Pontryagin–van Kampen duality and pcf theory

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Topological groups

Consider only Abelian groups $(G,+,0)$.

Topological group: $x+y$, $-x$ are continuous.

Examples:
$\mathbb{R}$ with the standard topology;
$\mathbb{Z}$ with the discrete topology;
1-dimensional torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}=[0,1)$.

G homogeneous.

$\therefore$ Suffices to study any nbd base at 0.

$\chi(G)$: Minimal cardinality of a nbd base at 0.

Examples: $\chi(\mathbb{R})=\chi(\mathbb{T})=\aleph_0$; $\chi(\mathbb{Z})=1$.

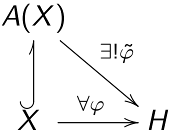
Birkhoff–Kakutani '36

G is metrizable $\iff$ $\chi(G)\leq\aleph_0$.

The free Abelian group

Markov '45: The free Abelian group $A(X)$:

$$A(X) = \{\ n_1 x_1 + \cdots + n_k x_k\ : k \in \mathbb{N}, n_i \in \mathbb{Z}, x_i \in X\}.$$



No simple direct definition for its topology. Hard to analyze.

$\chi(A(X)) = ?$

Goal: Find $\chi(A(X))$ ("hard") in terms of X ("easy").

The character of $A(X)$

$\chi(G) = \mathrm{cof}(\mathcal{N}_0, \supseteq)$.

$\mathfrak{d} := \mathrm{cof}(\mathbb{N}^{\mathbb{N}}, \leq)$. $f \leq g$ means $(\forall n)\ f(n) \leq g(n)$. $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$.

weight(X): Minimal cardinality of a base for the topology of X.

H–Ts–F–C '10

Assume $\exists$ cofinal $\{K_n\} \subseteq K(X)$, determining X's topology ("X is k_ω"). E.g., $X = \bigsqcup_n K_n$. Let $\kappa = \sup_n \mathrm{weight}(K_n)$.
Then $\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}([\kappa]^{\aleph_0}, \subseteq)$.

Example: $\chi(A(\mathbb{R})) = \mathfrak{d} \cdot \mathrm{cof}([\aleph_0]^{\aleph_0}, \subseteq) = \mathfrak{d} \cdot 1 = \mathfrak{d}$.

Nickolas–Tkachenko '05: Special cases (X compact, $mk_{\aleph_0}, \dots$).

Pontryagin–van Kampen duality

$\mathbb{T} = [0,1) \cong [-1/2, 1/2)$.

$\widehat{G}$: Continuous homomorphisms $\varphi : G \to \mathbb{T}$. AKA characters. (!)

Compact-open topology: $[K,\epsilon] = \{\varphi \in \widehat{G} : |\varphi[K]| \leq \epsilon\}$
$(K \in \mathrm{K}(X))$ nbd base at 0.

Examples: 1. $\widehat{\mathbb{Z}} = \mathbb{T}$: $1 \mapsto \alpha \implies n \mapsto n\alpha$.

2. $\widehat{\mathbb{R}} = \mathbb{R}$: $1 \mapsto \alpha \implies \frac{n}{m} \mapsto \frac{n}{m}\alpha \implies x \mapsto x\alpha$.

$$G \qquad \widehat{G} \qquad \widehat{\widehat{G}}$$

$$G \xhookrightarrow{\quad E \quad} \widehat{\widehat{G}}$$

$$E(g)(\varphi) := \varphi(g)$$

G reflexive: $G \overset{E}{\cong} \widehat{\widehat{G}}$.

Pontryagin '33: Compact/discrete $\implies$ reflexive.
van Kampen '34: Locally compact $\implies$ reflexive.

Subreflexive groups

"Most" groups are not locally compact (not even reflexive).

Even $\widehat{\text{metrizable}}$ need not be reflexive.

G is subreflexive if $G \overset{E}{\cong} E[G] \leq \widehat{\widehat{G}}$.

Proof steps

Assume $\exists$ cofinal $\{K_n\} \subseteq K(X)$ determining X's topology.

