Determinacy of Refinements to the Difference Hierarchy of Co-analytic sets

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3rd Münster Conference in Inner Model Theory, the Core Model Induction and Hod Mice
Theorems of the form:

$$\exists M (M \models T, M \text{ is iterable}) \implies \text{Det}(\omega^2-\Pi^1_1 + \Gamma)$$
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\[ \exists M (M \models T, M \text{ is iterable}) \iff \text{Det}(\omega^2-\Pi^1_1 + \Gamma) \]

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For the above to hold we would like to have:

$$T \vdash \text{Det}(\Gamma)$$
Goal

Theorems of the form:

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\[ T \subseteq \text{ZFC}, \Gamma \subseteq \Delta^1_1 \]

For the above to hold we would like to have:

\[ T \vdash \text{Det}(\Gamma) \]

Consider: \( T = \text{KP} + \Sigma^1_1\text{-Sep}, \Gamma = \Sigma^0_2 \).
\[ \Delta^1_2 \]

\[ \Pi^1_1 \]

\[ \Delta^1_1 \]

\[ \vdots \]

\[ \Sigma^0_2 \]

\[ \Sigma^0_1 \]

Borel, Projective hierarchy
Borel, Projective hierarchy

\( \Delta^1_2 \) \quad \omega^2-\Pi^1_1

\( \Pi^1_1 \) 

\( \Delta^1_1 \) 

\( \Sigma^0_2 \) 

\( \Sigma^0_1 \) 

\( (\omega^2 + 1)-\Pi^1_1 \)

Difference hierarchy on \( \Pi^1_1 \)
Borel, Projective hierarchy

\[ (\omega^2 + 1) - \Pi^1_1 \]

\( \Delta^1_2 \)

\( \omega^2 - \Pi^1_1 \)

\( \ldots \)

\( \Pi^1_1 \)

\( \ldots \)

\( \Delta^1_1 \)

\( \ldots \)

\( \Pi^1_1 \)

\( \ldots \)

\( \Sigma^0_2 \)

\( \Pi^1_1 \overset{\text{Det} \leftrightarrow 0^\#}{\rightleftharpoons} \Sigma^0_1 \)

Difference hierarchy on \( \Pi^1_1 \)

Classes we're interested in

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Borel, Projective hierarchy

$\Delta_2^1$

$\Delta_1^1$

$\Pi_1^1$

Difference hierarchy on $\Pi_1^1$

Refined difference hierarchy

Refined difference hierarchy

$\Sigma_2^0$

$\Sigma_1^0$

$\Omega^2 + 1 - \Pi_1^1 + \Delta_1^1$

$\omega^2 - \Pi_1^1 + \Delta_1^1$

$\omega^2 - \Pi_1^1 + \Pi_1^1$

$\omega^2 - \Pi_1^1 + \Sigma_2^0$

$\omega^2 - \Pi_1^1 + \Sigma_1^0$

$\omega^2 - \Pi_1^1$

Det $\leftrightarrow 0^+$

Det $\leftrightarrow 0^-$

Classes we're interested in

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\[ \Delta_2 \]
\[ \Pi_1 \]
\[ \Delta_1 \]
\[ : \]
\[ 3-\Pi_1 \]
\[ 2-\Pi_1 \]
\[ \Sigma_0 \]
\[ \Sigma_1 \]

\[ (\omega^2 + 1)-\Pi_1 \]
\[ \omega^2-\Pi_1 \]
\[ : \]
\[ \omega^2-\Pi_1 + \Sigma_0 \]
\[ \omega^2-\Pi_1 + \Sigma_0 \]
\[ \omega^2-\Pi_1 + \Sigma_0 \]

Classes we're interested in

Refined difference hierarchy

\[ \text{Det} \leftrightarrow 0^\dagger \]

\[ \text{Det} \leftrightarrow 0^\# \]

Difference hierarchy on \( \Pi_1 \)

Determinacy of \( \omega^2-\Pi_1 + \Gamma \)
**Definition**

Let $\Gamma$ be a pointclass closed under countable intersections (e.g. $\Pi^1_1$), $\alpha$ be a countable ordinal. We say a set $A$ is $\alpha$-$\Gamma$ if there is a sequence $\langle A_\beta \mid \beta \leq \alpha \rangle$ such that:

- Each $A_\beta \in \Gamma$;
- $A_\alpha = \emptyset$;
- $x \in A \iff$ the least $\beta$ such that $x \notin A_\beta$ is odd.

