Reflection and anti-reflection at the successor of a singular cardinal

joint work with Yair Hayut

Laura Fontanella

Hebrew University of Jerusalem, Einstein Institute of Mathematics
http://www.logique.jussieu.fr/∼fontanella
laura.fontanella@mail.huji.ac.il

25/10/2015

Reflection and compactness

Reflection: Given some structure $\mathcal{S}$ (e.g. a set of ordinals, a group, a topological space etc.), if the structure satisfies some property $\mathcal{P}$, then there is a substructure $\mathcal{S}'$ of smaller cardinality with the same property.

Compactness: Given some structure if every substructure of smaller cardinality satisfies a certain property, then the whole structure satisfies the same property.

When we have compactness for some property, then we have reflection for the negation of the property, and vice versa.

Example of compactness/reflection: König's lemma

Reflection beyond ZFC

Beyond ZFC:

- **Large cardinals** imply reflection properties
- **V=L** implies anti-reflection properties

Reflection of stationary sets

Reflection of stationary sets

Let κ be a regular cardinal,
Refl(κ): for every stationary subset S of κ, there exists $\alpha < \kappa$ of uncountable cofinality such that

$$S \cap \alpha \text{ is a stationary subset of } \alpha.$$

Applications: *Refl*(κ) is equiconsistent with "every κ-free abelian group is κ^+-free"

Reflection of stationary sets

Reflection of stationary sets

Let κ be a regular cardinal,
Refl(κ): for every stationary subset *S* of κ, there exists $\alpha < \kappa$ of uncountable cofinality such that

$$S \cap \alpha \text{ is a stationary subset of } \alpha.$$

Applications: *Refl*(κ) is equiconsistent with "every κ-free abelian group is κ^+-free"

Reflection of stationary sets

In ZFC:

- *Refl*(κ^+) fails if κ is a regular cardinal.

With large cardinals:

- If κ is weakly compact, then *Refl*(κ) holds.
- (Magidor '82) *Cons*($\exists(\kappa_n)_{n<\omega}$supercompact cardinals) $\rightarrow$ *Cons*(*Refl*($\aleph_{\omega+1}$))

If V=L:

- If $V = L$, then *Refl*(κ) fails at every regular uncountable cardinal κ which is not weakly compact.

Delta-reflection

Definition (Magidor, Shelah '94)

For $\kappa < \lambda$, $\Delta_{\kappa,\lambda}$ is the following statement:
given a stationary set $S \subseteq E^{\lambda}_{<\kappa}$ and an algebra $\mathcal{A}$ on λ with $< \kappa$ operations, there exists a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ such that the order type of $\mathcal{A}'$ is a regular cardinal $< \kappa$ and

$$S \cap \mathcal{A}' \text{ is stationary in } sup(\mathcal{A}')$$

We say that λ has the Delta-reflection if $\Delta_{\kappa,\lambda}$ holds for every $\kappa < \lambda$.

Applications of Delta-reflection

Applications (Magidor, Shelah)

Suppose that κ has the Delta-reflection, then

- *Refl*(κ) holds
- every $<\kappa$-free abelian group of size κ is free.
- Given a graph *G* of size κ. If every subgraph of *G* of size $<\kappa$ has coloring number $\leq \gamma < \kappa$, then *G* has coloring number $\leq \gamma$.
- Given *A* a family of κ sets all of size $<\kappa$, if every subfamily of size $<\kappa$ has a transversal, then *A* has a transversal.
- Given *X* a topological space locally of cardinality $<\kappa$, if *X* is $<\kappa$-collectionwise Hausdorff, then *X* is collectionwise Hausdorff

Consistency of the Delta-reflection at $\aleph_{\omega^2+1}$

If κ is weakly compact, then κ has the Delta-reflection.

Theorem (Magidor, Shelah '94)

Cons$(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow$ *Cons*$(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

Moreover, $\aleph_{\omega^2+1}$ is the smallest regular cardinal that can have the Delta-reflection.

Square principle

(Jensen) Square $\square_\kappa$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa^+)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $o.t.(C_\alpha) \leq \kappa$
3. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;

Square principle

Square is an anti-reflection principle

- (Solovay) $\square_\kappa$ implies $\neg$*Refl*(κ^+) (in particular it implies the failure of the Delta-reflection at κ^+).
- (Solovay) if κ is strongly compact, then $\square_\mu$ fails for every $\mu \geq \kappa$.

Todorčević square

(Todorčević) $\square(\kappa)$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;
3. there are no threads for the sequence, i.e. there is no club $C \subset \kappa$ such that $\beta \in \mathrm{Lim}(C)$ implies $C_\beta = C \cap \beta$;

Fact: $\square_\kappa$ implies $\square(\kappa^+)$

Todorčević square

$\square(\kappa)$ is an anti-reflection principle

- (Veličković) $\square(\kappa)$ implies the existence of two stationary subsets of E^κ_ω that do not reflect simultaneously (i.e. there is no α such that both reflect to α).
- (Rinot) $\square(\kappa)$ implies that every stationary subset of κ can be split into κ many disjoint stationary parts that do not reflect simultaneously
- (Solovay, Veličković) if κ is strongly compact, then $\square(\mu)$ fails for every $\mu \geq \kappa$.

