A microscopic approach to higher Souslin-tree constructions

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Boise Extravaganza in Set Theory
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This is joint work with Assaf Rinot, and still in progress.

Souslin Trees — History

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Is every ccc dense linear ordering necessarily separable?

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Theorem (Kurepa, 1935)

$$\exists \textit{ Souslin line } \iff \exists \textit{ Souslin tree.}$$

Definition

A tree T is Souslin if:

- it has height ω_1,
- every chain is countable, and
- every antichain is countable.

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Among other constructions:

$$\diamondsuit \implies \exists \text{ Souslin tree} \quad \text{(Jensen, 1972)}$$

$$MA_{\aleph_1} \implies \nexists \text{ Souslin tree} \quad \text{(Solovay \& Tennenbaum, 1971)}$$

Subsequent Progress

Once Souslin trees were known (consistently) to exist, efforts were made to extend the result in different directions:

1. κ-Souslin trees at higher cardinals $\kappa > \aleph_1$;
2. Souslin trees with additional properties;
3. Souslin trees from weaker axioms.

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In the direction of (1), constructions of κ-Souslin trees often require distinguishing different kinds of cardinals, depending on whether

- $\kappa = \lambda^+$ for λ regular;
- $\kappa = \lambda^+$ for $\aleph_0 = \mathrm{cf}(\lambda) < \lambda$;
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Subsequent Progress

Examples of extra properties that a Souslin trees have been constructed to satisfy:

- free;
- hard to specialize (remains non-special in any cofinality-preserving extension);
- specializable (can become special in some cofinality-preserving extension);
- complete (e.g. σ-closed);
- rigid;
- homogeneous.

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- homogeneous.

What happens when we try to combine these properties?

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For any infinite cardinals $\chi < \kappa$, a κ-Souslin tree T is said to be *χ-free* if for every nonzero $\tau < \chi$, any $\delta < \kappa$, and any sequence of distinct nodes $\langle w_\xi \mid \xi < \tau \rangle \in {}^{\tau} T_\delta$, the product tree $\bigotimes_{\xi<\tau} w_\xi\uparrow$ is again a κ-Souslin tree.

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We'll come back to this after we look at another property.

Ascent Paths

Definition

For any infinite cardinals $\theta < \kappa$, an *$\mathcal{F}_\theta^{\mathrm{bd}}$-ascent path* through a κ-Souslin tree $\langle T, <_T\rangle$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \kappa\rangle$, where:

1. $f_\alpha : \theta \to T_\alpha$ is a function for each $\alpha < \kappa$;
2. $\{i \in \theta \mid f_\alpha(i) <_T f_\beta(i)\}$ is a co-bounded subset of θ whenever $\alpha < \beta < \kappa$.

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Why is this useful?

Ascent Paths — First Application

For a tree T, consider the ω-reduced power tree ${}^{\omega}T/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on ω.

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Devlin constructed a consistent example of an $\aleph_2$-Souslin tree T with an $\mathcal{F}^{\mathrm{bd}}_{\aleph_0}$-ascent path.
Since T has an $\mathcal{F}^{\mathrm{bd}}_{\aleph_0}$-ascent path, it follows that ${}^{\omega}T/\mathcal{U}$ has a cofinal branch, that is, the ω-reduced power tree is not even Aronszajn.

Ascent Paths — Second Application

Shelah proved that if $\langle T, <_T \rangle$ is a special λ^+-tree that admits an $\mathcal{F}_\theta^{\mathsf{bd}}$-ascent path, then $\mathrm{cf}(\lambda) = \mathrm{cf}(\theta)$. This provides an approach to constructions of λ^+-trees that are impossible to specialize without changing cofinalities.

Conflicting Properties

Theorem

Suppose that $\theta < \kappa = \mathrm{cf}(\kappa)$ *are infinite cardinals, and that* $(T, <_T)$ *is a normal splitting* κ*-tree that admits an* $\mathcal{F}^{\mathrm{bd}}_{\theta}$*-ascent path.*
Then $(T <_T)$ *is not a* θ^{+}*-free* κ*-Souslin tree.*

The Proxy Principle

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$$P(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \varpi)$$

Tree-Indexed Ascent Paths

Definition

Suppose that $U \subseteq {}^{<\kappa}\kappa$ is a downward-closed κ-tree.
A *U-indexed $\mathcal{F}^{\mathrm{bd}}_{\theta}$-ascent path* through a κ-tree $\langle T, <_T \rangle$ is a sequence $\vec{f} = \langle f_u \mid u \in U \rangle$ such that:

1. $f_u : \theta \to T_{\mathrm{dom}(u)}$ is a function for each $u \in U$;
2. $\{i \in \theta \mid f_u(i) <_T f_v(i)\}$ is a co-bounded subset of θ whenever $u \sqsubset v$ are in U;
3. $\{i \in \theta \mid f_u(i) \neq f_v(i)\}$ is a co-bounded subset of θ whenever u, v are distinct elements of $U \cap {}^{\alpha}\kappa$ for some $\alpha < \kappa$.

Free Together with Ascent Paths

Theorem

Assume $V = L$. Then there exists an $\aleph_2$-Kurepa tree U, and an $\aleph_0$-free $\aleph_2$-Souslin tree that admits a U-indexed $\mathcal{F}^{\mathrm{bd}}_{\omega}$-ascent path.

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Assume $V = L$. Then there exists an $\aleph_2$-Kurepa tree U, and an $\aleph_0$-free $\aleph_2$-Souslin tree that admits a U-indexed $\mathcal{F}^{\mathrm{bd}}_{\omega}$-ascent path. In particular, there exists an $\aleph_0$-free $\aleph_2$-Souslin tree whose ω-reduced power tree is $\aleph_2$-Kurepa.