Chain conditions, layered partial orders and weak compactness

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Introduction

Productivity of chain conditions

The work presented in this talk is motivated by classical questions on the *productivity of chain conditions* in partial orders.

Definition

Given an uncountable regular cardinal κ, we let $\mathcal{C}_\kappa$ denote the statement that the product of two partial orders satisfying the κ-chain condition again satisfies the κ-chain condition

$\mathcal{C}_\kappa$ implies the non-existence of κ-Souslin trees and MA implies $\mathcal{C}_{\aleph_1}$.

In particular, the statement $\mathcal{C}_{\aleph_1}$ is independent from the axioms of $\mathbf{ZFC}$.

A folklore argument shows that $\mathcal{C}_\kappa$ holds if κ is weakly compact.

Proposition

If κ is a weakly compact cardinal, then $\mathcal{C}_\kappa$ holds.

Proof.

Assume that $\mathbb{P}_0$ and $\mathbb{P}_1$ are partial orders satisfying the κ-chain condition and $\langle\langle p_0^\alpha, p_1^\alpha\rangle \mid \alpha < \kappa\rangle$ is a bijective enumeration of an antichain in $\mathbb{P}_0 \times \mathbb{P}_1$.

Given $\alpha < \beta < \kappa$, there is $c(\alpha, \beta) < 2$ such that $p^\alpha_{c(\alpha,\beta)}$ and $p^\beta_{c(\alpha,\beta)}$ are incompatible in $\mathbb{P}_i$.

Using the weak compactness of κ, we can find a set $H \in [\kappa]^\kappa$ that is homogenous for the resulting coloring $c : [\kappa]^2 \longrightarrow 2$.

If $c \restriction [H]^2$ is constant with color i, then $\langle p_i^\alpha \mid \alpha \in H\rangle$ is a bijective enumeration of an antichain in $\mathbb{P}_i$, a contradiction. $\square$

Recall that, given an uncountable regular cardinal κ, a partial order is κ-*Knaster* if every κ-sized collection of conditions can be refined to a κ-sized set of pairwise compatible conditions.

A small modification of the above proof yields the following statement.

Proposition

If κ is weakly compact and $\nu < \kappa$, then ν-support products of partial orders satisfying the κ-chain condition are κ-Knaster.

A series of results of Todorčević, Shelah, Rinot and others shows that, for regular cardinals $\kappa > \aleph_1$, many consequences of weak compactness can be derived from the assumption $\mathcal{C}_\kappa$. For example:

- κ is weakly inaccessible.
- $\kappa \neq \mathrm{cof}(\beth_{\alpha+1})$ for every $\alpha \in \mathrm{On}$.
- Every stationary subset of κ reflects.
- $\square(\kappa)$ fails and κ is weakly compact in L.

These results suggest an affirmative answer to the following question.

Question (Todorčević)

Are the following statements equivalent for every regular cardinal $\kappa > \aleph_1$?

- κ is weakly compact.
- $\mathcal{C}_\kappa$ holds.

Motivated by this question, we want to consider properties of partial orders that ...

- ... imply the κ-chain condition,
- ... are preserved by forming products, and
- ... are equivalent to the κ-chain condition if κ is weakly compact.

It is now interesting to consider the question whether the κ-chain condition can be equivalent to such a property at non-weakly compact cardinals, because both possible answers yield interesting statements:

- A positive answer to this question would answer Todorčević's question in the negative.
- A negative answer leads to new characterizations of weak compactness using chain conditions.

Stationarily layered partial orders

Stationarily layered partial orders

We will now present examples of properties of partial orders that satisfy the above requirements.

Remember that, given a partial order $\mathbb{P}$, we say that $\mathbb{Q} \subseteq \mathbb{P}$ is a regular suborder if the inclusion map preserves incompatibility and maximal antichains in $\mathbb{Q}$ are maximal in $\mathbb{P}$.

