Many different values in Cichoń's diagram

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We are interested in *the meager ideal* $\mathcal{M}$ and *the null ideal* $\mathcal{N}$ on $\mathbb{R}$.

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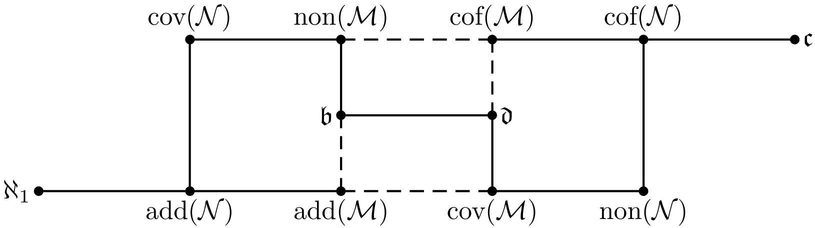
- $\mathfrak{c}$ denotes the size of the continuum.

Cichoń's diagram

Inequalities: **Bartoszyński, Fremlin, Miller, Rothberger, Truss**
Completeness: **Bartoszyński, Judah, Miller, Shelah**



Also $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

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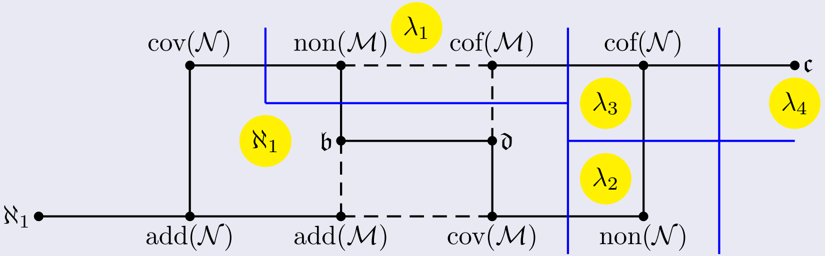
- *csi of proper forcing* only allows to assign $\aleph_1$ and $\aleph_2$.
- Many models are obtained from *fsi of ccc posets*, but such a poset forces $\mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$ where μ is the cofinality of the length of the iteration (when μ is uncountable).

A non fsi example

Theorem (A. Fischer, Goldstern, Kellner and Shelah)

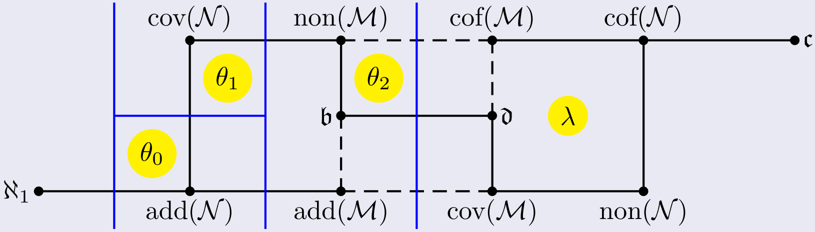
If $\lambda_1, \lambda_2 < \lambda_3 < \lambda_4$ *are pairwise distinct cardinals such that* $\lambda_i^{\aleph_0} = \lambda_i$ *for* $i = 1, 2, 3, 4$*, then it is consistent that*



Consistency examples (1)

Theorem (From Brendle, Judah and Shelah's fsi of ccc posets techniques 1990's)

If $\theta_0 \leq \theta_1 \leq \theta_2$ are uncountable regular cardinals and $\lambda^{<\theta_2} = \lambda$, then it is consistent that



A general description of cardinal invariants

Let X and Y be two uncountable Polish spaces of the form $\prod_{n<\omega} W_n$ with W_n discrete and at most countable.

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Let θ be a regular uncountable cardinal. A family $F \subseteq X$ is θ-$\sqsubset$-*unbounded* iff $|F| \geq \theta$ and, for any $y \in Y$,

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Fact

If $F \subseteq X$ is θ-$\sqsubset$-unbounded, then $\mathfrak{b}_\sqsubset \leq |F|$ and $\theta \leq \mathfrak{d}_\sqsubset$.

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(3) If $h \in \omega^\omega$, $h \geq^* 2$, consider the relation $\neq^*_h$ on $\mathbb{R}_h = \{x \in \omega^\omega : x < h\}$ defined as above. If $\sum_{i<\omega} \frac{1}{h(i)} < +\infty$ then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_{\neq^*_h}$ and $\mathfrak{d}_{\neq^*_h} \leq \mathrm{non}(\mathcal{N})$.

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Let A be a set. Say $x \in X$ is $\sqsubset$-*unbounded over* A iff $\forall_{y \in A \cap Y}(x \not\sqsubset y)$.

