Scales in Prikry extensions

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Introduction

Broad goal: study various properties of a singular cardinal $\kappa$. For example:

- $2^\kappa$.
- Cofinality of $[\kappa]^{<\omega}$ under inclusion ordering.
- Combinatorial properties, e.g., approachability property at $\kappa$.
- Mutual and tight stationarity at $\kappa$.

Each of these questions is closely related to pcf theory.
For simplicity, work with $\kappa$ singular of cofinality $\omega$.

- $\langle \kappa_n : n < \omega \rangle$ increasing and cofinal sequence of regular cardinals $< \kappa$.
- Consider $\prod_n \kappa_n$ under the eventual domination ordering: $f <^* g$ if $f(n) < g(n)$ for all large $n$.
- $\prod_n \kappa_n$ is said to have true cofinality $\lambda$ if there is a cofinal (linearly ordered) sequence of order-type $\lambda$.
- A scale is an increasing $<^*$-cofinal sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $(\prod_{n<\omega} \kappa_n, <^*)$. 
Pcf theory

Easy observations:

- Suppose $S_n$ are arbitrary linearly ordered sets. Then $\text{cf} \left( \prod_n S_n, <^* \right) = \text{cf} \left( \prod_n \kappa_n, <^* \right)$, where $\kappa_n = \text{cf} \left( S_n \right)$.

- Given a scale on $\prod_n \kappa_n$, can modify so that at limit $\alpha$, $f_\alpha$ is the least upper bound of $\langle f_\beta : \beta < \alpha \rangle$ if such exists. A scale with this property is called *continuous*. 
The basic theorems about scales are due to Shelah:

- There are $\kappa_n$ cofinal in $\kappa$ for which there exists a scale of length $\kappa^+$ in $(\prod_n \kappa_n, <^*)$.
- Any $\omega$-sequence of regular cardinals cofinal in $\kappa$ can be decomposed into finitely many pieces, each of which has true cofinality.

Scales are absolute witnesses to the possible cofinalities of products below a singular cardinal, and are the fundamental objects in pcf theory.
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There is essentially only one method known for forcing things at singular cardinals—Prikry forcing.
Broadly, there are two classes of Prikry forcing. In each case, no bounded subsets of $\kappa$ are added by the forcing.

- The first type changes a measurable cardinal $\kappa$ into a singular cardinal by adding some cofinal $\omega$-sequences.

- The second type adds new cofinal $\omega$-sequences to $\kappa$ which was a singular limit of measurable cardinals in the ground model. Prikry forcings of this type have a “diagonal” character: $n$th term of a generic cofinal sequence is controlled by a measure on the $n$th cardinal.
Scales in Prikry extensions

- A natural idea: analyze the pcf structure on all of products of $\omega$-sequences of regular cardinals which appear in the Prikry extension.

- Analysis of certain products was done by Jech (ordinary Prikry forcing), Sharon (extender-based forcing), Cummings–Foreman (supercompact diagonal forcing), and Lambie-Hanson.
• Fix a normal ultrafilter on a measurable cardinal \( \kappa \). Let \( j \) be the ultrapower embedding.

• Conditions of \( \mathbb{P} \) are of the form \( \langle p, A \rangle \), where \( p \) is a finite sequence and \( A \) is a measure one set in the ultrafilter. The ordering is by end-extension in the “\( p \)” coordinate and reverse inclusion in the “\( A \)” coordinate.

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• Any \( \omega \)-sequence \( \langle \mu_n : n < \omega \rangle \) in \( V[G] \) can be named by \( \langle \dot{\mu}_n : n < \omega \rangle \in V \) so that: there are \( \sigma : \omega \to \omega \) and \( F^n : [\kappa]^{\sigma(n)} \to \kappa \) so that for any \( n \), if \( \text{length}(p) = \sigma(n) \), then \( \langle p, \kappa \rangle \) decides \( \dot{\mu}_n = F^n(p) \).
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Observation: a normal name for $\mu_n$ represents an element in the $\sigma(n)$ iterated ultrapower. Example: suppose $\sigma(0) = 2$.