Proposition

Given a partial order $\mathbb{P}$, a suborder $\mathbb{Q}$ of $\mathbb{P}$ is regular if and only if the inclusion map preserves incompatibility and for every $p \in \mathbb{P}$, there is $q \in \mathbb{Q}$ such that for all $r \leq_{\mathbb{Q}} q$, the conditions p and r are compatible in $\mathbb{P}$.

In the setting of the proposition, the condition q is called a *reduct of p into $\mathbb{Q}$*.

Definition

Given a cardinal κ and a partial order $\mathbb{P}$, we let $\mathrm{Reg}_\kappa(\mathbb{P})$ denote the collection of all regular suborders of $\mathbb{P}$ of cardinality less than κ.

We consider properties of partial orders that imply that $\mathrm{Reg}_\kappa(\mathbb{P})$ is *large* in a certain sense. The following definition uses Jech's definition of stationarity in $\mathbb{P}_\kappa(A)$ to express largeness.

Definition

Given an uncountable regular cardinal κ, a partial order $\mathbb{P}$ is called *κ-stationarily layered* if $\mathrm{Reg}_\kappa(\mathbb{P})$ is stationary in $\mathbb{P}_\kappa(\mathbb{P})$.

Lemma (Cox)

If κ is an uncountable regular cardinal and $\mathbb{P}$ is a κ-stationarily layered partial order, then $\mathbb{P}$ is κ-Knaster.

To prove this lemma, we need the following observation that follows directly from a standard lifting argument.

Proposition

The following statements are equivalent for every uncountable regular cardinal κ and every partial order $\mathbb{P}$:

- $\mathbb{P}$ *is κ-stationarily layered.*
- *For every regular cardinal $\theta > \kappa$ with $\mathbb{P} \in \mathrm{H}(\theta)$, the collection of all elementary substructures M of $\mathrm{H}(\theta)$ with $|M| < \kappa$, $\kappa \cap M \in \kappa$ and $\mathbb{P} \cap M \in \mathrm{Reg}_\kappa(\mathbb{P})$ is stationary in $\mathbb{P}_\kappa(\mathrm{H}(\theta))$.*

Proof of the Lemma.

Let $\vec{p} = \langle p_\alpha \mid \alpha < \kappa \rangle$ be a sequence of conditions in $\mathbb{P}$ and let $\theta > \kappa$ be a regular cardinal with $\mathbb{P} \in \mathrm{H}(\theta)$.

Then there is a stationary subset S of $\mathbb{P}_\kappa(\mathrm{H}(\theta))$ consisting of elementary substructures M of $\mathrm{H}(\theta)$ with $\vec{p} \in M$, $\kappa \cap M \in \kappa$ and $\mathbb{P} \cap M \in \mathrm{Reg}_\kappa(\mathbb{P})$.

Given $M \in S$, there is a reduct $r(M)$ of $p_{\kappa \cap M}$ into $\mathbb{P} \cap M$. This defines a regressive function $r : S \longrightarrow \mathrm{H}(\theta)$ and we can find $q \in \mathbb{P}$ and $S' \subseteq S$ stationary in $\mathbb{P}_\kappa \mathrm{H}(\theta)$ such that $r(M) = q \in M$ for all $M \in S'$.

Pick $M, N \in S'$ with $\kappa \cap M < \kappa \cap N$. Then the conditions $p_{\kappa \cap M}$ and q are compatible in $\mathbb{P}$ and, since $p_{\kappa \cap M}, q \in N$, there is $q' \in \mathbb{P} \cap N$ extending $p_{\kappa \cap M}$ and q. Then the conditions $p_{\kappa \cap N}$ and q' are compatible in $\mathbb{P}$ and hence the conditions $p_{\kappa \cap M}$ and $p_{\kappa \cap N}$ are compatible in $\mathbb{P}$.

