, *On the Noetherian type of topological spaces*, Comment. Math. Univ. Carolin. **38** (1997), no. 3, 581–586.

A compact space is *dyadic* if it is a continuous image of some 2^κ.

If X is the quotient of $2^\omega \oplus 2^{\omega_1}$ induced by identifying $(0)_{n<\omega}$ and $(0)_{\alpha<\omega_1}$, then X is dyadic and $\mathrm{Nt}(X) = \aleph_2$.

More generally, $\mathrm{Nt}(X) > \kappa$ if κ is a regular cardinal, X is a space, $p \in X$, some local π-base at p is smaller than κ, and no local base at p is smaller than κ.

Recall that a local base (local π-base) at p is a coinitial family $\mathcal{U}$ of open neighborhoods of p. That is, $p \in U$ ($U \neq \varnothing$) and U is open for all $U \in \mathcal{U}$, and if $p \in O$ and O is open, then $U \subset O$ for some $U \in \mathcal{U}$.

A space H is *homogeneous* if for all $p, q \in H$ there exists a homeomorphism $f\colon H \to H$ such that $f(p) = q$.

Theorem (Milovich, 2008). $\mathrm{Nt}(X) = \aleph_0$ *for all homogeneous dyadic compact* X*.*

Corollary. $\mathrm{Nt}(G) = \aleph_0$ *for all compact groups* G*.*

Proof. All topological groups are homogeneous. By the Ivanovskiĭ–Kuz′minov Theorem (1959), compact groups are also dyadic. □

The *weight* $\mathsf{w}(X)$ of a space X is the least infinite cardinal κ such that X has a base not larger than κ.

The *π-character* $\pi\chi(p, X)$ of a point p in a space X is the least infinite cardinal κ such that p has a local π-base not larger than κ.

Theorem (Gerlits, 1976; Efimov, 1977). *If X is compact and dyadic, then $\sup_{p\in X} \pi\chi(p, X) = \mathsf{w}(X)$.*

Corollary. *If X is compact, homogeneous, and dyadic, then, for all $p \in X$, $\pi\chi(p, X) = \mathsf{w}(X)$.*

Theorem (Milovich–Spadaro, 2014). *If X is compact, κ is a regular uncountable cardinal, $\mathsf{w}(X) \geq \kappa$, and $\pi\chi(p, X) < \kappa$ on a dense set of $p \in X$, then $\mathsf{Nt}(X) > \kappa$.*

Every metric space has Noetherian type $\aleph_0$. Why? Take $\mathcal{B} = \bigcup_{n<\omega} \mathcal{R}_n$ where each $\mathcal{R}_n$ is a locally finite open cover refining the balls of diameter 2^{-n}.

A topological base $\mathcal{B}$ is called *efficient* if

- it has Noetherian type $\aleph_0$,
- $U \subsetneq V \Rightarrow \overline{U} \subset V$ for all $U, V \in \mathcal{B}$, and
- for all infinite $\mathscr{S} \subset \mathcal{B}$, the set $\{T \in \mathcal{B} : \exists S \in \mathscr{S} \;\; S \subsetneq T\}$ is infinite.

Lemma. *Every base of a compact metric space K contains an efficient base of K.*

Proof. Given a base $\mathcal{B}$ of K, we will choose a sequence $(\mathcal{A}_n)_{n<\omega}$ of finite open subcovers of $\mathcal{B}$ such that $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$ will be an efficient base.

Given $n < \omega$ and $(\mathcal{A}_m)_{m<n}$, choose, for each $p \in K$, an neighborhood N_p of p in $\mathcal{B}$ sufficently small that

1. $\text{diam}(N_p) \leq 2^{-n}$,
2. $\overline{N_p} \subset \bigcap\{A : p \in A \in \bigcup_{m<n} \mathcal{A}_m\}$,
3. $N_p \cap A = \varnothing$ or $N_p = A$ for all singleton $A \in \bigcup_{m<n} \mathcal{A}_m$, and
4. $\text{diam}(N_p) < \text{diam}(A)$ for all non-singleton $A \in \bigcup_{m<n} \mathcal{A}_m$.

Choose $\mathcal{A}_n$ to be a minimal (finite) subcover of $\{N_p : p \in K\}$.

Since $\max_{A \in \mathcal{A}_n} \operatorname{diam}(A) \leq 2^{-n}$, $\mathcal{A}$ will be a base.