So $$(A_0 \setminus A_1) \cup (A_2 \setminus A_3) \cup \cdots$$

**Fact**

If $\alpha > 1$ is a computable ordinal then $\Pi^1_1 \subsetneq \alpha$-$\Pi^1_1 \subsetneq (\alpha + 1)$-$\Pi^1_1 \subsetneq \Delta^1_2$. 

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Determinacy of $\omega^2$-$\Pi^1_1$ + $\Gamma$  
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- each $A_\beta \in \Gamma$;
- $A_\alpha = \emptyset$;
- $x \in A$ if and only if the least $\beta$ such that $x \not\in A_\beta$ is odd.

So $(A_0 \setminus A_1) \cup (A_2 \setminus A_3) \cup \cdots$.
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So $(A_0 \setminus A_1) \cup (A_2 \setminus A_3) \cup \cdots$.

Fact

*If $\alpha > 1$ is a computable ordinal then*

$$\Pi^1_1 \subsetneq \alpha\cdot \Pi^1_1 \subsetneq (\alpha + 1)\cdot \Pi^1_1 \subsetneq \Delta^1_2$$
We can refine the difference hierarchy by restricting the final set in the sequence.

**Definition**
For \( \Lambda \subseteq \Gamma \), we say

\[ \Lambda \in \alpha-\Gamma + \Lambda \]

if \( \Lambda \in (\alpha + 1)-\Gamma \), as witnessed by the sequence \( \langle \Lambda_\beta \mid \beta \leq \alpha + 1 \rangle \), but \( \Lambda_\alpha \in \Lambda \).
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**Definition**

For $\Lambda \subseteq \Gamma$, we say

$$A \in \alpha - \Gamma + \Lambda$$

if $A \in (\alpha + 1) - \Gamma$, as witnessed by the sequence $\langle A_\beta \mid \beta \leq \alpha + 1 \rangle$, but $A_\alpha \in \Lambda$.

Let $A \in \omega^2 - \Pi^1_1 + \Gamma$. In order to win the game for $A$, both players are trying not to be the first one to go out of an $A_\beta$ for $\beta < \omega^2$, and if they both succeed then I wins if he gets into $A_{\omega^2}$.
Auxiliary Game

The proof follows Martin’s “integration” method for proving $\alpha$-$\Pi^1_1$ determinacy from indiscernibles. The ingredients of that proof are:

▶ Characterise membership in $\Pi^1_1$ sets by well-orders
▶ Define an auxiliary game in which the players must confirm that they played into certain sets by exhibiting those well-orders
▶ The auxiliary game is constructed so as to be determined
▶ Using a winning strategy for the auxiliary game, a winning strategy for the original game is defined
▶ In moving from the auxiliary strategy to that for the original game, the players must “imagine” the auxiliary moves being played by their opponent; indiscernibility ensures that this is possible.
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Details of the Auxiliary Game

\[\langle a_0, \eta_0 \rangle \rightarrow \langle a_1, \eta_1 \rangle \rightarrow \langle a_2, \eta_2 \rangle \rightarrow \langle a_3, \eta_3 \rangle \rightarrow \ldots\]

The ordinal components \(\eta_i\) ∈ \(\mathbb{A}_\omega\) are partitioned so as to create \(\mathbb{A}_\omega^2\) many countable orderings. Each should witness that \(x = \langle a_0, a_1, a_2, \ldots \rangle \in A^{\beta}\) for some \(\beta < \omega^2\).

We say that the play is badly lost if one of these orderings witnesses that \(x \not\in A^{\beta}\). If the first such mistake occurs with \(\beta\) even then it is badly lost for I, otherwise for II.

II wins the auxiliary game if the play is not badly lost for either player; I wins if it is badly lost for II.
The ordinal components $\eta_i \in \aleph_\omega$ are partitioned so as to create $\omega^2$ many countable orderings. Each should witness that $x = \langle a_0, a_1, a_2, \ldots \rangle \in \Lambda_\beta$ for some $\beta < \omega^2$. 

▶ The play is badly lost if one of these orderings witnesses that $x / \in A_\beta$. If the first such mistake occurs with an even $\beta$ then it is badly lost for player I, otherwise for player II.