Theorem (F. , Hayut)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \Box(\aleph_{\omega^2+1}))$

- In particular the Delta-reflection does not imply the simultaneous reflection.
- $\Box(\kappa^+)$ implies the failure of the tree property at κ^+, so in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (see also F. , Magidor).
- The Delta-reflection at κ^+ is incompatible even with the weak square $\Box^*_\kappa$, so in a way this result is optimal.

Theorem (F. , Hayut)

$\mathit{Cons}(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to \mathit{Cons}(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1}))$

- In particular the Delta-reflection does not imply the simultaneous reflection.
- $\square(\kappa^+)$ implies the failure of the tree property at κ^+, so in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (see also F. , Magidor).
- The Delta-reflection at κ^+ is incompatible even with the weak square $\square^*_\kappa$, so in a way this result is optimal.

What is the idea of the proof?

Forcing a square sequence

We can force a $\square(\lambda^+)$-sequence with bounded approximations: a condition is a sequence of the form $\langle C_\alpha;\ \alpha \in \gamma + 1\rangle$ where $\gamma < \lambda^+$ and

- for every α, $C_\alpha \subseteq \alpha$ is a club (if α is a successor ordinal, then $C_\alpha = \{\alpha - 1\}$);
- for every α, β, if $\beta \in acc(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.

Given two conditions *s*, *t*, we say that *s* is stronger than *t* if $t \sqsubseteq s$.

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that
$M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$.
By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X \text{ subalgebra of } j(A) \text{ of order type } < j(\kappa) \text{ such that } j(S) \cap X \text{ is stationary in } \sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

Delta reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

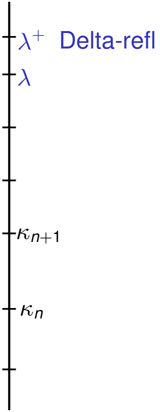
$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

Use a forcing $\mathbb{P}$ similar to diagonal Prikry forcing.

Delta reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

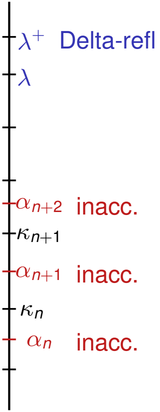


Use a forcing $\mathbb{P}$ similar to diagonal Prikry forcing.

Delta reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

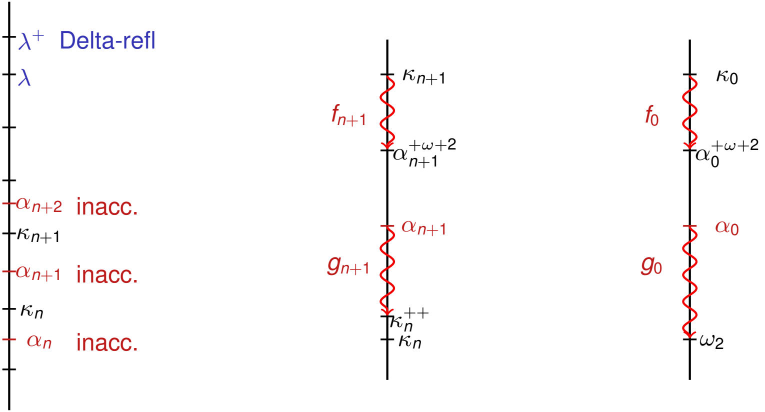


Use a forcing $\mathbb{P}$ similar to diagonal Prikry forcing.

Delta reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$



Use a forcing $\mathbb{P}$ similar to diagonal Prikry forcing.

Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\square(\aleph_{\omega^2+1})$.

Problem: if $\square(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\square(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1})$$

Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\Box(\aleph_{\omega^2+1})$.

Problem: if $\Box(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\Box(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \Box(\aleph_{\omega^2+1})$$

Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\square(\aleph_{\omega^2+1})$.

Problem: if $\square(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\square(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1})$$

Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\square(\aleph_{\omega^2+1})$.

Problem: if $\square(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\square(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1})$$

Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\square(\aleph_{\omega^2+1})$.

Problem: if $\square(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\square(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1})$$

Delta-reflection and square

New problem: $\mathbb{T}$ destroys stationary sets, so it may destroy stationary sets that do not reflect in $V^{\mathbb{S}*\mathbb{P}}$, thus the preservation lemma cannot be proven.

New solution: we do some preparation, namely we define an iteration $\mathbb{R}$ that preventively destroy all the stationary sets in $V^{\mathbb{S}\times\mathbb{P}}$ that would be destroyed by $\mathbb{T}$.

