A microscopic approach to higher Souslin-tree constructions

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Boise Extravaganza in Set Theory
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This is joint work with Assaf Rinot, and still in progress.
Souslin Trees — History

Souslin’s Problem (1920):

*Is every ccc dense linear ordering necessarily separable?*

A counterexample would be called a Souslin line.
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Theorem (Kurepa, 1935)

\[ \exists \text{Souslin line} \iff \exists \text{Souslin tree}. \]

Definition

A tree \( T \) is Souslin if:

- it has height \( \omega_1 \),
- every chain is countable, and
- every antichain is countable.
Souslin Trees — History

So... does there exist a Souslin tree?
Souslin Trees — History

So. . . does there exist a Souslin tree?
Yes?
Souslin Trees — History

So... does there exist a Souslin tree?
Yes?
No?

Theorem
Souslin's problem is independent of ZFC.
Among other constructions:
3 = \Rightarrow \exists \text{Souslin tree (Jensen, 1972)}
MA_1 = \Rightarrow \not\exists \text{Souslin tree (Solovay & Tennenbaum, 1971)}
Souslin Trees — History

So . . . does there exist a Souslin tree?
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Among other constructions:

\[ \Diamond \implies \exists \text{ Souslin tree} \quad (Jensen, 1972) \]
\[ \text{MA}_{\aleph_1} \implies \not\exists \text{ Souslin tree} \quad (Solovay & Tennenbaum, 1971) \]
Subsequent Progress

Once Souslin trees were known (consistently) to exist, efforts were made to extend the result in different directions:

1. \( \kappa \)-Souslin trees at higher cardinals \( \kappa > \aleph_1 \);
2. Souslin trees with additional properties;
3. Souslin trees from weaker axioms.
Subsequent Progress

Once Souslin trees were known (consistently) to exist, efforts were made to extend the result in different directions:

1. $\kappa$-Souslin trees at higher cardinals $\kappa > \aleph_1$;
2. Souslin trees with additional properties;
3. Souslin trees from weaker axioms.

In the direction of (1), constructions of $\kappa$-Souslin trees often require distinguishing different kinds of cardinals, depending on whether

- $\kappa = \lambda^+$ for $\lambda$ regular;
- $\kappa = \lambda^+$ for $\aleph_0 = \text{cf}(\lambda) < \lambda$;
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Examples of extra properties that a Souslin trees have been constructed to satisfy:

- free;
- hard to specialize (remains non-special in any cofinality-preserving extension);
- specializable (can become special in some cofinality-preserving extension);
- complete (e.g. $\sigma$-closed);
- rigid;
- homogeneous.
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- rigid;
- homogeneous.

What happens when we try to combine these properties?
Free Souslin Trees

If $S$ is a Souslin tree, its square $S \times S$ cannot be Souslin.
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Definition
For any infinite cardinals $\chi < \kappa$, a $\kappa$-Souslin tree $T$ is said to be $\chi$-free if for every nonzero $\tau < \chi$, any $\delta < \kappa$, and any sequence of distinct nodes $\langle w_\xi \mid \xi < \tau \rangle \in \tau T_\delta$, the product tree $\bigotimes_{\xi < \tau} w_\xi \uparrow$ is again a $\kappa$-Souslin tree.
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Interestingly, Jensen’s original $\aleph_1$-Sousin tree constructed from $\diamondsuit$ was $\aleph_0$-free.
Free Souslin Trees

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Interestingly, Jensen’s original $\aleph_1$-Souslin tree constructed from ♦ was $\aleph_0$-free. We’ll come back to this after we look at another property.
Ascent Paths

Definition
For any infinite cardinals $\theta < \kappa$, an $\mathcal{F}_{\theta}^{bd}$-ascent path through a $\kappa$-Souslin tree $\langle T, <_T \rangle$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \kappa \rangle$, where:

1. $f_\alpha : \theta \to T_\alpha$ is a function for each $\alpha < \kappa$;
2. $\{ i \in \theta \mid f_\alpha(i) <_T f_\beta(i) \}$ is a co-bounded subset of $\theta$ whenever $\alpha < \beta < \kappa$. 

Why is this useful?
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Ascent Paths — First Application

For a tree $T$, consider the $\omega$-reduced power tree $\omega T/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on $\omega$. 
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Devlin constructed a consistent example of an $\aleph_2$-Souslin tree $T$ with an $\mathcal{F}^{bd}_{\aleph_0}$-ascent path.
For a tree $T$, consider the $\omega$-reduced power tree $\omega T/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on $\omega$.

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Devlin constructed a consistent example of an $\aleph_2$-Souslin tree $T$ with an $\mathcal{F}_{\aleph_1}^{bd}$-ascent path.

Since $T$ has an $\mathcal{F}_{\aleph_0}^{bd}$-ascent path, it follows that $\omega T/\mathcal{U}$ has a cofinal branch, that is, the $\omega$-reduced power tree is not even Aronszajn.
Shelah proved that if \( \langle T, <_T \rangle \) is a special \( \lambda^+ \)-tree that admits an \( \mathcal{F}_\theta^{bd} \)-ascent path, then \( \text{cf}(\lambda) = \text{cf}(\theta) \). This provides an approach to constructions of \( \lambda^+ \)-trees that are impossible to specialize without changing cofinalities.
Theorem

Suppose that $\theta < \kappa = \text{cf}(\kappa)$ are infinite cardinals, and that $(T, <_T)$ is a normal splitting $\kappa$-tree that admits an $\mathcal{F}_\theta^{\text{bd}}$-ascent path. Then $(T <_T)$ is not a $\theta^+$-free $\kappa$-Souslin tree.
The Proxy Principle

We would like a unified principle $P$ that can be used to construct the various $\kappa$-Souslin trees regardless of the nature of $\kappa$, and this principle should follow from all the usual hypotheses that have been used in such constructions.
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$$P(\kappa, \mu, R, \theta, S, \nu, \sigma, \omega)$$
Tree-Indexed Ascent Paths

Definition
Suppose that \( U \subseteq <^\kappa \kappa \) is a downward-closed \( \kappa \)-tree.
A \( U \)-indexed \( \mathcal{F}^{bd}_\theta \)-ascent path through a \( \kappa \)-tree \( \langle T, <_T \rangle \) is a sequence \( \vec{f} = \langle f_u \mid u \in U \rangle \) such that:

1. \( f_u : \theta \to T_{\text{dom}(u)} \) is a function for each \( u \in U \);
2. \( \{ i \in \theta \mid f_u(i) <_T f_v(i) \} \) is a co-bounded subset of \( \theta \) whenever \( u \sqsubseteq v \) are in \( U \);
3. \( \{ i \in \theta \mid f_u(i) \neq f_v(i) \} \) is a co-bounded subset of \( \theta \) whenever \( u, v \) are distinct elements of \( U \cap \alpha \kappa \) for some \( \alpha < \kappa \).
Free Together with Ascent Paths

**Theorem**

Assume $V = L$. Then there exists an $\aleph_2$-Kurepa tree $U$, and an $\aleph_0$-free $\aleph_2$-Souslin tree that admits a $U$-indexed $\mathcal{F}^{\bd}_\omega$-ascent path.

In particular, there exists an $\aleph_0$-free $\aleph_2$-Souslin tree whose $\omega$-reduced power tree is $\aleph_2$-Kurepa.
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