A disjoint union theorem for trees

Konstantinos Tyros

University of Warwick
Mathematics Institute

Fields Institute, 2015
Theorem (Folkman)

For every pair of positive integers $m$ and $r$ there is integer $n_0$ such that for every $r$-coloring of the power-set $\mathcal{P}(X)$ of some set $X$ of cardinality at least $n_0$, there is a family $D = (D_i)_{i=1}^m$ of pairwise disjoint nonempty subsets of $X$ such that the family

\[ \mathcal{U}(D) = \left\{ \bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, \ldots, m\} \right\} \]

of non-empty unions is monochromatic.
Theorem (Carlson-Simpson)

For every finite Souslin measurable coloring of the power-set $\mathcal{P}(\omega)$ of $\omega$, there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of the natural numbers such that the set

$$\mathcal{U}(D) = \left\{ \bigcup_{n \in M} D_n : M \text{ is a non-empty subset of } \omega \right\}$$

is monochromatic.
A **tree** is a partially ordered set \((T, \leq_T)\) such that

\[
\text{Pred}_T(t) = \{ s \in T : s <_T t \}
\]

is finite and totally ordered for all \(t\) in \(T\).
We consider only **uniquely rooted and finitely branching trees with no maximal nodes.**
For $n < \omega$, the $n$-th level of $T$, is the set

$$T(n) = \{ t \in T : |\text{Pred}_T(t)| = n \}.$$
For a subset $D$ of $T$, we define its **level set**

$$L_T(D) = \{ n \in \omega : D \cap T(n) \neq \emptyset \}$$

$LT(D) = \{1, 3\}$
From now on, fix an integer $d \geq 1$.

A vector tree

$$T = (T_1, \ldots, T_d)$$

is a $d$-sequence of uniquely rooted and finitely branching trees with no maximal nodes.
Level products

For a vector tree \(\mathbf{T} = (T_1, \ldots, T_d)\) we define its level product as

\[
\bigotimes \mathbf{T} = \bigcup_{n < \omega} T_1(n) \times \ldots \times T_d(n)
\]

The \(n\)-th level of the level product of \(\mathbf{T}\) is

\[
\bigotimes \mathbf{T}(n) = T_1(n) \times \ldots \times T_d(n).
\]
Let $T = (T_1, \ldots, T_d)$ a vector tree.
For $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d)$ in $\otimes T$, set

\[ t \leq_T s \text{ iff } t_i \leq_{T_i} s_i \text{ for all } i = 1, \ldots, d. \]

For $t = (t_1, \ldots, t_d)$ in $\otimes T$, we define

\[ \text{Succ}_T(t) = \{ s \in \otimes T : t \leq_T s \} \]
A sequence $\mathbf{D} = (D_1, \ldots, D_d)$ is called a vector subset of $T$ if $D_i$ is a subset of $T_i$ for all $i = 1, \ldots, d$ and

$$LT_1(D_1) = \ldots = LT_d(D_d).$$

For a vector subset $\mathbf{D}$ of $T$ we define its level product

$$\otimes \mathbf{D} = \bigcup_{n < \omega} (T_1(n) \cap D_1) \times \ldots \times (T_d(n) \cap D_d).$$

For $t \in \otimes T$, a vector subset $\mathbf{D}$ of $T$ is $t$-dense, if

$$(\forall n)(\exists m)(\forall s \in \otimes T(n) \cap \text{Succ}_T(t) (\exists s' \in \otimes T(m) \cap \otimes \mathbf{D}) \ s \leq_T s').$$

$\mathbf{D}$ is called dense if it is root($\otimes T$)-dense.
A sequence $\mathbf{D} = (D_1, ..., D_d)$ is called a **vector subset** of $T$ if $D_i$ is a subset of $T_i$ for all $i = 1, ..., d$ and

$$L_{T_1}(D_1) = ... = L_{T_d}(D_d).$$

For a vector subset $\mathbf{D}$ of $T$ we define its **level product**

$$\otimes \mathbf{D} = \bigcup_{n<\omega} (T_1(n) \cap D_1) \times ... \times (T_d(n) \cap D_d).$$

For $t \in \otimes T$, a vector subset $\mathbf{D}$ of $T$ is $t$-dense,

$$(\forall n)(\exists m)(\forall s \in \otimes T(n) \cap \text{Succ}_T(t)(\exists s' \in \otimes T(m) \cap \otimes \mathbf{D}) \ s \leq_T s').$$

$\mathbf{D}$ is called **dense** if it is root($\otimes T$)-dense.
$(\forall n)(\exists m)(\forall s \in \otimes T(n) \cap \text{Succ}_T(t) (\exists s' \in \otimes T(m) \cap \otimes D) \ s \leq_T s')$. 