- There is $F^0 : [\kappa]^2 \to \kappa$ so that any condition with stem $\langle \zeta_0, \zeta_1, \ldots, \zeta_n \rangle$ decides $\dot{\mu}_0 = F^0(\zeta_0, \zeta_1)$. 
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- For fixed $\zeta_0$, the function $F^0(\zeta_0, -) : \kappa \to \kappa$ represents an ordinal $< j(\kappa)$ in the ultrapower, $j(F^0)(\zeta_0, \kappa)$. 
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- The function $\zeta_0 \mapsto j(F^0)(\zeta_0, \kappa)$ then represents an ordinal $< j(j(\kappa))$ in an ultrapower of the ultrapower. This ordinal is $j(j(F^0))(\kappa, j(\kappa))$. 
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- Set $N_0 = V$. 

Recall that we have $U$ a normal $\kappa$-complete ultrafilter on $\kappa$ in $V$. The ultrapower is well-founded, so we identify it as a transitive inner model of $V$, and have the ultrapower embedding $j: V \rightarrow \text{Ult}(V, U)$. Define $j^{(0)} = j$, $U^{(0)} = U$, and $\kappa^{(0)} = \kappa$. Set $N_1 = \text{Ult}(V, U)$.

In general, define $N_{n+1}$ to be the ultrapower of $N_n$ by $j^{(n)}(U^{(n)})$. This can be computed in $N_n$. This gives a commutative system of embeddings $i_{m,n}: N_m \rightarrow N_n$, where $i_{m,m+1} = j^{(m)}$ and the other maps are obtained by composition. Finally, define $N_\omega$ to be the direct limit of this system.
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- Finally, define $N_\omega$ to be the direct limit of this system.
Our observation from before: a normal name for $\mu_n$ represents an element in the $\sigma(n)$ iterated ultrapower. (In general, an element $x$ of the $n$th iterated ultrapower can be written $x = i_{0,n}(F)(\kappa^{(0)}, \kappa^{(1)}, \ldots, \kappa^{(n-1)})$. We say that $F$ represents $x$.)
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(In general, an element $x$ of the $n$th iterated ultrapower can be written $x = i_{0,n}(F)(\kappa^{(0)}, \kappa^{(1)}, \ldots, \kappa^{(n-1)})$. We say that $F$ represents $x$.)
It seems that the Prikry points are being replaced in the $j$-images of the normal name by the $\kappa^{(n)}$. This can be made precise by a theorem due independently to Bukovsky and Dehornoy from the 1970s:

The critical sequence $\langle \kappa^{(0)}, \kappa^{(1)}, \ldots \rangle$ is $i_{0,\omega}(\mathbb{P})$-generic over $N_\omega$, and the generic extension $M_\omega$ is $\bigcap_n N_n$. 
If $\langle \dot{\mu}_n : n < \omega \rangle$ is the name of an $\omega$-sequence cofinal in $\kappa$ forced to have true cofinality, define 
\[ \nu_n = i_{0,\sigma(n)}(F^n)(\kappa^{(0)}, \ldots, \kappa^{(\sigma(n)-1)}) \].

That is, $\langle \nu_n : n < \omega \rangle$ is the evaluation of the $i_{0,\omega}$-image of $\langle \dot{\mu}_n : n < \omega \rangle$ using the critical sequence as the Prikry sequence.
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That is, $\langle \nu_n : n < \omega \rangle$ is the evaluation of the $i_{0,\omega}$-image of $\langle \dot{\mu}_n : n < \omega \rangle$ using the critical sequence as the Prikry sequence.

A $\mathbb{P}$-name for a scale on $\langle \dot{\mu}_n : n < \omega \rangle$ can be evaluated with the critical sequence in the $\omega$th iterated ultrapower to get a scale on $\prod_n \nu_n$. A scale in $M_\omega$ on $\prod_n \nu_n$ can be used to obtain a name for a scale on $\prod_n \mu_n$. 
The point is that properties of the $M_\omega$ scale on $\prod_n \nu_n$ transfer over to the scale in the Prikry extension.