Since also each $\mathcal{A}_n$ is finite, $\operatorname{Nt}(\mathcal{A}) = \aleph_0$.

Since $\operatorname{diam}(A) < \operatorname{diam}(B)$ for all $m > n$, $A \in \mathcal{A}_m$, and $B \in \mathcal{A}_n \setminus [K]^1$,

$$\text{if } \mathcal{A}_i \ni U \supsetneq V \in \mathcal{A}_j, \text{ then } i \leq j.$$

Since also each $\mathcal{A}_n$ is a minimal cover,

$$\text{if } \mathcal{A}_i \ni U \supsetneq V \in \mathcal{A}_j, \text{ then } i < j.$$

Since also $\mathcal{A}_i \ni U \supsetneq V \in \mathcal{A}_j$ and $i < j$ imply $U \supset \overline{V}$,

$$U \supsetneq V \Rightarrow U \supset \overline{V} \text{ for all } U, V \in \mathcal{A}.$$

Finally, given a finite $\mathcal{F} \subset \mathcal{A}$ and a non-repeating sequence $(U_n)_{n<\omega}$ of elements of $\mathcal{A}$, it suffices to find some U_n with a strict superset in $\mathcal{A} \setminus \mathcal{F}$.

Since $(U_n)_{n<\omega}$ is non-repeating and each $\mathcal{A}_n$ is finite, we may pass to a subsequence $(V_n)_{n<\omega}$ of $(U_n)_{n<\omega}$ that diam$(V_n) \to 0$.

We may then pass to a subsequence $(W_n)_{n<\omega}$ such that $(\overline{W_n})_{n<\omega}$ converges to a singleton $\{p\}$ (in the (compact) Vietoris hyperspace).

Since $(W_n)_{n<\omega}$ is non-repeating, p is not an isolated point.

Hence, p has a neighborhoods $Y, Z \in \mathcal{A} \setminus \mathcal{F}$ such that $Y \subsetneq Z$.

For m sufficiently large, $W_m \subset Y \subsetneq Z$.

Let X be a compact space of uncountable weight κ. Without loss of generality, X is a subspace of $[0,1]^\kappa$.

Let $\mathcal{A}$ be a base of X of size κ and consisting only of nonempty open F_σ sets.

(To find such a base, take any base $\mathcal{Z}$ of size κ and, for each finite subcover of $\mathcal{Z}$, choose a refining finite cover by open F_σ sets; take $\mathcal{A}$ to be the union these refinements.)

Given a function f and a set I, let $f \restriction I$ denote the restriction of f to $\mathrm{dom}(f) \cap I$. Given a set E of functions, let $E \restriction I$ denote $\{f \restriction I : f \in E\}$. Given a set $\mathcal{J}$ of sets of functions, let $\mathcal{J} \restriction I = \{E \restriction I : E \in \mathcal{J}\}$.

We say that $E \subset X$ is *supported* on a set I if, for all $p, q \in X$, if $p \restriction I = q \restriction I$, then $p \in E \Leftrightarrow q \in E$.

By compactness of X, every open F_σ set has a countable support.

Assume that there is a continuous surjection $h\colon 2^\lambda \to X$.

Let $(M_\alpha)_{\alpha<\kappa}$ be a long ω_1-approximation sequence with $\mathcal{A}, h \in M_0$.

Letting $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$, each $U \in \mathcal{A}_\alpha$ is supported on M_α.
Why? Each $U \in \mathcal{A} \cap M_\alpha$ is supported on some countable C. M_α knows this; hence, we may choose $C \in M_\alpha$; hence, $C \subset M_\alpha$.

For each $\alpha < \kappa$, $\mathcal{A}_\alpha \restriction M_\alpha$ is a base of $X \restriction M_\alpha$.

Why? Given $p \in X$, if R is an open product of rational intervals such that $p \in R$ and $R \cap X$ is supported on M_α, then $R \cap X$ is supported on a finite $F \subset M_\alpha$ and there is a closed product Q of rational intervals such that $p \in Q \subset R$ and $Q \cap X$ is suppported on F. M_α knows about a finite cover of $Q \cap X$ by elements of $\mathcal{A}$ with union contained in $R \cap X$. Hence, $p \in A \subset R \cap X$ and $A \in \mathcal{A} \cap M_\alpha$ for some A in this cover. Hence, $p \restriction M_\alpha \in A \restriction M_\alpha \subset (R \cap X) \restriction M_\alpha$ and $A \restriction M_\alpha$ is open in $X \restriction M_\alpha$ because A is supported on M_α.

