Cardinal Invariants of Generalized Continua

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June 12, 2015
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What is a Cardinal Invariant?

- A cardinal $n$ which gives structural information about the continuum.
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- Generally cardinal invariants $n$ satisfy $\aleph_1 \leq n \leq c$. 

'The continuum' often means $\omega_\omega$, but can mean $\omega_2$, $\mathcal{P}(\omega)$, $\left[\omega\right]^{<\omega}$, etc.
What is a Cardinal Invariant?

- A cardinal \( n \) which gives structural information about the continuum.
- Generally cardinal invariants \( n \) satisfy \( \aleph_1 \leq n \leq c \).
- ‘The continuum’ often means \( \omega \omega \), but can mean \( \omega^2 \), \( \mathcal{P} \omega \), \( [\omega]^{\omega} \), etc.
Examples of Cardinal Invariants

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Examples of Cardinal Invariants

- $c$, the cardinality of the continuum, is perhaps the simplest example of a cardinal invariant of the continuum.
- $b$, the minimal cardinality of an unbounded family in $\langle \omega \omega, \leq^* \rangle$, where $\leq^*$ is the relation of eventual dominance.
- $d$, the cofinality of $\langle \omega \omega, \leq^* \rangle$.
Examples of Cardinal Invariants

- $\mathfrak{c}$, the cardinality of the continuum, is perhaps the simplest example of a cardinal invariant of the continuum.
- $\mathfrak{b}$, the minimal cardinality of an unbounded family in $\langle \omega \omega, \leq^* \rangle$, where $\leq^*$ is the relation of eventual dominance.
- $\mathfrak{d}$, the cofinality of $\langle \omega \omega, \leq^* \rangle$.
- $\mathfrak{a}$, the minimal cardinality of an infinite maximal almost disjoint family.
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Examples of Cardinal Invariants

- $c$, the cardinality of the continuum, is perhaps the simplest example of a cardinal invariant of the continuum.
- $b$, the minimal cardinality of an unbounded family in $\langle \omega \omega, \leq^* \rangle$, where $\leq^*$ is the relation of eventual dominance.
- $\delta$, the cofinality of $\langle \omega \omega, \leq^* \rangle$.
- $\alpha$, the minimal cardinality of an infinite maximal almost disjoint family.
- $u$, the minimal cardinality of an ultrafilter base.
- $s$, the minimal cardinality of a splitting family (a family which splits all members of $[\omega]^{\aleph_0}$, where $A$ splits $B$ iff $|A \cap B| = |A \setminus B| = \aleph_0$).
Relations between Cardinal Invariants
A standard and extensive reference for cardinal invariants of the continuum is Blass’s chapter of the Handbook of Set Theory, edited by Foreman and Kanamori.
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How does the Hasse diagram of cardinal invariants change when we generalize like this?

Are there any cardinals where the same system of inequalities which hold for invariants of the continuum is found to hold again?
Generalized Continua

\[ \omega \omega \quad \kappa \kappa \]
Generalized Continua

\[ \omega \omega \quad \kappa \kappa \]

\[ \omega^2 \quad \kappa^2 \]
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Cardinal Invariants of Generalized Continua
Generalized Continua

\[ \omega_\omega \quad \kappa_\kappa \]

\[ \omega_2 \quad \kappa_2 \]

\[ \mathcal{P}_\omega \quad \mathcal{P}_\kappa \]

\[ [\omega]^\omega \quad [\kappa]^\kappa \]
Introduction

We begin by generalizing the bounding and dominating numbers $b$ and $\delta$. 
Cummings and Shelah generalized the classical invariants $b$ and $\d$ to uncountable cardinals.
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They completely characterize possible values of $\langle b_\kappa, d_\kappa, 2^\kappa \rangle$ for $\kappa$ regular.
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They completely characterize possible values of $\langle b_\kappa, d_\kappa, 2^\kappa \rangle$ for $\kappa$ regular.