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Definition (Judah and Shelah 1990)

A poset $\mathbb{P}$ is θ-$\sqsubset$*-good* iff for any $\mathbb{P}$-name $\dot{y}$ of a member of Y there is $A \subseteq Y$ non-empty of size $< \theta$ such that, whenever $x \in X$ is unbounded over A then $\Vdash x \not\sqsubset \dot{y}$.

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(3) *Let $\delta \geq \theta$ be an ordinal and $\mathbb{P} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \rangle_{\xi<\delta}$ a fsi of θ-$\sqsubset$-good non-trivial ccc posets.*

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Lemma

Any poset of size $< \theta$ is θ-$\sqsubset$-good. In particular, Cohen forcing is $\sqsubset$-good.

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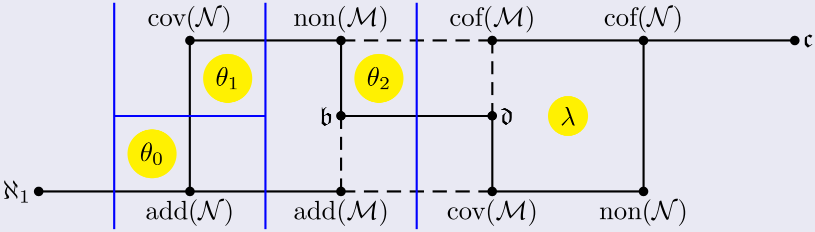

Consistency examples (2)

Theorem (From Brendle, Judah and Shelah's techniques 1990's)

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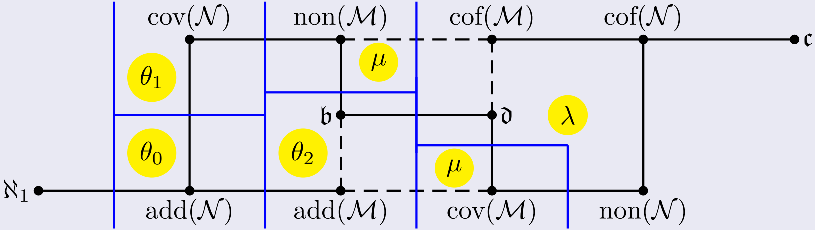

Consistency examples (3)

Theorem (M. 2013)

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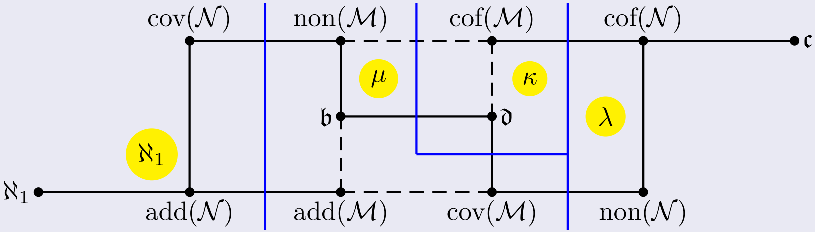

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- $\mathbb{P} \lessdot_M \mathbb{P}$ for any poset $\mathbb{P} \in M$.
- $\mathbb{S}^M \lessdot_M \mathbb{S}^N$ for any Suslin ccc poset $\mathbb{S}$ coded in M and $M \subseteq N$ transitive ZFC models.

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Lemma

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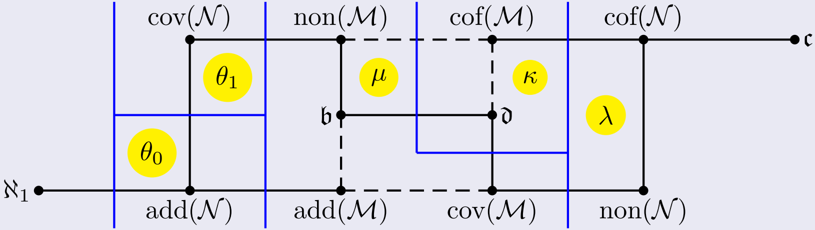
Lemma (Blass and Shelah 1984)

$\sqsubset$-unbounded reals are preserved at limit steps of parallel fsi's.