We may choose $\mathcal{Y}_\alpha \subset \mathcal{A}_\alpha \restriction M_\alpha$ to be an efficient base of $X \restriction M_\alpha$. (Why? Every compact space with countable weight is metrizable.)

Because each $A \in \mathcal{A}_\alpha$ is supported on M_α, there is a unique $\mathcal{W}_\alpha \subset \mathcal{A}_\alpha$ such that $\mathcal{Y}_\alpha = \mathcal{W}_\alpha \restriction M_\alpha$.

Given E a subset of a poset P, let $\uparrow E = \bigcup\{p\uparrow : p \in E\}$.

Let $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$ where $\mathcal{W}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{W}_\beta$.

Let $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \ \ \overline{U} \subset V\}$.

Assume that $\min_{p \in X} \pi\chi(p, X) = \kappa$.

We claim that $\mathcal{U} = \mathcal{U}_{<\kappa}$ is a base of X with Noetherian type $\aleph_0$.

First, we show that $\mathcal{U}$ is a base.

Given $p \in A \in \mathcal{A}$, we need to find $U \in \mathcal{U}$ such that $p \in U \subset A$. Choose $\alpha < \kappa$ such that $A \in M_\alpha$. Then A is supported on M_α just as each $U \in \mathcal{U}_\alpha$ is, so it suffices to show that $\mathcal{U}_\alpha \restriction M_\alpha$ is a base of $X \restriction M_\alpha$.

$\mathcal{U}_\alpha$ is a downward-closed subset of $\mathcal{W}_\alpha$. Therefore, $\mathcal{U}_\alpha \restriction M_\alpha$ is a downard-closed subset of the base $\mathcal{W}_\alpha \restriction M_\alpha$. Hence, it suffices to show that $\mathcal{U}_\alpha \restriction M_\alpha$ covers $X \restriction M_\alpha$.

Because $\mathcal{A}_{<\alpha}$ is too small to contain a local π-base, M_α knows about a finite cover of X by elements of $\mathcal{A} \setminus \uparrow \mathcal{A}_{<\alpha}$. We have $p \in T \in M_\alpha$ for some T in this cover.

$T \restriction M_\alpha$ is open, so we may choose $R, S \in \mathcal{W}_\alpha \restriction M_\alpha$ such that

$$p \restriction M_\alpha \in \overline{R} \subset S \subset T \restriction M_\alpha.$$

R meets all the requireements for being in $\mathcal{U}_\alpha \restriction M_\alpha$.

It remains to show that $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

For this, we must actually use the continuous surjection $h\colon 2^\lambda \to X$.

Let $\mathcal{B}$ denote the clopen algebra $\mathrm{Clop}(2^\lambda)$.

Since $\mathcal{W}_\alpha \restriction M_\alpha$ is an efficient base, for each α and $W \in \mathcal{W}_\alpha$, there is an $E_{\alpha,W} \in \mathcal{B} \cap M_\alpha$ such that

$$h^{-1}[\overline{W}] \subset E_{\alpha,W} \subset \bigcap\{h^{-1}[Z] : \overline{W} \subset Z \in \mathcal{W}_\alpha\}$$

because only there are only finitely many Z as above.

Letting $\mathcal{E}_\alpha = \{E_{\alpha,W} : W \in \mathcal{W}_\alpha\}$, we have $\mathrm{Nt}(\mathcal{E}_\alpha) = \aleph_0$.

Why? If $E_{\alpha,R} \subsetneq E_{\alpha,S_m} \neq E_{\alpha,S_n}$ for all $m < n < \omega$, then, for all $m < \omega$ and $\overline{S_m} \subset T \in \mathcal{W}_\alpha$, we have $R \subset T$. By the definition of efficient base, there are infinitely many T as above, in contradiction with $\mathrm{Nt}(\mathcal{W}_\alpha) = \aleph_0$.

Let $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$. We have $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ because $\mathcal{D}_\alpha \subset \mathcal{E}_\alpha$.

Let $\mathcal{C} = \mathcal{B} \cap \uparrow \{h^{-1}[U] : U \in \mathcal{U}\}$.

Let $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$. Note that $\mathcal{D}_\alpha \subset \mathcal{C}_\alpha$.