Nothing about the case of $\kappa$ singular is said.
Definition

Let $\mathbb{P} = \langle P, \leq \rangle$ be a preorder.
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- $\mathcal{B} \subseteq P$ is called **bounded** iff there exists $p \in P$ with $b \leq p$ for every $b \in \mathcal{B}$.
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- \( B \subseteq P \) is called \textit{bounded} iff there exists \( p \in P \) with \( b \leq p \) for every \( b \in B \).
- \( b_\mathbb{P} \) is the minimal cardinality of an \textit{un}bounded subset of \( \mathbb{P} \) (or zero if all subsets of \( \mathbb{P} \) are bounded).
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- $b_\mathbb{P}$ is the minimal cardinality of an **unbounded** subset of $\mathbb{P}$ (or zero if all subsets of $\mathbb{P}$ are bounded).
- $\mathcal{D} \subseteq P$ is called **dominating** iff it is cofinal in $\mathbb{P}$.
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- \( \mathcal{B} \subseteq P \) is called *bounded* iff there exists \( p \in P \) with \( b \leq p \) for every \( b \in \mathcal{B} \).
- \( b_P \) is the minimal cardinality of an *unbounded* subset of \( \mathbb{P} \) (or zero if all subsets of \( \mathbb{P} \) are bounded).
- \( \mathcal{D} \subseteq P \) is called *dominating* iff it is cofinal in \( \mathbb{P} \).
- \( d_P \) is the minimal cardinality of a dominating subset of \( \mathbb{P} \).
Lemma

For any preorder $\mathbb{P}$,

\[ cf \ b_\mathbb{P} = b_\mathbb{P} \leq cf \ d_\mathbb{P} \leq d_\mathbb{P} \leq |\mathbb{P}|. \]
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Proof.

Straightforward.
Lemma

Let \( P, Q \) be preorders. If \( f : P \rightarrow Q \) embeds \( P \) cofinally into \( Q \), then \( b_P = b_Q \) and \( \delta_P = \delta_Q \).
Lemma

Let $P, Q$ be preorders. If $f : P \to Q$ embeds $P$ cofinally into $Q$, then $b_P = b_Q$ and $\mathfrak{d}_P = \mathfrak{d}_Q$.

Proof.

Easy application of definitions.
Recall that for $x, y \in \omega \omega$, $x \leq^* y$ iff
\[ |\{ n < \omega \mid x(n) > y(n)\}| < \aleph_0. \]
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For $x, y \in \kappa^\kappa$ and $\lambda \leq \kappa$ a cardinal, $x \leq_\lambda y$ iff
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**Definition**

For $x, y \in \kappa\kappa$ and $\lambda \leq \kappa$ a cardinal, $x \leq_\lambda y$ iff
$$|\{\alpha < \kappa \mid x(\alpha) > y(\alpha)\}| < \lambda.$$ 

The assumed ordering of $\kappa\kappa$, when another is not specified, is $\leq_\kappa$. 
Eventual Dominance

- In the case of $\omega \omega$, $\leq_\omega$ (previously denoted $\leq^*$) is equivalent to eventual dominance.
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- This continues to hold in the case of $\kappa$ regular.
- However, for $\kappa$ singular eventual dominance is strictly stronger than $\leq_\kappa$.

Example Consider $f \in \aleph_\omega \cdot \aleph_\omega$ given by $f(\alpha) = \aleph_n$ where $\alpha = \beta + n$ and $\beta$ limit.

$f$ is not $\leq_\kappa$-below any increasing function, and does not eventually dominate any increasing unbounded function.
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- $b_\kappa = b_{P_\kappa}$.
- $d_\kappa = d_{P_\kappa}$.
Theorem (Cummings-Shelah (1995))

Let $V$ be a model of GCH, and $\kappa \mapsto \langle b(\kappa), d(\kappa), m(\kappa) \rangle$ be a class function from the class of regular functions to the class of triples of cardinals and satisfying

- $\kappa^+ \leq \text{cf } b(\kappa) = b(\kappa) \leq \text{cf } d(\kappa) = d(\kappa) \leq m(\kappa)$
- $\text{cf } m(\kappa) > \kappa$

for every regular $\kappa$. 
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- $V^\mathbb{P} \models b_\kappa = b(\kappa)$
- $V^\mathbb{P} \models d_\kappa = d(\kappa)$
- $V^\mathbb{P} \models 2^\kappa = m(\kappa)$. 
What if $\kappa$ is Singular?