This shows that the sequence $\langle p_{\kappa \cap M} \mid M \in S' \rangle$ consists of pairwise compatible conditions. □

$\mathcal{F}$-layered partial orders

In the spirit of the approach outlined above, we are interested in classes of stationarily layered partial orders that are closed under products. We present classes with the property that any two members are layered on a *common stationary set*.

Definition

Let κ be an uncountable regular cardinal, let $\lambda \geq \kappa$ be a cardinal and let $\mathcal{F}$ be a normal filter on $\mathbb{P}_\kappa(\lambda)$.

- A partial order $\mathbb{P}$ is $\mathcal{F}$-layered, if it has cardinality at most λ and

$$\{a \in \mathbb{P}_\kappa(\lambda) \mid s[a] \in \mathrm{Reg}_\kappa(\mathbb{P})\} \in \mathcal{F}$$

  holds for every surjection $s : \lambda \longrightarrow \mathbb{P}$.

- A partial order $\mathbb{P}$ is *completely $\mathcal{F}$-layered* if every subset of $\mathbb{P}$ of cardinality at most λ is contained in a regular suborder of $\mathbb{P}$ of cardinality at most λ and every regular suborder of $\mathbb{P}$ of size at most λ is $\mathcal{F}$-layered.

Let κ be an uncountable regular cardinal, let $\lambda \geq \kappa$ be a cardinal and let $\mathcal{F}$ be a filter on $\mathbb{P}_\kappa(\lambda)$.

Lemma

If $\lambda = \lambda^{<\kappa}$ holds, then every completely $\mathcal{F}$-layered partial order is κ-stationarily layered and therefore κ-Knaster.

Lemma

The class of completely $\mathcal{F}$-layered partial order is closed under products.

Moreover, for many interesting filters $\mathcal{F}$, the class of completely $\mathcal{F}$-layered partial orders is also closed under products with larger supports.

Layering at weakly compact cardinals

Let κ be a weakly compact cardinal. The *weakly compact filter* on κ is the filter generated by sets of the form

$$R_{\Phi,A,a} \;=\; \{\alpha < \kappa \mid \mathrm{V}_\alpha \models \Phi(A \cap \mathrm{V}_\alpha, a)\},$$

where Φ is a Π^1_1-formula, $A \subseteq \mathrm{V}_\kappa$, $a \in \mathrm{V}_\kappa$ and $\mathrm{V}_\kappa \models \Phi(A, a)$.

This filter is normal and contains the collection of all inaccessible cardinals less than κ as an element.

We let $\mathcal{F}_{wc}$ denote the filter on $\mathbb{P}_\kappa(\kappa)$ induced by the weakly compact filter. This filter is again normal.

Theorem

If κ is weakly compact, then every partial order of cardinality at most κ that satisfies the κ-chain condition is $\mathcal{F}_{wc}$-layered.

The above theorem directly implies the following result.

Theorem

Given a weakly compact cardinal κ, the following statements are equivalent for every partial order $\mathbb{P}$:

- $\mathbb{P}$ *satisfies the κ-chain condition.*
- $\mathbb{P}$ *is κ-Knaster.*
- $\mathbb{P}$ *is κ-stationarily layered.*
- $\mathbb{P}$ *is completely $\mathcal{F}_{wc}$-layered.*

Moreover, it can be shown that the class of completely $\mathcal{F}_{wc}$-layered partial orders is closed under ν-support products for every $\nu < \kappa$.

These results show that complete $\mathcal{F}_{wc}$-layeredness satisfies the demands listed above and it is interesting to ask whether the existence of a normal filter $\mathcal{F}$ on $\mathbb{P}_\kappa(\lambda)$ with the property that the κ-chain condition is equivalent to complete $\mathcal{F}$-layeredness implies the weak compactness of κ.

It turns out that the weak compactness of κ already follows from the weaker assumption that the κ-chain condition is equivalent to stationary layeredness.

This leads to the following new characterization of weakly compact cardinals.