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“Being badly lost” is an open condition because a play is badly lost iff there is an initial position where the orderings for one player are wrong. Altogether this means that the above auxiliary game is open, and so it is determined.
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Now consider extending the proof to our situation: we don’t have a \(\omega^2-\Pi^1_1\) set, but a \(\omega^2-\Pi^1_1 + \Gamma\) set, so we modify the win condition to be: I wins if the play is badly lost for II or it is not badly lost for either player and \(x \in A_{\omega^2}\).
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$A_{\omega^2}$ is an element of $\Gamma$, so this condition is no longer open; to find a winning strategy we need to analyse the complexity of this condition.
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We will need a lightface condition, so the first task is to work out what “lightface $\Sigma^0_n$” should mean for a subset of $(\omega \times \aleph_\omega)^\omega$. 
Effective Descriptive Set Theory on $\kappa^\omega$

**Definition**

Call a subset $R$ of $\kappa^\omega$ *generalised lightface open* if there is a $\Sigma_1(L_\kappa)$ set $X \subseteq \kappa^{<\omega}$ such that:

$$x \in R \iff \exists p \in X (p \subseteq x)$$

One can also define lightface open subsets of $\kappa^\omega$ in the obvious way.

If we replace $L_\kappa$ with $\langle L_\kappa, \vec{c} \rangle$ for some countable set of ordinals $\vec{c}$, then we can make the same definition to get the lightface in $\vec{c}$ open sets.
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Determinacy of $\omega^2$-$\Pi^1_1 + \Gamma$
Münster, July 2015
The Generalised Lightface Borel Hierarchy

Let $P$ be a relation on $\kappa^\omega$, then:

- $P$ is called $\Sigma^0_{1+1}$ if $P$ is generalised lightface open;
- $P$ is $\Sigma^0_{n+1}$ iff there is a $\Pi^0_n$ predicate $R \subseteq \kappa^\omega \times \omega$ such that $x \in P \iff \exists a \in \omega (R(x,a))$;
- $P$ is $\Pi^0_n$ iff $\neg P$ is $\Sigma^0_n$;
- $P$ is $\Delta^0_n$ iff it is $\Sigma^0_n$ and $\Pi^0_n$.

Note that we go up by $\omega$ unions, not $\kappa$ unions.

If we replace "lightface" with "lightface in $\vec{c}$" then we get the $\Sigma^0_n(\vec{c})$ hierarchy on $\kappa^\omega$. 
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If we replace “lightface” with “lightface in $\vec{c}$” then we get the $\Sigma^0_n (\vec{c})$ hierarchy on $\kappa^\omega$. 
We can prove the analogue of the Kleene Basis theorem in this context: If $X \subseteq \kappa^\omega$ is $\Sigma^1_1$ and non-empty, it has an element definable over any admissible set $M$ with $L_\kappa \in M$. The idea is that $\Sigma^1_1$ relations are $\Pi^1_1$ over any admissible containing $L_\kappa$. The leftmost path through the corresponding tree is then a definable element. This allows us to reduce the complexities of properties in the determinacy arguments, and hence prove determinacy of the auxiliary games in weak models.
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This allows us to reduce the complexities of properties in the determinacy arguments, and hence prove determinacy of the auxiliary games in weak models.
A basic fact is that if $A \subseteq \omega^\omega$ is $\Sigma^0_n$, it is also $\Sigma^0_n$ in this sense, considered as a subset of each $\kappa^\omega$. A quick calculation then shows that, if the main game is $\omega^2-\Pi^1_1 + \Sigma^0_n$ for $n > 1$ then the auxiliary game is $\Sigma^0_n(\langle \aleph_i \mid i < \omega \rangle)$ on $(\omega \times \aleph_\omega)^\omega$, a pointclass we abbreviate to $\hat{\Sigma}^0_n$. 

Example

If $A \in \omega^2-\Pi^1_1 + \Sigma^0_2$ then the auxiliary winning set $A^*$ is $\hat{\Sigma}^0_2$ and, if $M$ is a transitive model of KP + $\Sigma^1_1$-Sep containing $\langle \aleph_i \mid i < \omega \rangle$ then there is a $\Sigma^1_1$-definable winning strategy for $A^*$ in $M$. 