Letting $\mathcal{D} = \mathcal{D}_{<\kappa}$, we claim that $\mathrm{Nt}(\mathcal{D}) = \aleph_0$.

To prove this, it suffices to show that, for all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,

2. $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and

3. $H \uparrow \cap \mathcal{D}_\alpha = \varnothing$.

To be continued...

Topological applications of long ω_1-approximation sequences III

David Milovich
Texas A&M International University

2015 Winter School in Abstract Analysis
Hejnice, Czech Republic

Outline of a proof of $\mathrm{Nt}(X) = \aleph_0$ where $h\colon 2^\lambda \to [0,1]^\kappa$ is continuous, $X = h[2^\lambda]$, and $\pi\chi(p, X) = \mathrm{w}(X) = \kappa$ for all $p \in X$:

1. $\mathcal{A}$ is a base of X of size κ consisting of F_σ sets.
2. $(M_\alpha)_{\alpha<\kappa}$ is a long ω_1-approximation sequence with $h, \mathcal{A} \in M_0$.
3. $\mathcal{W}_\alpha \restriction M_\alpha \subset \mathcal{A}_\alpha \restriction M_\alpha$ is an efficient base of $X \restriction M_\alpha$.
4. $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$.
5. $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \ \ \overline{U} \subset V\}$.
6. $\mathcal{U} = \mathcal{U}_{<\kappa}$ is a base of X.
7. $h^{-1}[\overline{U}] \subset E_{\alpha,U}$ clopen $\subset \bigcap\{h^{-1}[W] : \overline{U} \subset W \in \mathcal{W}_\alpha\}$.
8. $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$.

---

9. $\mathrm{Nt}(\mathcal{D}) = \aleph_0$ where $\mathcal{D} = \mathcal{D}_{<\kappa}$.
10. $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

Let $\mathcal{B} = \mathrm{Clop}(2^\lambda)$.

Let $\mathcal{C} = \mathcal{B} \cap \uparrow \{h^{-1}[U] : U \in \mathcal{U}\}$.

Let $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$. Note that $\mathcal{D}_\alpha \subset \mathcal{C}_\alpha$.

To prove $\mathrm{Nt}(\mathcal{D}) = \aleph_0$, it suffices to show that, for all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,

2. $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and

3. $H \uparrow \cap \mathcal{D}_\alpha = \varnothing$.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
(2) $H\uparrow \cap \mathcal{D}_{<\alpha}$ *is finite, and*
(3) $H\uparrow \cap \mathcal{D}_\alpha = \varnothing$*:*

To prove $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$, suppose that $K \in \mathcal{C}_\alpha$.

Then M_α knows that $h^{-1}[A] \subset K$ for some $A \in \mathcal{A}$.

So, choosing A as above in $\mathcal{A}_\alpha$, we then find $\overline{U} \subset W \subset A$ where $U \in \mathcal{U}_\alpha$ and $W \in \mathcal{W}_\alpha$, using the fact that $\mathcal{W}_\alpha \upharpoonright M_\alpha$ is a base and $\mathcal{U}_\alpha$ is a downward-closed subset of $\mathcal{W}_\alpha$.

We then have $\mathcal{D}_\alpha \ni E_{\alpha,U} \subset h^{-1}[W] \subset h^{-1}[A] \subset K$.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
(2) $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and
(3) *$H \uparrow \cap \mathcal{D}_\alpha = \varnothing$:*

To prove $H \uparrow \cap \mathcal{D}_\alpha = \varnothing$, we suppose $H \subset E_{\alpha,U} \in \mathcal{D}_\alpha$ and deduce a contradiction.

By definition of $\mathcal{U}_\alpha$, we have $\overline{U} \subset V$ for some $V \in \mathcal{V}_\alpha$.

Inductively assuming $\mathcal{C}_{<\alpha} \subset \uparrow \mathcal{D}_{<\alpha}$, there exist $\beta < \alpha$ and $E_{\beta,T} \in \mathcal{D}_\beta$ such that $E_{\beta,T} \subset H$. Hence,

$$h^{-1}[T] \subset E_{\beta,T} \subset H \subset E_{\alpha,U} \subset h^{-1}[V].$$

Hence, $T \subset V$. But $T \in \mathcal{U}_\beta \subset \mathcal{W}_{<\alpha}$ and $V \in \mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$. Contradiction.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow\mathcal{D}_\alpha$,
(2) *$H\uparrow\cap\mathcal{D}_{<\alpha}$ is finite, and*
(3) $H\uparrow\cap\mathcal{D}_\alpha = \varnothing$:

To prove that every $H\uparrow\cap\mathcal{D}_{<\alpha}$ is finite, proceed by induction on α. (3) makes limit steps trivial.