Theorem

The following statements are equivalent for every uncountable regular cardinal κ:

- *κ is weakly compact.*
- *Every partial order satisfying the κ-chain condition is κ-stationarily layered.*

This equivalence is established by showing that the second statement implies that κ is inaccessible and has the tree property.

In the following, we outline the proof of this implication.

Given an uncountable regular cardinal κ and a tree $\mathbb{T}$ of height κ, we let $\mathbb{P}(\mathbb{T})$ denote the partial order whose conditions are finite partial functions $s : \mathbb{T} \longrightarrow \omega$ that are injective on chains in $\mathbb{T}$ and whose ordering is given by reversed inclusion.

Lemma (Baumgartner)

If κ is an uncountable regular cardinal and $\mathbb{T}$ is a κ-Aronszajn tree, then the partial order $\mathbb{P}(\mathbb{T})$ satisfies the κ-chain condition.

Lemma

If κ is an uncountable regular cardinal and $\mathbb{T}$ is a κ-Aronszajn tree, then the partial order $\mathbb{P}(\mathbb{T})$ is not κ-stationarily layered.

Proof.

Assume toward a contradiction that $\mathbb{P}(\mathbb{T})$ is κ-stationarily layered.

Then there is an elementary submodel M of $\mathrm{H}(\kappa^+)$ such that $|M| < \kappa$, $\kappa \cap M \in \kappa$, $\mathbb{P} \in M$ and $\mathbb{P}(\mathbb{T}) \cap M \in \mathrm{Reg}_\kappa(\mathbb{P}(\mathbb{T}))$.

Pick $t \in \mathbb{T}(\kappa \cap M)$ and set $p = \{\langle t, 0\rangle\}$. Then p is a condition in $\mathbb{P}(\mathbb{T})$ and there is a reduct q of p into $\mathbb{P}(\mathbb{T}) \cap M$. Since the conditions p and q are compatible in $\mathbb{P}(\mathbb{T})$, we have $q(s) \neq 0$ for all $s \in \mathrm{dom}(q)$ with $s <_{\mathbb{T}} t$.

Let $\beta < \kappa$ be minimal with $\mathrm{dom}(q) \subseteq \mathbb{T}_{<\beta}$. Then $\beta < \kappa \cap M$ and elementarity implies that $\mathbb{T}(\beta) \subseteq M$, because $\mathbb{T}$ is a κ-Aronszajn tree.

Let u denote the unique element of $\mathbb{T}(\beta)$ with $u <_{\mathbb{T}} t$. Set $r = q \cup \{\langle u, 0\rangle\}$.

By the above remarks, r is a condition in $\mathbb{P}(\mathbb{T}) \cap M$ below q. This implies that the conditions p and r are compatible in $\mathbb{P}(\mathbb{T})$, a contradiction. □

The second part of the above implication is a consequence of the following characterization of inaccessible cardinals using stationary layeredness.

Definition

Given an uncountable cardinal κ, a partial order $\mathbb{P}$ is $<\kappa$-*linked* if there is $\lambda < \kappa$ and a function $c : \mathbb{P} \longrightarrow \lambda$ that is injective on antichains in $\mathbb{P}$.

Lemma

The following statements are equivalent for every uncountable regular cardinal κ*:*

- κ *is strongly inaccessible.*
- *Every* $<\kappa$*-linked partial order is* κ*-stationarily layered.*

Stationarily layered partial orders and the Knaster property

Since $<\kappa$-linked partial orders are κ-Knaster, the above characterization of inaccessible cardinals naturally leads to the question if there is a large cardinal property that corresponds to the statement that every κ-Knaster partial order is κ-stationarily layered.

The following results show that the question whether this equality characterizes weak compactness is independent from the axioms of $\mathbf{ZFC}$.

Theorem

Let κ be an uncountable regular cardinal with the property that every κ-Knaster partial order is κ-stationarily layered. Then κ is a Mahlo cardinal and every stationary subset of κ reflects.

In particular, this property characterizes weak compactness in canonical inner models.

The proof of this result uses notions introduced by Todorčević.