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Determinacy of the Auxiliary Game

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We can then prove that the auxiliary game is determined using arguments analogous to those used to establish ordinary \( \Sigma^0_n \) determinacy.
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**Example**

If $A \in \omega^2-\Pi^1_1 + \Sigma^2_2$ then the auxiliary winning set $A^*$ is $\hat{\Sigma}^0_2$ and, if $M$ is a transitive model of $KP + \Sigma_1$-Sep containing $\langle \aleph_i \rangle$ then there is a $\Sigma_1$-definable winning strategy for $A^*$ in $M$. 
Having shown that the auxiliary game is determined, we need a form of indiscernibility to perform the “integration” part of Martin’s method.
Generating Indiscernibles

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The kind of indiscernibility we use is as follows:

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A closed-unbounded class of ordinals $C$ is a class of $\Sigma_n$ generating indiscernibles for the theory $T$ if,
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A closed-unbounded class of ordinals $C$ is a class of $\Sigma_n$ generating indiscernibles for the theory $T$ if, letting $A_T[\vec{c}]$ be the least transitive model of the theory $T$ (in the language including a predicate for $\vec{c}$) containing the sequence $\vec{c}$,

$$A_T[\vec{c}] \equiv_{\Sigma_n} A_T[\vec{d}]$$
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So, taking $T = KP + \Sigma_1$-Sep, $n = 1$, this principle would imply that the winning strategy for the $\hat{\Sigma}^0_2$ auxiliary game behaves the same when defined over any of these models.
Generating Indiscernibles

We obtain indiscernibles by starting with a mouse $M \models T$. ($M$ must satisfy $T$ in the language with its $M$-ultrafilter $F$ as a predicate)

\[\langle p, X \rangle \leq \langle q, Y \rangle \iff q \text{ is an initial segment of } p \land X \cup (p \setminus q) \subseteq Y\]
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**Definition**

Let $M_\lambda$ be the $\lambda$th iterate in the iteration of $M$ by $F$, with measurable $\kappa_\lambda$, and let $\mathbb{P}$ be the Prikry forcing for $F_\lambda$:

$$\mathbb{P} = \{ \langle p, X \rangle \mid p \in [\kappa_\lambda]^{<\omega}, X \in F_\lambda \cap M_\lambda \}$$

$$\langle p, X \rangle \leq \langle q, Y \rangle \iff q \text{ is an initial segment of } p \land X \cup (p \setminus q) \subseteq Y$$

**Theorem (L.S.)**

If $M \models KP + \Sigma^1_n$-Sep and $V = L[F]$, then the class of iteration points is a class of generating indiscernibles for $KP + \Sigma^1_n$-Sep.

This requires us to show that, although $\mathbb{P}$ is a class forcing over $M_\lambda$, $KP + \Sigma^1_n$-Sep holds in the generic extension.
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We need to show that the forcing behaves nicely, but don’t want to show it’s pre-tame. Let $\theta = \text{On} \cap M_\lambda$. 

Define a set of “names,” which can be thought of as $L_\theta[\vec{c}]$ where $\vec{c}$ is a constant symbol for a Prikry-generic sequence. We give each name a rank according to where it appears in $L_\theta[\vec{c}]$. 

After we fix a generic $\vec{c} \subseteq \kappa_\lambda$, the interpretation of any name in $L_\theta[\vec{c}]$ is just the corresponding set in $L_\theta[\vec{c}]$. 

Define a ranked language $L_P$ which in addition to everything from $L_{\{\in\}}$ contains ranked variables $v_{\alpha\iota}$ for $\alpha < \theta$ and all the names from $L_\theta[\vec{c}]$. A sentence of $L_P$ is ranked if all variables in it are ranked.
Ramified Forcing Language

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Ramified Forcing

We can now define the (weak) forcing relation.

\[ p \Vdash ^* x \in y \iff x \in L_{\alpha}[\vec{c}], y \in L_{\beta}[\vec{c}] \text{ and:} \]