Konstantinos Tyros
A disjoint union theorem for trees
The Halpern–Läuchli Theorem

Theorem (Halpern–Läuchli)

Let $T$ be a vector tree. Then for every dense vector subset $D$ of $T$ and every subset $P$ of $\otimes D$, there exists a vector subset $D'$ of $D$ such that either

(i) $\otimes D'$ is a subset of $P$ and $D'$ is a dense vector subset of $T$, or

(ii) $\otimes D'$ is a subset of $P^c$ and $D'$ is a $t$-dense vector subset of $T$ for some $t$ in $\otimes T$. 

Konstantinos Tyros

A disjoint union theorem for trees
Let $T$ be a vector tree. We define

$$\mathcal{U}(T) = \{ U \subseteq \bigotimes T : U \text{ has a minimum} \}.$$ 

We let $\mathcal{U}(T)$ take its topology from $\{0, 1\} \bigotimes T$.

Let $D$ be a vector subset of $T$.

A **$D$-subspace** of $\mathcal{U}(T)$ is a family

$$U = (U_t)_{t \in \bigotimes D}$$

such that

1. $U_t \in \mathcal{U}(T)$ for all $t \in \bigotimes D$,
2. $U_s \cap U_t = \emptyset$ for $s \neq t$,
3. $\min U_t = t$ for all $t \in \bigotimes D$. 

Konstantinos Tyros

A disjoint union theorem for trees
For a subspace $U = (U_t)_{t \in \otimes D(U)}$ we define its span by

$$[U] = \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(U) \right\} \cap \mathcal{U}(T)$$

$$= \left\{ \bigcup_{t \in \Gamma} U_t : \Gamma \subseteq \otimes D(U) \text{ and } \Gamma \in \mathcal{U}(T) \right\}.$$

If $U$ and $U'$ are two subspaces of $\mathcal{U}(T)$, we say that $U'$ is a subspace of $U$, and write $U' \leq U$, if $[U'] \subseteq [U]$.

**Remark**

$U' \leq U$ implies that $D(U')$ is a vector subset of $D(U)$.
Let $T$ be a vector tree and $\mathcal{P}$ a Souslin measurable subset of $U(T)$. Also let $D$ be a dense vector subset of $T$ and $U$ a $D$-subspace of $U(T)$. Then there exists a subspace $U'$ of $U(T)$ with $U' \leq U$ such that either

(i) $[U']$ is a subset of $\mathcal{P}$ and $D(U')$ is a dense vector subset of $T$, or

(ii) $[U']$ is a subset of $\mathcal{P}^c$ and $D(U')$ is a $t$-dense vector subset of $T$ for some $t$ in $\otimes T$. 

Konstantinos Tyros

A disjoint union theorem for trees
Corollary (Carlson–Simpson)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $U(D)$ is monochromatic.

Let $\Lambda$ be a finite alphabet. We view the elements of $\Lambda^\omega$ as infinite constant words over $\Lambda$. Also let $(v_n)_n$ be a sequence of distinct symbols that do not occur in $\Lambda$. An infinite dimensional variable word is a map $f : \omega \rightarrow \Lambda \cup \{v_n : n \in \mathbb{N}\}$ such that for every $n$ we have that $f^{-1}(v_n) \neq \emptyset$ and $\max f^{-1}(v_n) < \min f^{-1}(v_{n+1})$. If $(a_n)_n \in \Lambda^\omega$ then by $f((a_n)_n)$ we denote the constant word resulting by substituting each occurrence of $v_n$ by $a_n$.

Theorem

Let $\Lambda$ be a finite alphabet. Then for every Souslin measurable coloring of $\Lambda^\omega$ there exists an infinite dimensional word such that the set $\{f((a_n)_n) : (a_n)_n \in \Lambda^\omega\}$ is monochromatic.
Corollary (Carlson–Simpson)

For every finite Souslin measurable coloring of \( \mathcal{P}(\omega) \) there is a sequence \( D = (D_n)_{n<\omega} \) of pairwise disjoint subsets of \( \omega \) such that the set \( U(D) \) is monochromatic.

Let \( \Lambda \) be a finite alphabet. We view the elements of \( \Lambda^\omega \) as infinite constant words over \( \Lambda \). Also let \((v_n)_n\) be a sequence of distinct symbols that do not occur in \( \Lambda \). An infinite dimensional variable word is a map \( f : \omega \to \Lambda \cup \{v_n : n \in \mathbb{N}\} \) such that for every \( n \) we have that \( f^{-1}(v_n) \neq \emptyset \) and \( \max f^{-1}(v_n) < \min f^{-1}(v_{n+1}) \). If \((a_n)_n \in \Lambda^\omega\) then by \( f((a_n)_n) \) we denote the constant word resulting by substituting each occurrence of \( v_n \) by \( a_n \).