Consistency examples (4)

Theorem (M. 2013)

Let $\theta_0 \leq \theta_1 \leq \mu \leq \kappa$ *be uncountable regular cardinals,* $\lambda \geq \kappa$ *with* $\lambda^{<\theta_1} = \lambda$. *Then, it is consistent that*



Consistency examples (5)

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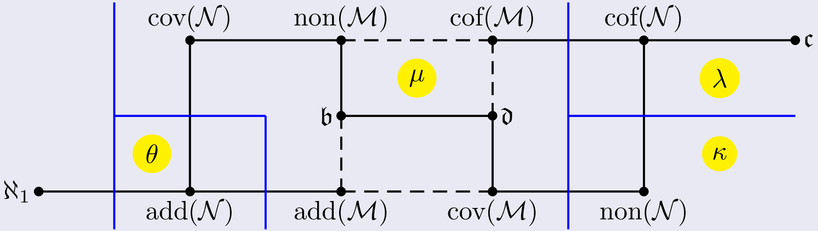
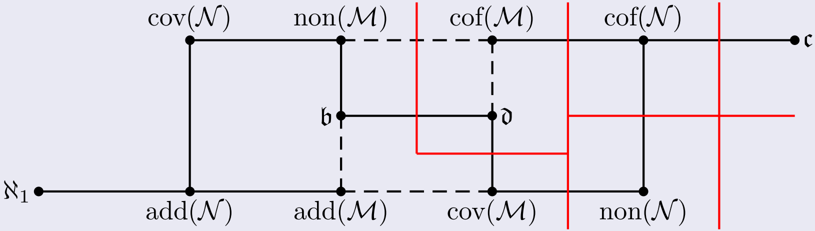


Question (1)

Question

Is it consistent that $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$?



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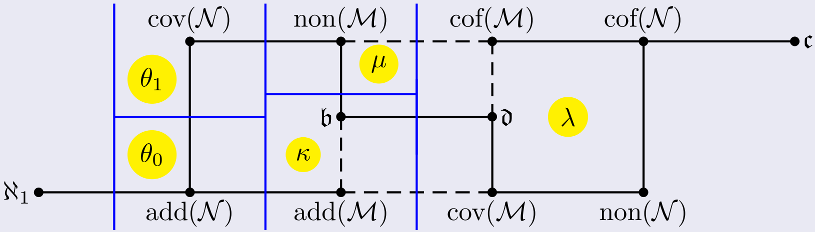
Theorem (Goldstern and M. and Shelah)

Let $\theta_0 \leq \theta_1 \leq \kappa = \kappa^{\aleph_0} \leq \mu = \mu^{\aleph_0}$ *be uncountable regular cardinals,* $\mu < \lambda = \lambda^{<\mu} \leq 2^\kappa$*. Then, there is a ccc poset forcing*



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However, *nice* subforcings of $\mathbb{E}$ may add dominating reals **(Pawlikowski 1992)**.

Ultrafilter limits

Fix D a non-principal ultrafilter on ω.

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If $\langle p_n \rangle_{n<\omega}$ is an uniform sequence in $\mathbb{E}$ then $\lim_n^D p_n$ forces that $\{n < \omega : p_n \in \dot{G}\}$ is infinite. Moreover, there is an $\mathbb{E}$-name $\dot{D}^*$ of an ultrafilter on ω containing D such that $\lim_n^D p_n$ forces $\{n < \omega : p_n \in \dot{G}\} \in \dot{D}^*$.

The iteration towards $\mathfrak{b} = \kappa = \aleph_1 \leq \mathrm{non}(\mathcal{M}) = \mu < \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$

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Theorem (Engelking and Karłowicz 1965)

If $\kappa^{\aleph_0} = \kappa$ and $\delta < (2^\kappa)^+$ then there is a set $\{h_\epsilon : \epsilon < \kappa\} \subseteq \kappa^\delta$ such that every countable partial function from δ to κ is contained in some h_ϵ.

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Theorem (Goldstern and M. and Shelah)

There is an iteration candidate that is nice*, that is, for any* $\epsilon < \omega_1$ *there is a sequence* $\mathbf{D}^\epsilon = \langle \dot{D}^\epsilon_\alpha \rangle_{\alpha \leq \lambda\mu}$ *such that*

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Theorem (Goldstern and M. and Shelah)

Any nice iteration candidate preserves $\aleph_1$-$\leq^*$-unbounded families.

Sketch of the proof: Assume $F \subseteq \omega^\omega$ is a $\aleph_1$-$\leq^*$-unbounded family in V and that there are a $\mathbb{P}$-name $\dot{g}$ of a real and a condition $p \in \mathbb{P}$ that forces $|\{f \in F : f \leq^* \dot{g}\}| \geq \aleph_1$. We can find $\langle (f_\xi, p_\xi, m_\xi)\rangle_{\xi<\omega_1}$ such that

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By the Δ-system lemma, there is $K \subseteq \omega_1$ uncountable such that $\langle p_\xi \rangle_{\xi \in K}$ forms a uniform Δ-system, wlog, there is an $m < \omega$ such that $m_\xi = m$ for all $\xi \in K$.

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Question (2)

Question

Is it consistent that $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$?