1. \[ \alpha = \beta = 0 \text{ and } x \in y \in \kappa \text{ or } x \in p \land y = \{ z \gamma | \varphi(z \gamma) \} \] or
2. \[ \alpha < \beta, y = \{ z \gamma | \varphi(z \gamma) \} \text{ and } p \Vdash ^* \varphi(x) ; \] or else
3. \[ \alpha \geq \beta \text{ and } \exists z \in L_{\gamma}[\vec{c}] \text{ for some } \gamma, \] either \[ \beta > \gamma \text{ or } \beta = \gamma = 0 \text{ and } p \Vdash ^* z = x \land z \in y \]

\[ p \Vdash ^* x = y \iff p \Vdash ^* \forall z \alpha(z \in x \iff z \in y) \text{ for } \alpha \text{ the maximum of the ranks of } x \text{ and } y. \]

4. \[ p \Vdash ^* \varphi \land \psi \iff p \Vdash ^* \varphi \land p \Vdash ^* \psi. \]

5. \[ p \Vdash ^* \neg \varphi \iff \forall q \in P(q \leq p \Rightarrow q \Vdash ^* \neg \varphi) \]

6. \[ p \Vdash ^* \exists x \alpha(\varphi(x \alpha)) \text{ iff there is some } t \in L_{\alpha}[\vec{c}] \text{ such that } p \Vdash ^* \varphi(t) \]

\[ p \Vdash ^* \exists x(\varphi(x)) \text{ iff there is some } t \in \bigcup_{\alpha} L_{\alpha}[\vec{c}] \text{ such that } p \Vdash ^* \varphi(t) \]
Ramified Forcing

We can now define the (weak) forcing relation.

1. If $p \Vdash^* x \in y$ iff $x \in L_{\alpha}[\dot{c}], y \in L_{\beta}[\dot{c}]$ and:
   - $\alpha = \beta = 0$ and either $x \in y \in \kappa$ or $x \in p \land y = \dot{c}$; or
   - $\alpha < \beta$, $y = \{z^\gamma \mid \varphi(z^\gamma)\}$ and $p \Vdash^* \varphi(x)$; or else
   - $\alpha \geq \beta$ and $\exists z \in L_{\gamma}[\dot{c}]$ for some $\gamma$, either $\beta > \gamma$ or $\beta = \gamma = 0$ and $p \Vdash^* z = x \land z \in y$

2. If $p \Vdash^* x = y$ iff $p \Vdash^* \forall z^{\alpha}(z \in x \iff z \in y)$ for $\alpha$ the maximum of the ranks of $x$ and $y$.

3. $p \Vdash^* \varphi \land \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

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Forcing Theorems

- This forcing relation is definable, and in fact if \( \varphi \) is a \( \Sigma_n \) sentence of the forcing language, then \( p \models^* \varphi \) is \( \Sigma_n^M \).

This is proved first for \( \Delta^1_1 \) formulæ simultaneously with the Prikry lemma, and relies on the fact that we don't quantify over \( P \) in the definition of \( \models^* \) for atomic formulæ.
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- This slide was a bit empty, so here’s a picture of mice playing a game:
Forcing Theorems

The definition of genericity is odd due to the weak setting:

**Definition**

$G \subseteq \mathbb{P}$ is $M_{\lambda}$-generic if it *both* meets all $\Sigma^M_\lambda$ dense subclasses of $\mathbb{P}$ *and* decides every $\Sigma^n$ sentence of $L_{\mathbb{P}}$.
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In this case, the generic extension $M_\lambda[G]$ is $L_\theta[\vec{c}]$, where

$$\vec{c} = \bigcup \{ p \mid \langle p, X \rangle \in G \}.$$
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From now on we denote $M[G]$ as $M_\lambda[\vec{c}]$ as above.

Note that, as in modern forcing, the generic extension is the class of names interpreted by the generic.
Forcing Theorems

**Theorem (L.S.)**

The Truth Lemma holds for any such generic, and we can always find one by taking the Prikry sequence $\vec{c}$ a countable sequence of critical points cofinal in $\kappa_\lambda$.

In fact more is true. By the Prikry property, we have the following:

**Theorem (L.S.)**

For any $p = \{c_0, \ldots, c_l\}$ and $y$ an arbitrary constant $\Sigma_n$-definable in $M_\lambda[\vec{c}]$ (without indiscernible parameters above $c_l$). Suppose $\psi$ is $\Pi_{n-1}$. Then:

$$M_\lambda[\vec{c}] \models \exists z \psi(z, y) \iff M_\lambda \models \exists Y \langle p, Y \rangle \models \exists z \psi(z, y)$$

Now we can show that

$M_\lambda[\vec{c}] \models \text{KP} + \Sigma_n\text{-Sep.}$
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Now it’s easy to see that all $M_\lambda[\vec{c}]$ models are $\Sigma_n$ elementary equivalent, minimal and models of $KP + \Sigma_n$-Sep with $\vec{c} \in M_\lambda[\vec{c}]$. 