Suppose that $K \in \mathcal{D}_{<\alpha+1}$. We will show that $K\uparrow\cap\mathcal{D}_{<\alpha+1}$ is finite.

If $K \in \mathcal{D}_{<\alpha}$, then $K\uparrow\cap\mathcal{D}_{<\alpha+1}$ equals $K\uparrow\cap\mathcal{D}_{<\alpha}$, which is finite by our induction hypothesis.

So, assume that $K \in \mathcal{D}_\alpha$. Since $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$, the set $K\uparrow\cap\mathcal{D}_\alpha$ is finite.

Therefore, it suffices to show that $K\uparrow\cap\mathcal{D}_{<\alpha}$ is finite.

Recall that $\daleth(\alpha)$ is finite, $M_{<\alpha} = \bigcup_{i\in\daleth(\alpha)} N^i_\alpha$, and $N^i_\alpha \prec H(\theta)$.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
(2) *$H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and*
(3) $H \uparrow \cap \mathcal{D}_\alpha = \varnothing$:

It suffices to show that each $K \uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$ is finite.

By our induction hypothesis, it suffices to find $H \in \mathcal{C}_{<\alpha}$ such that $K \uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha = H \uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$.

Since $\mathcal{B}$ is just Clop(2^λ), $H = \{p \in 2^\lambda : p \restriction N^i_\alpha \in K \restriction N^i_\alpha\}$ satisfies $K \subset H \in \mathcal{B} \cap N^i_\alpha$ and $K \uparrow \cap \mathcal{B} \cap N^i_\alpha = H \uparrow \cap \mathcal{B} \cap N^i_\alpha$.

Since $K \in \mathcal{C}$ and $\mathcal{C}$ is upward closed in $\mathcal{B}$, we have $H \in \mathcal{C} \cap N^i_\alpha \subset \mathcal{C}_{<\alpha}$.

Since $\mathcal{D}_{<\alpha} \subset \mathcal{C}_{<\alpha} \subset \mathcal{B}$, we have $K \uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha = H \uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$.

Outline of a proof of $\mathrm{Nt}(X) = \aleph_0$ where $h\colon 2^\lambda \to [0,1]^\kappa$ is continuous, $X = h[2^\lambda]$, and $\pi\chi(p, X) = \mathrm{w}(X) = \kappa$ for all $p \in X$:

1. $\mathcal{A}$ is a base of X of size κ consisting of F_σ sets.
2. $(M_\alpha)_{\alpha<\kappa}$ is a long ω_1-approximation sequence with $h, \mathcal{A} \in M_0$.
3. $\mathcal{W}_\alpha \restriction M_\alpha \subset \mathcal{A}_\alpha \restriction M_\alpha$ is an efficient base of $X \restriction M_\alpha$.
4. $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$.
5. $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \ \overline{U} \subset V\}$.
6. $\mathcal{U} = \mathcal{U}_{<\kappa}$ is a base of X.
7. $h^{-1}[\overline{U}] \subset E_{\alpha,U}$ clopen $\subset \bigcap\{h^{-1}[W] : \overline{U} \subset W \in \mathcal{W}_\alpha\}$.
8. $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$.
9. $\mathrm{Nt}(\mathcal{D}) = \aleph_0$ where $\mathcal{D} = \mathcal{D}_{<\kappa}$.

---

10. $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

Seeking a contradiction, suppose that

$T \subset U_m \neq U_n$ and $T, U_m, U_n \in \mathcal{U}$ for all $m < n < \omega$.

Let $T \in \mathcal{U}_\alpha$ and let $U_m \in \mathcal{U}_{\beta_m}$ for all $m < \omega$.

Choose $S \in \mathcal{U}_\alpha$ such that $\overline{S} \subset T$. Then, for all m, we have

$$\mathcal{D} \ni E_{\alpha,S} \subset h^{-1}[T] \subset h^{-1}[U_m] \subset E_{\beta_m,U_m} \in \mathcal{D}.$$

Since $\mathrm{Nt}(\mathcal{D}) = \aleph_0$, we may thin out $(\beta_m)_{m<\omega}$ such that,

for some $\beta < \kappa$ and $U \in \mathcal{U}_\beta$, we have $\forall m\ E_{\beta_m,U_m} = E_{\beta,U}$.