$M_\omega$ strongly resembles $V$: for example, $M_\omega$ has the same $\omega$-sequences as $V$, so the $M_\omega$ scale is actually a scale in the sense of $V$ also (although the $\nu_n$ need not be cardinals in $M_\omega$).

The hope is that this gives “absoluteness” results which say that certain scale properties cannot be forced by Prikry forcing.
Actually, there is a special case where we come close to achieving this.
Define $\tau(\gamma)$ to be largest so that $\gamma \in \text{image}(i_0, \tau(\gamma))$. Intuitively, this is the least $k < \sigma(n)$ so that $F^n$ in the normal name really depends on $k$. 

**Definition**

A function $f: \omega \rightarrow \text{ON}$ is forgetful if $\lim_{n \rightarrow \omega} \tau(f(n)) = \infty$, i.e., for every $m < \omega$ there is $k < \omega$ so that $\tau(f(n)) > m$ for all $n > k$.

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The special case is when \( n \mapsto \nu_n \) is forgetful.
For each $k < \omega$ and ordinal $\gamma$, choose $G^k_\gamma$ representing $\gamma$ in the $k$th iterated ultrapower. Choose $G^k_\gamma$ so that it doesn’t depend on the first $\tau(\gamma)$-coordinates, where $\tau$ is defined as in the previous slide. Now let $\langle f_\alpha : \alpha < \lambda \rangle$ be a scale on $\prod_n \nu_n$ in $V$.

**Theorem**

There is a scale $\langle g_\alpha : \alpha < \lambda \rangle$ on $\prod_n \mu_n$ defined by

$$g_\alpha(n) = G^\sigma_{f_\alpha(n)}(\zeta_0, \ldots, \zeta_{\sigma(n)} - 1).$$
Remarks

- If the ground model scale $\vec{f}$ used is continuous, then $\vec{g}$ will be continuous at points of cofinality $< \kappa$.
- An ordinal $\alpha$ is good in $\vec{g}$ iff it was good in $\vec{f}$.
- Slightly weaker special case works: Each $f_\alpha$ is forgetful.
For the diagonal Prikry forcing, one can define the iterated ultrapowers up to $\omega$, show that the $\omega$ iterated ultrapower is well-founded, and prove a version of the theorem of Bukovsky and Dehornoy. The special case is also straightforward to adapt.

In the argument for the diagonal Prikry forcing, the ultrafilters used are not required to be normal. Thus, the theorem immediately generalizes to the extender-based forcing which adds many $\omega$-sequences to a limit of strong cardinals.

Straightforward to adapt the arguments to the case of supercompact and diagonal supercompact Prikry forcings of Magidor and Gitik–Sharon.
Examples

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- What is the cofinality of $\prod_n \zeta_n$?
- We can compute $\nu_n = i_{0,n+1}(\kappa^{(n)}) = j^{n+1}(\kappa)$.
- Recall that $j$ is continuous at points of $V$-cofinality different from $\kappa$, and $\text{cf} (j(\kappa)) > \kappa$, so $\text{cf} (\nu_n) = \text{cf} (j(\kappa))$ for $k \geq 1$.
- So $\prod_n \nu_n$ has the same cofinality structure as $\prod_n \text{cf} (j(\kappa))$, and hence $\prod_n \zeta_n$ has cofinality $\text{cf} (j(\kappa))$ in $V[G]$. 
Now assume $2^\kappa > \kappa^+\omega$. Suppose there is an interesting scale $\vec{f}$ in $V$ on $\prod_n \kappa^+ n$. 
Examples

Now assume $2^\kappa > \kappa^+\omega$. Suppose there is an interesting scale $\vec{f}$ in $V$ on $\prod_n \kappa^+$. 

- Let $(\kappa^+)^V$ be represented in the ultrapower by the function $F_n : \kappa \to \kappa$.

- The theorem defines a scale $\vec{g}$ in $\prod_n F_n(\zeta_n)$ with similar properties as $\vec{f}$. 
Thank you!