$\mathcal{F}$-Mathias reals and generic filters

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Institute of Mathematics CAS
Outline

Mathias–Prikry forcing $\mathbb{M}(\mathcal{F})$

Properties of $\mathbb{M}(\mathcal{F})$

Generic filters
D. Chodounský, O. Guzmán, M. Hrušák, *Mathias–Prikry and Laver type forcing; Summable ideals, coideals, and $\pm$-selective filters*, submitted

Mathias–Prikry forcing $\mathcal{M}(\mathcal{F})$

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Generic filters
Definition (Mathias forcing)

\[ \mathcal{M} = \{ \langle a, F \rangle : a \in [\omega]^{<\omega}, F \in [\omega]^\omega \} \]

\[ \langle a, F \rangle < \langle b, H \rangle \quad \text{if} \quad b \subseteq a, F \subseteq H, \text{and} \ a \setminus b \subseteq H. \]
Definition (Mathias forcing)

Let \( \mathcal{F} \) be a filter on \( \omega \).

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\mathbb{M}(\mathcal{F}) = \{ \langle a, F \rangle : a \in [\omega]^{<\omega}, F \in \mathcal{F} \}
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Definition (Mathias real for $\mathcal{F}$)
$$x = \bigcup \{ a : \langle a, F \rangle \} \in \mathbb{G}, \text{where } \mathbb{G} \text{ is an } \mathbb{M}(\mathcal{F}) \text{ generic filter}.$$  

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A Mathias real is a pseudo-intersection of $\mathcal{F}$ ($x \subseteq^* F$ for each $F \in \mathcal{F}$).
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\( \langle a, F \rangle < \langle b, H \rangle \) if \( b \subseteq a, F \subset H \), and \( a \setminus b \subset H \).

Definition (Mathias real for \( \mathcal{F} \))
\( x = \bigcup \{ a : \langle a, F \rangle \} \in G \), where \( G \) is an \( \mathcal{M}(\mathcal{F}) \) generic filter.

Fact
A Mathias real is a pseudo-intersection of \( \mathcal{F} \) (\( x \subseteq^* F \) for each \( F \in \mathcal{F} \)).

Definition
\( U \subset [\omega]^{<\omega} \) is an \( \mathcal{F} \)-universal set if \([F]^{<\omega} \cap U \neq \emptyset \) for each \( F \in \mathcal{F} \).

Fact
\([x]^{<\omega} \cap U \neq \emptyset \) for each \( \mathcal{F} \)-universal set \( U \).
Definition (Mathias like real for $\mathcal{F}$)

Let $\mathcal{F}$ be a filter on $\omega$. A set $m \subset \omega$ is a Mathias like real for $\mathcal{F}$ if

1. $m \subseteq^* F$ for each $F \in \mathcal{F}$,
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If $m \subset \omega$ is a Mathias like real for $\mathcal{F}$ and $c \subset \omega$ is a Cohen real, then $m \cap c$ is a Mathias real for $\mathcal{F}$. 


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What are the properties of $\mathbb{M}(\mathcal{F})$?

**Fact**
$\mathbb{M}(\mathcal{F})$ is not $\omega^\omega$ bounding.
Theorem
$\mathbb{M}(\mathcal{F})$ does not add dominating reals iff $\mathcal{F}$ is Menger.

Theorem
$\mathbb{M}(\mathcal{F})$ is almost $\omega^\omega$ bounding iff $\mathcal{F}$ is Hurewicz.

Let $X$ be a topological space.

Definition
$X$ is Menger if no continuous image of $X$ in $\omega^\omega$ is dominating.

Definition
$X$ is Hurewicz if every continuous image of $X$ in $\omega^\omega$ is bounded.
Properties of $\mathbb{M}(\mathcal{F})$

Let $\mathcal{F}$ be a filter on $\omega$. The following are equivalent:

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Applications

Hurewicz and Menger classes are closed with respect to closed subsets, countable unions, products with compacts, continuous images, ...
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Proposition

Let $\mathcal{F}$ be an analytic filter on $\omega$. $\mathbb{M}(\mathcal{F})$ does not add a dominating real if and only if $\mathcal{F}$ is $F_\sigma$.

Theorem

It is consistent that $\mathfrak{b} = \omega_1$ and every Tychonoff space $X$ of size $\omega_1$ is a $\gamma$-space, provided that $X^n$ is Hurewicz for all $n \in \omega$.