The preceding result shows that the values of $b_\kappa$ and $d_\kappa$ are essentially independent for different $\kappa$ so long as all cardinals are regular.
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What if $\kappa$ is Singular?

- The preceding result shows that the values of $b_{\kappa}$ and $d_{\kappa}$ are essentially independent for different $\kappa$ so long as all cardinals are regular.
- In analogy with the famed result of Easton and the ensuing singular cardinal problem, we might expect the situation for singular $\kappa$ to be more complicated.
- Indeed it is, and the independence between different cardinals breaks down in the case of singular cardinals.
What if $\kappa$ is Singular?

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- In analogy with the famed result of Easton and the ensuing singular cardinal problem, we might expect the situation for singular $\kappa$ to be more complicated.

- Indeed it is, and the independence between different cardinals breaks down in the case of singular cardinals.

- We shall prove that for any cardinal $\kappa$, $b_{\text{cf } \kappa} \leq b_\kappa$ and $d_{\text{cf } \kappa} \leq d_\kappa$ (a result of the author).
To prove $b_{\text{cf} \kappa} \leq b_{\kappa}$ and $d_{\text{cf} \kappa} \leq d_{\kappa}$, it will be useful to consider bounding and dominating numbers of more general preorders.
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**Definition**

Let $\kappa, \lambda, \mu$ be cardinals with $\lambda \leq \kappa$, and define $\mathbb{P}_{\kappa \lambda \mu} = \langle \kappa \mu, \leq \lambda \rangle$. 
To prove $b_{\text{cf}} \kappa \leq b_\kappa$ and $d_{\text{cf}} \kappa \leq d_\kappa$, it will be useful to consider bounding and dominating numbers of more general preorders.

**Definition**

Let $\kappa, \lambda, \mu$ be cardinals with $\lambda \leq \kappa$, and define $P_{\kappa\lambda\mu} = \langle \kappa^\mu, \leq \lambda \rangle$.

- $b_{\kappa\lambda\mu} = b_{P_{\kappa\lambda\mu}}$. 
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- $b_{\kappa\lambda\mu} = b_{P_{\kappa\lambda\mu}}$.
- $d_{\kappa\lambda\mu} = d_{P_{\kappa\lambda\mu}}$. 
First Parameter Inequality

Lemma

Suppose $\kappa_1 \leq \kappa_2$. Then $b_{\kappa_1 \lambda \mu} \leq b_{\kappa_2 \lambda \mu}$ and $d_{\kappa_1 \lambda \mu} \leq d_{\kappa_2 \lambda \mu}$.
Lemma

Suppose \( \kappa_1 \leq \kappa_2 \). Then \( b_{\kappa_1 \lambda \mu} \leq b_{\kappa_2 \lambda \mu} \) and \( d_{\kappa_1 \lambda \mu} \leq d_{\kappa_2 \lambda \mu} \).

Proof.

We prove the result for dominating numbers; the case of bounding numbers is similar. Let \( \Pi \) be a partition of \( \kappa_2 \) into disjoint subsets of cardinality \( \kappa_1 \), and let \( D \) be a dominating family in \( \langle \kappa_2 \mu, \leq \lambda \rangle \). For all but \( < \kappa \) many elements \( A \) of \( \Pi \), \( D \upharpoonright A \) must be a dominating family on \( \langle A \mu, \leq \lambda \rangle \), and clearly \( \langle A \mu, \leq \lambda \rangle \simeq \langle \kappa_1 \mu, \leq \lambda \rangle \).
Lemma

Suppose $\lambda_1 \leq \lambda_2$. Then $b_{\kappa \lambda_1 \mu} \leq b_{\kappa \lambda_2 \mu}$ and $d_{\kappa \lambda_1 \mu} \leq d_{\kappa \lambda_2 \mu}$. 
Lemma

Suppose $\lambda_1 \leq \lambda_2$. Then $b_{\kappa \lambda_1 \mu} \leq b_{\kappa \lambda_2 \mu}$ and $d_{\kappa \lambda_1 \mu} \leq d_{\kappa \lambda_2 \mu}$.