1. Chasco/Galindo–Hernández '99: $A(X)$ is subreflexive.
2. G subreflexive $\Longrightarrow \chi(G) = \mathrm{cof}(\mathrm{K}(\widehat{G}), \subseteq)$:

$A^\circ := \{\chi \in \widehat{G} : |\chi[A]| \le 1/4\}$.

$$\begin{array}{ccccccc} \mathcal{N}_0(G) & \stackrel{\circ}{\longrightarrow} & \mathrm{K}(\widehat{G}) & \stackrel{\circ}{\longrightarrow} & \mathcal{N}_0(\widehat{\widehat{G}}) & \longrightarrow & \mathcal{N}_0(G) \\ U & \longmapsto & K = U^\circ & \longmapsto & K^\circ & \longmapsto & K^\circ \cap G \end{array}$$

3. Pestov '95: $\widehat{A(X)} = C(X, \mathbb{T})$ with compact-open topology.

$\therefore \chi(A(X)) = \mathrm{cof}(\mathrm{K}(\widehat{A(X)}), \subseteq) = \mathrm{cof}(\mathrm{K}(C(X, \mathbb{T})), \subseteq)$.

In $C(X, \mathbb{T})$, $[K_n, 1/n]$ a nbd base at 0 $\Longrightarrow$ metrizable!

$C(X, \mathbb{T})$ is complete, not locally compact.

5. $C(X, \mathbb{T})$ has (stable) density $\kappa = \sup_n \mathrm{weight}(K_n)$ (direct).

Proof steps (final)

6. $\Gamma := C(X, \mathbb{T})$ is metrizable, complete, not locally compact, stable density κ.

Claim: $\operatorname{cof}(\mathrm{K}(\Gamma), \subseteq) = \operatorname{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}, \text{pointwise} \subseteq)$

($\leq$) Fix: Closed nbd base U_n at 0; dense $D \subseteq \Gamma$, $|D| = \kappa$.

$$(F_1, F_2, \dots \subseteq D) \longmapsto \bigcap_n F_n U_n.$$

Compact. A monotone cofinal map.

($\geq$) $\exists$ "grids" $A_n \subseteq U_n$ with $|A_n| \nearrow \text{density}(\Gamma) = \kappa$.

$$\mathrm{K}(\Gamma) \ni K \longmapsto f_K = (K \cap A_1, K \cap A_2, \dots) \in \prod_n \mathrm{Fin}(A_n) \approx \mathrm{Fin}(\kappa)^{\mathbb{N}}.$$

Cofinal: $(F_1, F_2, \dots) \in \prod_n \mathrm{Fin}(A_n) \Longrightarrow K = \bigcup_n F_n \cup \{0\} \longrightarrow 0$, thus compact.

7. $\operatorname{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}, \subseteq) = \mathfrak{d} \cdot \operatorname{cof}([\kappa]^{\aleph_0}, \subseteq)$. □

What is $\mathrm{cof}([\kappa]^{\aleph_0}, \subseteq)$?

$$\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}([\kappa]^{\aleph_0}, \subseteq).$$

κ	$\mathrm{cof}([\kappa]^{\aleph_0}, \subseteq)$
$\aleph_0$	1
$\aleph_1$	$\aleph_1$
$\aleph_2$	$\aleph_2$
$\vdots$	$\vdots$
$\aleph_n$	$\aleph_n$
$\vdots$	$\vdots$
$\aleph_\omega = \sup_n \aleph_n$	?

König: $\mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq) > \aleph_\omega$.

$\mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq)$ is the central object of study in Shelah's pcf theory.

An upper bound on $\mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq)$

$$\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$$

Shelah's Archive

This is the archive of Saharon Shelah's mathematical papers, now located at
`http://shelah.logic.at/`.

$$\aleph_\omega^{\aleph_0} = |[\aleph_\omega]^{\aleph_0}| = \mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq) \cdot |[\aleph_0]^{\aleph_0}| = \mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq) \cdot 2^{\aleph_0}.$$

Shelah '91: $\mathrm{cof}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}$.