Definition

Let κ be uncountable and regular, $S \subseteq \kappa$ and $\mathbb{T}$ be a tree of height κ.

- A map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ is *regressive* if $r(\mathbb{T}) <_{\mathbb{T}} t$ holds for $t \in \mathbb{T} \upharpoonright S$ that is not minimal in $\mathbb{T}$.
- We say that S *is nonstationary with respect to* $\mathbb{T}$ if there is a regressive map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ with the property that for every $t \in \mathbb{T}$ there is a function $c_t : r^{-1}\{t\} \longrightarrow \lambda_t$ such that $\lambda_t < \kappa$ and c_t is injective on $\leq_{\mathbb{T}}$-chains.
- The tree $\mathbb{T}$ is *special* if κ is nonstationary with respect to $\mathbb{T}$.

Lemma

Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a κ-Aronszajn tree that does not split at limit levels. If there is a stationary subset S of κ such that S is nonstationary with respect to $\mathbb{T}$, then the partial order $\mathbb{P}(\mathbb{T})$ is κ-Knaster.

Theorem (Todorčević)

The following statements are equivalent for every inaccessible cardinal κ:

- *κ is a Mahlo cardinal.*
- *There are no special κ-Aronszajn trees.*

Now, assume that κ is an uncountable regular cardinal with the property that every κ-Knaster partial order is κ-stationarily layered.

Since $<\kappa$-linked partial orders are κ-Knaster, this assumption implies that κ is inaccessible.

By the above lemma, there are no special κ-Aronszajn trees and Todorčević's theorem implies that κ is a Mahlo cardinal.

Next, assume that there is a stationary subset S of κ that does not reflect. Then there is a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ that avoids S.

Let $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$ be the corresponding tree consisting of *full codes of walks through* $\vec{C}$. Then $\mathbb{T}$ is a κ-Aronszajn tree and S is nonstationary with respect to $\mathbb{T}$. Hence $\mathbb{P}(\mathbb{T})$ is κ-Knaster and not κ-stationarily layered.

In contrast, it it also consistent that every κ-Knaster partial order is κ-stationarily layered and κ is not not weakly compact. The proof of the following theorem shows that such cardinals exist in a model constructed by Kunen.

Theorem

If κ is a weakly compact cardinal, then there is a partial order $\mathbb{P}$ such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V.

- *κ is inaccessible and not weakly compact.*
- *Every κ-Knaster partial order is κ-stationarily layered.*

In Kunen's model, there is an inaccessible cardinal κ and a normal κ-Souslin tree $\mathbb{T}$ such that forcing with $\mathbb{T}$ makes κ weakly compact. Define

$$\mathcal{F} \ = \ \{F \in \mathbb{P}_\kappa(\kappa) \mid \mathbb{1}_{\mathbb{T}} \Vdash \text{“}\check{F} \in \mathcal{F}_{wc}\text{''}\}.$$

Then $\mathcal{F}$ is a normal filter on $\mathbb{P}_\kappa(\kappa)$ and every κ-Knaster partial order is completely $\mathcal{F}$-layered. In particular, every κ-Knaster partial order is κ-stationarily layered.

Open questions

Question

Assume that κ is an inaccessible cardinal with the property that every κ-Knaster partial order is κ-stationarily layered. Is κ weakly compact in L?

Question

Are there other natural instances of pairs (Φ, Γ) with

- $\Phi(\kappa)$ is a large cardinal property weaker than weak compactness of κ.
- $\Gamma(\kappa)$ is a class of partial orders satisfying the κ-chain condition.

so that $\mathbf{ZFC}$ proves that for every inaccessible cardinal κ, the statement $\Phi(\kappa)$ is equivalent to the statement that every partial order in $\Gamma(\kappa)$ is κ-stationarily layered.

In particular, is there a class of partial orders satisfying the κ-chain condition that corresponds to Mahlo cardinals in this way?

Thank you for listening!