Proposition

There exists a MAD family $A$ on $\omega$ such that $\mathbb{M}(\mathcal{F}(A))$ adds a dominating real.

Proposition

If $\mathfrak{d} = \mathfrak{c}$, then there exists an infinite MAD family $A$ such that $\mathbb{M}(\mathcal{F}(A))$ does not add a dominating real.
Hurewicz and Menger topological spaces

In the following a cover of $X$ is countable open cover of $X$.

**Definition**

$\mathcal{U}$ is a $\gamma$-cover of $X$ if $\mathcal{U}$ is a cover of $X$ and for every $x \in X$ the family $\{ U \in \mathcal{U} : x \notin U \}$ is finite.
Hurewicz and Menger topological spaces

In the following a \textit{cover of X} is countable open cover of $X$.

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$\mathcal{U}$ is a $\gamma$-\textit{cover} of $X$ if $\mathcal{U}$ is a cover of $X$ and for every $x \in X$ the family $\{U \in \mathcal{U} : x \notin U\}$ is finite.

**Definition**

$X$ is \textit{Menger} if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of covers of $X$ there is $\{\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} : n \in \omega\}$ such that $\bigcup \mathcal{V}_n : n \in \omega$ is a cover of $X$.

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\textbf{\textuparrow\text{-covers of filters}}

For $a \subseteq \omega$ denote $\uparrow a = \{x \subseteq \omega : a \subseteq x\}$.

\textbf{Fact}

\begin{itemize}
  \item $\uparrow a$ is compact
  \item $a$ is finite $\Rightarrow$ $\uparrow a$ is open
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$\mathcal{U}$ is an $\uparrow$-cover of $X \subset 2^\omega$ if $\mathcal{U}$ is a cover of $X$ consisting of sets of the form $\uparrow a$, $a \in [\omega]^{<\omega}$. 
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Let \(\mathcal{F}\) be a filter on \(\omega\). \(U \subset [\omega]^{<\omega}\) is an \(\mathcal{F}\)-universal set iff \(\mathcal{U} = \{\uparrow a : a \in U\}\) is an \(\uparrow\text{-cover}\) of \(\mathcal{F}\).
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Let $\mathcal{F}$ be a filter on $\omega$, $\mathcal{O}$ a cover of $\mathcal{F}$ (consisting of open subsets of $2^\omega$). There is an $\uparrow$-cover $\mathcal{U}$ of $\mathcal{F}$, such that $\mathcal{F} \subset \bigcup \mathcal{U} \subset \bigcup \mathcal{O}$.
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$X$ is Menger if for every sequence $\{U_n : n \in \omega\}$ of covers of $X$ there is $\{V_n \in [U_n]^{<\omega} : n \in \omega\}$ such that $\bigcup V_n : n \in \omega$ is a cover of $X$.

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Corollary

If $X$ is filter on $\omega$, we can replace “($\gamma$-)cover” by “($\gamma$-)\raisebox{-0.5pt}{$\uparrow$}-cover” in the definitions of Menger and Hurewicz properties.
Mathias–Prikry forcing $\mathbb{M}(\mathcal{F})$

Properties of $\mathbb{M}(\mathcal{F})$

Generic filters
Special ultrafilters

Definition
An ultrafilter $\mathcal{F}$ on $\omega$ is a $P$-point if for each $\mathcal{C} \in [\mathcal{F}]^\omega$ there is a pseudo-intersection $P \in \mathcal{F}$ such that $P \subseteq^* F$ for each $F \in \mathcal{C}$.

Definition
An ultrafilter $\mathcal{F}$ on $\omega$ is selective if for each $\{A_i : i \in \omega\}$, a partition of $\omega$ disjoint with $\mathcal{F}$ there is a selector $S \in \mathcal{F}$ such that $|S \cap A_i| = 1$ for each $i \in \omega$. 