Theorem

Let \( \Lambda \) be a finite alphabet. Then for every Souslin measurable coloring of \( \Lambda^\omega \) there exists an infinite dimensional word such that the set \( \{f((a_n)_n) : (a_n)_n \in \Lambda^\omega \} \) is monochromatic.
Corollary (Carlson–Simpson)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $U(D)$ is monochromatic.

Let $\Lambda$ be a finite alphabet. We view the elements of $\Lambda^\omega$ as infinite constant words over $\Lambda$. Also let $(v_n)_n$ be a sequence of distinct symbols that do not occur in $\Lambda$. An infinite dimensional variable word is a map $f : \omega \to \Lambda \cup \{v_n : n \in \mathbb{N}\}$ such that for every $n$ we have that $f^{-1}(v_n) \neq \emptyset$ and $\max f^{-1}(v_n) < \min f^{-1}(v_{n+1})$. If $(a_n)_n \in \Lambda^\omega$ then by $f((a_n)_n)$ we denote the constant word resulting by substituting each occurrence of $v_n$ by $a_n$.

Theorem

Let $\Lambda$ be a finite alphabet. Then for every Souslin measurable coloring of $\Lambda^\omega$ there exists an infinite dimensional word such that the set $\{f((a_n)_n) : (a_n)_n \in \Lambda^\omega\}$ is monochromatic.
**Corollary (Carlson–Simpson)**

*For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $\mathcal{U}(D)$ is monochromatic.*

Let $\Lambda$ be a finite alphabet. We view the elements of $\Lambda^\omega$ as infinite constant words over $\Lambda$. Also let $(v_n)_n$ be a sequence of distinct symbols that do not occur in $\Lambda$. An infinite dimensional variable word is a map $f : \omega \to \Lambda \cup \{v_n : n \in \mathbb{N}\}$ such that for every $n$ we have that $f^{-1}(v_n) \neq \emptyset$ and $\max f^{-1}(v_n) < \min f^{-1}(v_{n+1})$. If $(a_n)_n \in \Lambda^\omega$ then by $f((a_n)_n)$ we denote the constant word resulting by substituting each occurrence of $v_n$ by $a_n$.

**Theorem**

*Let $\Lambda$ be a finite alphabet. Then for every Souslin measurable coloring of $\Lambda^\omega$ there exists an infinite dimensional word such that the set $\{f((a_n)_n) : (a_n)_n \in \Lambda^\omega\}$ is monochromatic.*
Corollary (Carlson–Simpson)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $D = (D_n)_{n<\omega}$ of pairwise disjoint subsets of $\omega$ such that the set $U(D)$ is monochromatic.

Let $\Lambda$ be a finite alphabet. We view the elements of $\Lambda^\omega$ as infinite constant words over $\Lambda$. Also let $(v_n)_n$ be a sequence of distinct symbols that do not occur in $\Lambda$. An infinite dimensional variable word is a map $f : \omega \to \Lambda \cup \{v_n : n \in \mathbb{N}\}$ such that for every $n$ we have that $f^{-1}(v_n) \neq \emptyset$ and $\max f^{-1}(v_n) < \min f^{-1}(v_{n+1})$. If $(a_n)_n \in \Lambda^\omega$ then by $f((a_n)_n)$ we denote the constant word resulting by substituting each occurrence of $v_n$ by $a_n$.

Theorem

Let $\Lambda$ be a finite alphabet. Then for every Souslin measurable coloring of $\Lambda^\omega$ there exists an infinite dimensional word such that the set $\{f((a_n)_n) : (a_n)_n \in \Lambda^\omega\}$ is monochromatic.
We fix a vector tree $T$.
Fix a finite alphabet $\Lambda$.
For $m < n < \omega$, set

$$W(\Lambda, T, m, n) = \Lambda \otimes T \upharpoonright [m, n],$$

where $\otimes T \upharpoonright [m, n] = \bigcup_{j=m}^{n-1} \otimes T(j)$. We also set

$$W(\Lambda, T) = \bigcup_{m \leq n} W(\Lambda, T, m, n).$$
A disjoint union theorem for trees
Let \((v_s)_{s \in \otimes T}\) be a collection of distinct variables, set of symbols disjoint from \(\Lambda\).