Thin out $(\beta_m)_{m<\omega}$ again to make it constant or strictly increasing.

In the case $\beta_0 < \beta_1$, we have $\overline{U_1} \subset V$ for some $V \in \mathcal{V}_{\beta_1}$, so

$$h^{-1}[U_0] \subset E_{\beta,U} \subset h^{-1}[V],$$

in contradiction with $U_0 \in \mathcal{U}_{\beta_0} \subset \mathcal{W}_{<\beta_1}$ and $V \in \mathcal{V}_{\beta_1} = \mathcal{W}_{\beta_1} \setminus \uparrow\mathcal{W}_{<\beta_1}$.

So, we are in the other case, $\beta_0 = \beta_m$ for all $m < \omega$.

Since $\mathcal{W}_{\beta_0} \restriction M_{\beta_0}$ is an efficient base, each U_m a finite set $\mathcal{F}_m$ of strict supersets in $\mathcal{W}_{\beta_0}$, but $\bigcup_{m<\omega} \mathcal{F}_m$ is infinite.

Given an arbitrary $i < \omega$, choose $j > i$ such that $\mathcal{F}_j \nsubseteq \mathcal{F}_i$.

Choose $W \in \mathcal{F}_j \setminus \mathcal{F}_i$. Since $\mathcal{W}_\alpha \restriction M_\alpha$ is an efficient base, $\overline{U_j} \subset W$.

Hence, $h^{-1}[\overline{U_i}] \subset E_{\beta,U} \subset h^{-1}[W]$; hence, $\overline{U_i} \subset W$. But $\neg(U_i \subsetneq W)$.

Hence $U_i = \overline{U_i} = W$; hence, $h^{-1}[U_i] = E_{\beta,U}$.

Thus, $U_i = h[E_{\beta,U}]$ for all $i < \omega$. Contradiction. $\square$

An *FN-map* on a boolean algebra B is a function $f\colon B \to [B]^{<\aleph_0}$ such that, for all weakly increasing pairs $x \leq y$ in B, there exists $z \in f(x) \cap f(y)$ such that $x \leq z \leq y$.

B has the Freese-Nation (FN) property if it has an FN map.

A boolean subalgebra A of B is *relatively complete* if, for every $b \in B$, there exists $a \in A$ such that $A \cap \uparrow b = A \cap \uparrow a$. In this case we write $A \leq_{\mathrm{rc}} B$.

(Fuchino, 1994) The following are equivalent.
(1) B has the FN.
(2) $B \cap M \leq_{\mathrm{rc}} B$ for all countable $M \prec H(\theta)$ with $B \in M$.
(3) $B \cap M \leq_{\mathrm{rc}} B$ for all $M \prec H(\theta)$ with $B \in M$.

(Fuchino, 1994) The following are equivalent.
(1) B has the FN.
(2) $B \cap M \leq_{\mathrm{rc}} B$ for all countable $M \prec H(\theta)$ with $B \in M$.
(3) $B \cap M \leq_{\mathrm{rc}} B$ for all $M \prec H(\theta)$ with $B \in M$.

Proof of (3)⇒(1) using a long ω_1-approximation sequence:

Let $(M_\alpha)_{\alpha<|B|}$ be a long ω_1-approximation sequence with $B \in M_0$. For each $x \in B$, let $\rho(x) = \min\{\alpha : x \in M_\alpha\}$.

For each $\alpha < |B|$, choose a well-ordering $\sqsubseteq_\alpha$ of $\{x \in B : \rho(x) = \alpha\}$ with length at most ω. Set $\sqsubseteq = \bigcup_{\alpha<|A|} \sqsubseteq_\alpha$

For each α, $i < \daleth(\alpha)$, and x with $\alpha = \rho(x)$, since $B \cap N^i_\alpha \leq_{\mathrm{rc}} B$, there exist $\pi^i_+(x) = \min(B \cap N^i_\alpha \cap \uparrow x)$ and $\pi^i_-(x) = \max(B \cap N^i_\alpha \cap \downarrow x)$.