Proof.

Trivial.
Lemma

\[ b_{\kappa\lambda\mu} = b_{\kappa\lambda}(cf\ \mu) \text{ and } d_{\kappa\lambda\mu} = d_{\kappa\lambda}(cf\ \mu). \]
Lemma

\[ b_{\kappa\lambda\mu} = b_{\kappa\lambda}(cf\ \mu) \quad \text{and} \quad d_{\kappa\lambda\mu} = d_{\kappa\lambda}(cf\ \mu). \]

Proof.

A cofinal embedding \( f : \langle \kappa(cf\ \mu), \leq \lambda \rangle \rightarrow \langle \kappa\mu, \leq \lambda \rangle \) is readily constructed from an increasing cofinal function \( g : cf\ \mu \rightarrow \mu \).
Theorem

For any cardinal \( \kappa \), \( \mathfrak{b}_{\text{cf} \, \kappa} \leq \mathfrak{b}_\kappa \) and \( \mathfrak{d}_{\text{cf} \, \kappa} \leq \mathfrak{d}_\kappa \).
Theorem

For any cardinal $\kappa$, $b_{cf \kappa} \leq b_\kappa$ and $d_{cf \kappa} \leq d_\kappa$.

Proof.

Using the inequalities for the first two parameters and the equality for the last parameter, we have

$$b_{cf \kappa} = b(cf \kappa)(cf \kappa)_\kappa \leq b(cf \kappa)_{\kappa \kappa} \leq b_\kappa,$$

and similarly for the case of $d_{cf \kappa} \leq d_\kappa$. 

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Cardinal Invariants of Generalized Continua
Splitting Numbers

We now move on to generalizing the splitting number \( s \) to uncountable cardinals.
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He focuses on showing what is entailed by \( s_\kappa \) being large relative to \( \kappa \), in particular concluding large cardinal properties of \( \kappa \) under apparently mild assumptions about \( s_\kappa \).
Zapletal, following Kamo, Suzuki, and others, generalized the classical invariant $\mathfrak{s}$ to uncountable cardinals.

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He focuses on showing what is entailed by $\mathfrak{s}_\kappa$ being large relative to $\kappa$, in particular concluding large cardinal properties of $\kappa$ under apparently mild assumptions about $\mathfrak{s}_\kappa$.

The case of $\aleph_\omega$ is discussed as representative of the singular case.
Definition

- $\mathcal{F} \subseteq [\kappa]^{\kappa}$ is called a *splitting family* iff for every $B \in [\kappa]^{\kappa}$ there exists $A \in \mathcal{F}$ with

\[|A \cap B| = |A \setminus B| = \kappa.\]

Such an $A$ is said to *split* $B$. 
**Definition**

- \( \mathcal{F} \subseteq [\kappa]^{\kappa} \) is called a *splitting family* iff for every \( B \in [\kappa]^{\kappa} \) there exists \( A \in \mathcal{F} \) with

\[
|A \cap B| = |A \setminus B| = \kappa.
\]

Such an \( A \) is said to *split* \( B \).

- \( s_\kappa \) is the minimal cardinality of a splitting family in \( [\kappa]^{\kappa} \).
$s_\kappa$, $\kappa$ Regular

\[ b_\kappa, d_\kappa \geq \kappa^+, \text{ regardless of the value of } \kappa. \]
$s_{\kappa}, \kappa$ Regular

- $b_\kappa, d_\kappa \geq \kappa^+$, regardless of the value of $\kappa$.

- Surprisingly, for $s_\kappa$ to be large relative to $\kappa$ requires $\kappa$ to possess large cardinal properties.

\( \kappa \text{ is strongly inaccessible iff } s_\kappa \geq \kappa \) (here we count \( \aleph_0 \) as strongly inaccessible).

\( \kappa \) is strongly inaccessible iff \( s_\kappa \geq \kappa \) (here we count \( \aleph_0 \) as strongly inaccessible).

Proof.

Omitted.

\( \kappa \) is weakly compact iff \( s_\kappa > \kappa \) (here we count \( \aleph_0 \) as weakly compact).

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Proof.