In general, $\mathrm{cof}([\aleph_\alpha]^{|\alpha|}) < \aleph_{\alpha^{++++}}$, $\forall \alpha < \sup\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$.

Consequences of Shelah's Theorem

$\mathrm{cof}([\aleph_\alpha]^{\aleph_0}) < \aleph_{\alpha^{++++}}$, $\forall\alpha < \sup\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$.

For κ with $\aleph_0 < \mathrm{cof}(\kappa) < \kappa < \sup\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$:
$\mathrm{cof}([\aleph_\kappa]^{\aleph_0}, \subseteq) = \aleph_\kappa$.

Example. $\mathrm{cof}([\aleph_{\aleph_{\aleph_n}}]^{\aleph_0}, \subseteq) = \aleph_{\aleph_{\aleph_n}}$.

Shelah, Gitik: If $\nexists$ "large cardinals" (in the DJ core model):

$$\mathrm{cof}([\kappa]^{\aleph_0}, \subseteq) = \begin{cases} \kappa & \mathrm{cof}(\kappa) > \aleph_0 \\ \kappa^+ & \mathrm{cof}(\kappa) = \aleph_0 \end{cases}$$

Excursion to worlds with large cardinals

Bonanzinga–Matveev Problem '09:

1. $(\forall \kappa \leq \mathfrak{c})$ $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}, \text{pointwise} \subseteq) = \mathfrak{d} \cdot \kappa$?
2. $(\forall \kappa \leq \mathfrak{d})$ $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}, \text{pointwise} \subseteq) = \mathfrak{d}$?

YES for $\kappa = \aleph_n$ $(\forall n)$ and for $\kappa = \mathfrak{c}$.

$\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}, \subseteq) = \mathfrak{d} \cdot \mathrm{cof}([\kappa]^{\aleph_0}, \subseteq)$.

$\therefore$ **NO for (1)** if $\mathfrak{d} < \kappa$ and $\mathrm{cof}(\kappa) = \aleph_0$.

Ts '10 (w/large cardinals)

NO for (2): $(\forall 1 \leq \gamma < \aleph_1)$. Consistently,
$\aleph_\omega < \mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1} < \mathrm{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1} = \mathfrak{c}$.

Proof:

1. Gitik–Magidor '92 "force" (w/LC) $\mathrm{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1}$.
2. Add $\aleph_{\omega+1}$ dominating reals (FSI), get $\mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1}$.

Exercise: Why does Cohen forcing fail?

When does $\mathrm{cof}([\kappa]^{\aleph_0}) = \kappa$ (in ZFC)?

Known ones:

1. $\kappa^{\aleph_0}$,
2. $\aleph_n$,
3. $\aleph_\kappa$ for $\aleph_0 < \mathrm{cof}(\kappa) < \kappa < \min\{\lambda : \aleph_\lambda = \lambda\}$.

Find fixed points of $\mathrm{cof}([\kappa]^{\aleph_0})$ above $\sup\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$?

Unavoidable problems for the set theorist, 2/2

A_α $(\alpha \in I)$: abelian metric groups.

$\bigoplus_{\alpha\in I} A_\alpha$, with each $A_\beta \subseteq \bigoplus_{\alpha\in I} A_\alpha$ continous.

Lukacz–Ts ∞

$\kappa :=$ # non-discrete A_α's. Then $\chi(\bigoplus_{\alpha\in I} A_\alpha) = \mathrm{cof}(\mathbb{N}^\kappa, \leq)$.

$$f(\kappa) := \mathrm{cof}(\mathbb{N}^\kappa, \leq) = \mathrm{cof}(\mathbb{N}^\kappa, \leq^*).$$

f increasing, $\kappa^+ \leq f(\kappa) \leq 2^\kappa$.

When does $f(\kappa) = 2^\kappa$?

Examples: $\kappa^{\aleph_0}$.

More examples? Is $\mathrm{cof}(\mathbb{N}^{\aleph_1}, \leq) < 2^{\aleph_1}$ consistent (w/LC)?