$\rho(\pi^i_+(x)), \rho(\pi^i_-(x)) < \rho(x)$ for all $i < \daleth(\alpha)$. (There is no $i < \daleth(0)$.)

Recursively define $f\colon B \to [B]^{<\aleph_0}$ by

$$f(x) = \{y : y \sqsubseteq x\} \cup \bigcup_{i < \beth(\rho(x))} \left(f(\pi^i_+(x)) \cup f(\pi^i_-(x)) \right).$$

Suppose $x \leq y$. We verify that $S = [x, y] \cap f(x) \cap f(y)$ is nonempty by induction on $\max\{\rho(x), \rho(y)\}$.

If $\rho(x) = \rho(y)$, then
$x \sqsubseteq y$, in which case $x \in S$, or
$y \sqsubseteq x$, in which case $y \in S$.

If $\rho(x) < \rho(y)$, then $x \in N^i_{\rho(y)}$ for some i, in which case $[x, \pi^i_-(y)] \cap f(x) \cap f(\pi^i_-(y))$ is a nonempty subset of S.

If $\rho(y) < \rho(x)$, then $y \in N^i_{\rho(x)}$ for some i, in which case $[\pi^i_+(x), y] \cap f(\pi^i_+(x)) \cap f(y)$ is a nonempty subset of S. $\square$

All free boolean algebras (*i.e.*, algebras isomorphic to some Clop(2^λ)) and their retracts (*i.e.*, projective boolean algebras) have the FN.

All countable boolean algebras are retracts of Clop(2^ω).

All $\aleph_1$-sized boolean algebras with the FN are retracts of Clop(2^{ω_1}).

If $\kappa \geq \omega_2$, then the clopen algebra exp(Clop(2^{ω_2})) of the Vietoris hyperspace exp(2^κ) of nonempty closed subsets of 2^κ has the FN but is not a retract of a free boolean algebra and not even a subalgebra of a free boolean algebra.

Topologically speaking, exp(2^κ) is openly generated but is not Dugundji and not even dyadic.

Our theorem about homogeneous dyadic compacta generalizes a bit:

If X is a homogeneous continuous image of the Stone space Ult(B) of a boolean algebra B with the FN, then Nt(X) $= \aleph_0$.

Two boolean subalgebras $A, B \subset C$ *commute* if, for all pairs $A \ni x \leq y \in B$, there exists $z \in A \cap B$ such that $x \leq z \leq y$.

(Heindorf–Shapiro, 1994)

- A boolean algebra has the *strong Freese-Nation property* (SFN) if it has a pairwise commuting cofinal family of finite subalgebras.
- Retracts of free boolean algebras have the SFN.
- exp(Clop(2^{ω_2})) has SFN.
- The SFN implies the FN.
- Does the FN imply the SFN?

Theorem (Milovich, 2014). There is a boolean algebra of size $\aleph_2$ with the FN but not the SFN.

The proof uses a long ω_1-approximation sequence and uses almost all of coherence properties mentioned in Part I.

Lajos Soukup has recently announced a σ-closed version of long ω_1-approximation sequences:

Assume GCH and $\square^{**}_\mu$ for all regular uncountable μ. Then, for every cardinal κ and set x, there exist $(M_\alpha)_{\alpha<\kappa}$ and $(N^i_\alpha)_{i<\omega;\alpha<\kappa}$ such that

- $\kappa \subset \bigcup_{\alpha<\kappa} M_\alpha$.
- $x \in M_\alpha$,
- $|M_\alpha| = \aleph_1$,
- $M_{<\alpha} = \bigcup_{i<\omega} N^i_\alpha$,
- $[M_\alpha]^\omega \subset M_\alpha \prec H(\theta)$, and
- $[N^i_\alpha]^\omega \subset N^i_\alpha \prec H(\theta)$.

References

L. Heindorf and L. B. Shapiro, *Nearly Projective Boolean Algebras*, with an appendix by S. Fuchino, Lecture Notes in Mathematics **1596**, Springer-Verlag, Berlin, 1994.

D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, *Topology and its Applications* **156** (2008), 443–464.

D. Milovich, *On the strong Freese-Nation property* (2014), arXiv:1412.7443.

D. Soukup, *Davies-trees in infinite combinatorics* (2014), arXiv:1407.3604.

L. Soukup, *On properties of families of sets (Part 3)* (2014), `http://bcc.impan.pl/14Young/index.php/slides`.