Omitted.
Lemma (Ben Neria-Gitik 2014)

Suppose $\kappa$, $\lambda$ are regular cardinals and $\kappa^+ < \lambda$. The statement $s_\kappa = \lambda$ is equiconsistent with the statement $o(\kappa) = \lambda$. 
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Suppose $\kappa$, $\lambda$ are regular cardinals and $\kappa^+ < \lambda$. The statement $s_\kappa = \lambda$ is equiconsistent with the statement $o(\kappa) = \lambda$.

Proof.

Omitted; it should be noted that this proof is significantly more technically difficult than the proof of any other result discussed in this talk.
Lemma

If $\kappa$ is not strongly inaccessible, then $\log \kappa \leq s_\kappa \leq 2^{< \log \kappa} < \kappa$, where

$$\log \kappa = \min\{\lambda \mid 2^\lambda \geq \kappa\}.$$  

Proof.

To see that $\log \kappa \leq s_\kappa$, we note that the rows of a binary $\kappa \times \lambda$ matrix can be taken to represent subsets of $\kappa$, and such a matrix represents a separating family (like a splitting family but with splitting weakened to $A$ separates $B$ iff $|A \cap B|, |A \setminus B| \geq 1$) iff it has distinct columns. The minimum number of rows required to produce a binary $\kappa \times \lambda$ matrix with distinct columns is clearly $\lambda = \log \kappa$, which implies $\log \kappa \leq s_\kappa$. 
Lemma

If $\kappa$ is not strongly inaccessible, then $\log \kappa \leq s_{\kappa} \leq 2^{\log \kappa} < \kappa$, where

$$\log \kappa = \min\{\lambda \mid 2^\lambda \geq \kappa\}.$$ 

Proof.

The construction used by Zapletal in his proof that if $\kappa$ is not strongly inaccessible then $s_{\kappa} < \kappa$ gives a splitting family of size $2^{\log \kappa}$, and so $s_{\kappa} \leq 2^{\log \kappa}$. 

$\blacksquare$
Lemma (Zapletal 1995)

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$s_\kappa$, $\kappa$ Singular
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Let $\kappa$ be a cardinal.

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- $s_{\kappa} \leq \max \{ pcf(N_\alpha) \mid \alpha < \text{cf } \kappa \}$.
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Let $\kappa$ be a cardinal.

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- $s_\kappa \leq s_{\text{cf } \kappa}$.
- If $\kappa$ is strong limit, $s_\kappa = s_{\text{cf } \kappa}$.
- $s_\kappa \leq \text{maxpcf}\{\aleph_\alpha \mid \alpha < \text{cf } \kappa\}$.

Proof.

Omitted.
A Generalized Classical Inequality

- It turns out that the classical inequality $s \leq d$ generalizes to all cardinals, regular or singular.
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A Generalized Classical Inequality

- It turns out that the classical inequality $\mathfrak{s} \leq \mathfrak{d}$ generalizes to all cardinals, regular or singular.
- The method of proof still does distinguish the regular and singular cases.
- This result does not appear to exist in the literature.
Theorem

For any infinite cardinal \( \kappa \), \( s_\kappa \leq d_\kappa \).
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Proof.

The case of regular cardinals is a straightforward generalization of the classical proof that $s \leq d$ (see Blass 2010 Handbook article).
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Proof.

The case of regular cardinals is a straightforward generalization of the classical proof that $s \leq d$ (see Blass 2010 Handbook article). For singular $\kappa$, we combine previous results to obtain

$$s_\kappa \leq s_{cf \kappa} \leq d_{cf \kappa} \leq d_\kappa.$$
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- Much work remains to be done to see what happens to the classical partial order of cardinal invariants when these are extended into the uncountable.
- In particular, much less is known about the singular case than the regular case.
Acknowledgements

- Thank you to Ernest Schimmerling for advising my master’s thesis on this topic.
- Thank you to Samuel Coskey for introducing me to cardinal invariants of the continuum and encouraging me to pursue their generalization.
- Thank you to Liljana Babinkostova, Marion Scheepers, and the members of the 2014 Boise State REU for providing stimulating discussions and a great mathematical work environment!
References