Fix a vector level subset \(D\) of \(T\). Let

\[ W_v(\Lambda, T, D, m, n) \]

be the set of all functions

\[ f : \otimes T \upharpoonright [m, n) \to \Lambda \cup \{ v_s : s \in \otimes D \} \]

such that

- The set \(f^{-1}(\{v_s\})\) is nonempty and admits \(s\) as a minimum in \(\otimes T\), for all \(s \in \otimes D\).
- For every \(s\) and \(s'\) in \(\otimes D\), we have
  \[ L_{\otimes T}(f^{-1}(\{v_s\})) = L_{\otimes T}(f^{-1}(\{v_{s'}\})). \]
A disjoint union theorem for trees
For \( f \in W_v(\Lambda, T, D, m, n) \), set
\[
ws(f) = D, \ bot(f) = m \text{ and } top(f) = n.
\]

Moreover, we set
\[
W_v(\Lambda, T) = \bigcup \{ W_v(\Lambda, T, D, m, n) : m \leq n \text{ and } D \text{ is a vector level subset of } T \text{ with } L_T(D) \subset [m, n) \}.
\]

The elements of \( W_v(\Lambda, T) \) are viewed as \textbf{variable words over the alphabet} \( \Lambda \).
For variable words \( f \) in \( W_v(\Lambda, T) \) we take \textbf{substitutions}:

For every family \( a = (a_s)_{s \in \otimes ws(f)} \subseteq \Lambda \), let

\( f(a) \in W(\Lambda, T) \) be the result of substituting for every \( s \) in \( \otimes ws(f) \) each occurrence of \( v_s \) by \( a_s \).

Moreover, we set

\[
[f]_\Lambda = \{ f(a) : a = (a_s)_{s \in \otimes ws(f)} \subseteq \Lambda \},
\]

the constant span of \( f \).
An infinite sequence \( X = (f_n)_{n < \omega} \) in \( W_v(\Lambda, T) \) is a \textbf{subspace}, if:

1. \( \text{bot}(f_0) = 0 \).
2. \( \text{bot}(f_{n+1}) = \text{top}(f_n) \) for all \( n < \omega \).
3. Setting \( D_i = \bigcup_{n < \omega} ws_i(f_n) \) for all \( i = 1, \ldots, d \), where \( ws(f_n) = (ws_1(f_n), \ldots, ws_d(f_n)) \), we have that \( (D_1, \ldots, D_d) \) forms a dense vector subset of \( T \).

For a subspace \( X = (f_n)_{n < \omega} \) we define

\[
[X]_\Lambda = \left\{ \bigcup_{q=0}^{n} g_q : n < \omega \text{ and } g_q \in [f_q]_\Lambda \text{ for all } q = 0, \ldots, n \right\}.
\]

For two subspaces \( X \) and \( Y \), we write \( X \leq Y \) if \( [X]_\Lambda \subseteq [Y]_\Lambda \).
An infinite sequence $X = (f_n)_{n<\omega}$ in $W_v(\Lambda, T)$ is a **subspace**, if:

1. bot$(f_0) = 0$.
2. bot$(f_{n+1}) = \text{top}(f_n)$ for all $n < \omega$.
3. Setting $D_i = \bigcup_{n<\omega} \text{ws}_i(f_n)$ for all $i = 1, \ldots, d$, where $\text{ws}(f_n) = (\text{ws}_1(f_n), \ldots, \text{ws}_d(f_n))$, we have that $(D_1, \ldots, D_d)$ forms a dense vector subset of $T$.

For a subspace $X = (f_n)_{n<\omega}$ we define

$$[X]_\Lambda = \left\{ \bigcup_{q=0}^{n} g_q : n < \omega \text{ and } g_q \in [f_q]_\Lambda \text{ for all } q = 0, \ldots, n \right\}.$$

For two subspaces $X$ and $Y$, we write $X \leq Y$ if $[X]_\Lambda \subseteq [Y]_\Lambda$. 
Theorem

Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite coloring of the set of the constant words $W(\Lambda, T)$ over $\Lambda$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_{\Lambda}$ is monochromatic.
Let $W^\infty(\Lambda, T)$, be the set of all sequences $(g_n)_{n<\omega}$ in $W(\Lambda, T)$ such that:

1. $\text{bot}(g_0) = 0$ and
2. $\text{bot}(g_{n+1}) = \text{top} g_n$ for all $n < \omega$.

For a subspace $X$, we set

$$[X]_\Lambda^\infty = \{(g_n)_{n<\omega} \in W^\infty(\Lambda, T) : (\forall n < \omega) \bigcup_{q=0}^{n} g_q \in [X]_\Lambda \}.$$

**Theorem**

Let $\Lambda$ be a finite alphabet and $T$ a vector tree. Then for every finite Souslin measurable coloring of the set $W^\infty(\Lambda, T)$ and every subspace $X$ of $W(\Lambda, T)$ there exists a subspace $X'$ of $W(\Lambda, T)$ with $X' \leq X$ such that the set $[X']_\Lambda^\infty$ is monochromatic.