I.e. $M_\lambda[\vec{c}] = AT[\vec{c}]$. 
We thus have the ingredients required to mimic Martin's proof: definable winning strategies for the auxiliary game, and suitable indiscernibles.
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I.e. $\mathcal{M}_\lambda[\vec{c}] = \mathcal{A}_T[\vec{c}]$. 

Indiscernibility
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- I.e. $M_{\lambda}[\vec{c}] = A_T[\vec{c}]$.

- We thus have the ingredients required to mimic Martin’s proof: definable winning strategies for the auxiliary game, and suitable indiscernibles.
To win the original game, the player must be able to ignore the components of the auxiliary game that are not played in the original.
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They “imagine” their opponent has played indiscernibles $\vec{c}$.

They then move according to the auxiliary strategy’s output, as computed by any model $M_\lambda [\vec{c}] = A_T [\vec{c}]$.

This strategy is winning in $V$ because any counterexample would be a real existing by Shoenfield absoluteness in a suitable $M_\lambda [\vec{c}]$. 
Theorem (L.S.)

The implication:

\[ \exists M (M \models T, M \text{ is iterable}) \implies \text{Det}(\omega^2-\Pi_1^1 + \Gamma) \]

holds for the following values of \( T \) and \( \Gamma \):

- \( \Sigma_0^1 \) and \( \Sigma_0^2 \) for KP + \( \Sigma_0^1 \) and KP + \( \Sigma_0^2 \) respectively
- \( \Sigma_0^3 \) and \( \Sigma_0^{n+1} \) for ZFC − \( +P\alpha(\kappa) \) and ZFC − \( P\alpha(\kappa) \) respectively
- \( \Delta_{11} \) for \( \alpha \) and \( \alpha < \omega_{CK}^1 \)
Theorem (L.S.)

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<table>
<thead>
<tr>
<th>$T$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;'cleverness' + $\exists a \ 'clever mouse'&quot;</td>
<td>$\Sigma^0_1$</td>
</tr>
<tr>
<td>$\text{KP} + \Sigma^1_1\text{-Sep}$</td>
<td>$\Sigma^0_2$</td>
</tr>
<tr>
<td>$\text{KP} + \Sigma^2_2\text{-Sep}$</td>
<td>$\Sigma^0_3$</td>
</tr>
<tr>
<td>$\text{KP} + \Sigma^\alpha_{n+1}\text{-Sep}$</td>
<td>$n-\Pi^0_3$</td>
</tr>
<tr>
<td>$\text{ZFC}^- + \mathcal{P}^\alpha(\kappa) \text{ exists}$</td>
<td>$\Sigma^0_{1+\alpha+3}$ ($\alpha &lt; \omega^1_{CK}$)</td>
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Open Questions

1. What other combinations of $T, \Gamma$ can we find proofs of?

C. M. Le Sueur.
Determinacy of refinements to the difference hierarchy of co-analytic sets.
submitted.
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2. Are there reverse implications, or at least limitations?

3. Does the generalised lightface hierarchy generate interesting effective descriptive set theory?

4. Is the specialised forcing useful for anything else?

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Thanks

**Definition**

Let $M$ be a mouse, $Q^M_\kappa$ the $Q$-structure of $M$ at $\kappa$ and $\theta = \text{On} \cap Q^M_\kappa$. $M$ is *clever* if, for any $\Sigma_1$ formula $\varphi(x, y)$ and parameter $p \in [\kappa]^{<\omega}$,

$$\{\xi < \kappa \mid Q^M_\kappa \models \varphi(\xi, p)\} \in F^M \implies \exists \tau < \theta \left(\{\xi < \tau \mid J^\mathcal{F}_\tau \models \varphi(\xi, p)\} \in F^\kappa \cap Q^M_\kappa\right)$$

This implies Rowbottom’s theorem holds for partitions $\Sigma_1$ definable over $M$.