Delta-reflection and square

New problem: $\mathbb{T}$ destroys stationary sets, so it may destroy stationary sets that do not reflect in $V^{\mathbb{S}*\mathbb{P}}$, thus the preservation lemma cannot be proven.

New solution: we do some preparation, namely we define an iteration $\mathbb{R}$ that preventively destroy all the stationary sets in $V^{\mathbb{S}\times\mathbb{P}}$ that would be destroyed by $\mathbb{T}$.

Factorising $\mathbb{P}$

$$\mathbb{C}_n := \prod_{m \geq n} Coll(\kappa_m^{++}, < \kappa_{m+1})$$

For $c, c' \in \mathbb{C}_0$, let

- $c \sim c' \iff \exists n \forall m \geq n\ c(m) = c'(m)$
- $c \leq^* c' \iff \exists n \forall m \geq n\ c(m) \leq c'(m)$

$$\mathbb{C}_{fin} := (\mathbb{C}_0 / \sim, \leq^*)$$

$\mathbb{P}$ can be factorised like this

$$\mathbb{P} \equiv \mathbb{C}_{fin} * \mathbb{P}^*$$

The preparation

In $V^{\mathbb{C}_{fin} \times \mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}}$, then $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

For every $n < \omega$, $(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}$ contains a κ_n^+-directed closed dense subsets, thus

$$V^{(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}} \models \kappa_n \text{ is supercompact}$$

In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)}$$

The preparation

In $V^{\mathbb{C}_{fin}\times\mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$, then $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

For every $n < \omega$, $(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}$ contains a κ_n^+-directed closed dense subsets, thus

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}*\mathbb{T}} \models \kappa_n \text{ is supercompact}$$

In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)}$$

The preparation

In $V^{\mathbb{C}_{fin} \times \mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}}$, then $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

For every $n < \omega$, $(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}$ contains a κ_n^+-directed closed dense subsets, thus

$$V^{(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}} \models \kappa_n \text{ is supercompact}$$

In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)}$$

The preparation

In $V^{\mathbb{C}_{fin} \times \mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}}$, then $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

For every $n < \omega$, $(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}$ contains a κ_n^+-directed closed dense subsets, thus

$$V^{(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}} \models \kappa_n \text{ is supercompact}$$

In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)}$$

The idea of the proof

Part 1:
$V^{\mathbb{S}} \models \square(\lambda^+)$

A forcing $\mathbb{B}$ does not add a thread to a $\square(\lambda^+)$-sequence if $\mathbb{B} \times \mathbb{B}$ does not change the cofinality of λ^+.

$\mathbb{C}_{fin}, \mathbb{R}$ and $\mathbb{P}^*$ satisfy this requirement, thus

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)} \models \square(\lambda^+)$$

The idea of the proof

Part 1:
$V^{\mathbb{S}} \models \Box(\lambda^+)$

A forcing $\mathbb{B}$ does not add a thread to a $\Box(\lambda^+)$-sequence if $\mathbb{B} \times \mathbb{B}$ does not change the cofinality of λ^+.

$\mathbb{C}_{fin}, \mathbb{R}$ and $\mathbb{P}^*$ satisfy this requirement, thus

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)} \models \Box(\lambda^+)$$

The idea of the proof

Part 1:
$V^{\mathbb{S}} \models \Box(\lambda^+)$

A forcing $\mathbb{B}$ does not add a thread to a $\Box(\lambda^+)$-sequence if $\mathbb{B} \times \mathbb{B}$ does not change the cofinality of λ^+.

$\mathbb{C}_{fin}, \mathbb{R}$ and $\mathbb{P}^*$ satisfy this requirement, thus

$$V^{(\mathbb{C}_{fin} \times \mathbb{S}) * (\mathbb{R} \times \mathbb{P}^*)} \models \Box(\lambda^+)$$

The idea of the proof

Part 2:

Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_{n}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_{n}\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_{n}\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ “fake versions” S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

The idea of the proof

Part 2:
Suppose that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)} \models \dot{S} \subseteq E^{\lambda^+}_{<\kappa_n} \text{ stationary, } \dot{A} \text{ algebra on } \lambda^+ \text{ with } < \kappa_n\text{-many operations}$$

Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary}$$

Forcing with $\mathbb{C}_n/\mathbb{C}_{fin}$, we still have

$$V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T) \models S^* \text{ is stationary.}$$

Moreover κ_n is supercompact in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}(G_T)$, so here S^* reflects on a subalgebra B^* of A^* of order type $< \kappa_n$. By the distributivity of $\mathbb{T}$, the subalgebra B^* already existed in $V^{(\mathbb{C}_n\times\mathbb{S})*\mathbb{R}}$.

This gives us a subalgebra B of the real algebra A where the real stationary set S reflects, so we have the conclusion.

Thank you