Theorem (Zapletal)
An ultrafilter $\mathcal{F}$ is a $P$-point iff for each analytic ideal $I \subseteq \mathcal{F}$ there is an $\mathcal{F}$-ideal $\mathcal{C}$ such that $I \subseteq \mathcal{C} \subseteq \mathcal{F}$.
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Let $\mathcal{F}$ be an ultrafilter in $\omega$. The following properties are equivalent:

1. $\mathcal{F}$ is selective,
2. For each $c: [\omega]^2 \rightarrow 2$ there exists a $c$-homogeneous set $F \in \mathcal{F}$,
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An ultrafilter $\mathcal{F}$ on $\omega$ is a *P-point* if for each $C \in [\mathcal{F}]^\omega$ there is a pseudo-intersection $P \in \mathcal{F}$ such that $P \subseteq^* F$ for each $F \in C$.

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An ultrafilter $\mathcal{F}$ on $\omega$ is *selective* if for each $\{A_i : i \in \omega\}$, a partition of $\omega$ disjoint with $\mathcal{F}$ there is a selector $S \in \mathcal{F}$ such that $|S \cap A_i| = 1$ for each $i \in \omega$.

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**Theorem (Zapletal)**
An ultrafilter $\mathcal{F}$ is a P-point iff for each analytic ideal $\mathcal{I} \subset \mathcal{F}^*$ there is an $F_\sigma$ ideal $\mathcal{C}$ such that $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{F}^*$.
Generic filters

Theorem (folklore?)

*The generic filter on the poset \((\mathcal{P}(\omega) \setminus \text{Fin}, \subseteq^*)\) is a selective ultrafilter.*
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\((\text{LC})^1\) An ultrafilter is selective iff it is a generic filter on \(\mathcal{P}(\omega) \setminus \text{Fin}\) over \(L(\mathbb{R})\).

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\(^1\) (\(\text{LC}\)) denotes the assumption that there exist sufficiently large cardinals in \(V\). In this talk infinitely many Woodin’s and a measurable above them.
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Let \( \mathcal{I} \) be an \( F_\sigma \) ideal on \( \omega \). Denote by \( \mathbb{Q}_\mathcal{I} \) the forcing \( (\mathcal{P}(\omega) \setminus \mathcal{I}, \subset^*) \).

Theorem (Zapletal, Ch.)

\((\text{LC})\) *\( \mathcal{F} \) is a \( \mathbb{Q}_\mathcal{I} \)-generic filter over \( L(\mathbb{R}) \) iff*

\begin{itemize}
  \item \( \mathcal{F} \) is a P-point disjoint with \( \mathcal{I} \), and
  \item for each closed set \( C \subset \mathcal{P}(\omega) \) disjoint with \( \mathcal{F} \) there is \( e \in \mathcal{F}^* \) such that \( C \subseteq \langle \mathcal{I}, \{e\} \rangle \).
\end{itemize}

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In this talk infinitely many Woodin’s and a measurable above them.
Definition (Mathias like real for $\mathcal{F}$)

Let $\mathcal{F}$ be a filter on $\omega$. A set $m \subset \omega$ is a Mathias like real for $\mathcal{F}$ if

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Lemma
Let $U$ be a P-point. Assume there is (in some extension of $V$) an elementary embedding $j: V \rightarrow M$ such that $\mathbb{R} \cap V$ is countable in $M$. Then there is a Mathias real $g \in j(\mathcal{F})$ (over $V$).
**proof**

**Definition (Mathias like real for \( \mathcal{F} \))**

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**Theorem**

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**Lemma**

Let \( \mathcal{U} \) be a P-point. Assume there is (in some extension of \( V \)) an elementary embedding \( j : V \to M \) such that \( \mathbb{R} \cap V \) is countable in \( M \). Then there is a Mathias real \( g \in j(\mathcal{F}) \) (over \( V \)).

**Lemma**

Let \( \mathcal{F} \) be as in the theorem. Suppose \( D \in L(\mathbb{R}) \) is open dense in \( \mathcal{Q}_I \). Then \( M(\mathcal{F}) \models g \in D^V[\dot{g}] \).
proof continued

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Let $\mathcal{U}$ be a P-point. Assume there is (in some extension of $V$) an elementary embedding $j: V \to M$ such that $\mathbb{R} \cap V$ is countable in $M$. Then there is a Mathias real $g \in j(\mathcal{F})$ (over $V$).

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Pass to an extension $V[G]$ where $j: V \rightarrow M$ exists. There is a Mathias real $g \in j(\mathcal{F})$. Now $g \in D^V[G]$, i.e. $g \in j(D) \cap j(\mathcal{F})$. 