Chromatic number of infinite graphs

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$[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$
$[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$.

A *graph* is (V, X) or simply X where V is the set of *vertices*, $X \subseteq [V]^2$ is the set of *edges*.

$N(x) = \{y : \{x, y\} \in X\}$,
$N^-(x) = \{y < x : \{x, y\} \in X\}$, $d_-(x) = |N^-(x)|$

A *good coloring* is a function $f : V \to \mu$ such that if $\{x, y\} \in X$ then $f(x) \neq f(y)$.

The *chromatic number* of X, $\mathrm{Chr}(X)$ is the minimal μ such that a good coloring $f : V \to \mu$ exists.

Theorem. (Galvin–K) The statement that each graph has a chromatic number is equivalent to the Axiom of Choice.

Theorem. (de Bruijn–Erdős) If $n < \omega$, X is a graph, for every finite $W \subseteq V$, $\mathrm{Chr}(X|W) \leq n$, then $\mathrm{Chr}(X) \leq n$.

Proof. (1) Ultrafilter.
Let U be an ultrafilter on $[V]^{<\omega}$ such that $\{s \in [V]^{<\omega} : x \in s\} \in U$ $(x \in V)$.
For each $s \in [V]^{<\omega}$ let $f_s : s \to n$ be a good coloring of $X|s$. For $x \in V$ let $g(x) < n$ be the unique color such that $\{s : f_s(x) = g(x)\} \in U$. $g : V \to n$ is a good coloring: if $\{x, y\} \in X$, then $\{s : x, y \in s\}$, $\{s : f_s(x) = g(x)\}$, $\{s : f_s(y) = g(y)\}$ are all in U and for any s like that, $f_s(x) \neq f_s(y)$.

In other words: X can be embedded into the ultraproduct of its finite subgraphs

$$X \subseteq \prod_s (X|s)/U.$$

The good colorings of the factors give a good coloring of the ultraproduct.

(2) Gödel's compactness theorem.

Theorem. (Rado's selection principle) *Assume that* $A_v \neq \emptyset$ *is finite for* $v \in V$*, and for every* $\emptyset \neq s \in [V]^{<\omega}$ *there is a family*

$$\emptyset \neq \mathcal{F}_s \subseteq \prod\{A_v : v \in s\}$$

of functions such that if $s \subseteq t \in [V]^{<\omega}$*,* $f \in \mathcal{F}_t$*, then* $f|s \in \mathcal{F}_s$*. Then there is* $g \in \prod\{A_v : v \in V\}$ *such that* $g|s \in \mathcal{F}_s$ *for* $\emptyset \neq s \in [V]^{<\omega}$*.*

The *coloring number* of a graph (V, X), $\mathrm{Col}(V, X)$, is the least cardinal μ such that there is a well ordering $<$ of V such that each $v \in V$ is joined into less than μ smaller vertices. (A μ-good well ordering.)

We have $\mathrm{Chr}(X) \leq \mathrm{Col}(X)$.

Examples. *If κ, λ are infinite cardinals,*

(a) $\mathrm{Col}(K_\kappa) = \kappa$,

(b) $\mathrm{Col}(K_{\kappa,\kappa}) = \kappa$,

(c) *if $\kappa < \lambda$, then* $\mathrm{Col}(K_{\kappa,\lambda}) = \kappa^+$.

Theorem. *Let $\mu \geq \omega$ be a cardinal. If X is a graph on the vertex set V, then the following are equivalent.*

(a) $\mathrm{Col}(X) \leq \mu$,

(b) *there is an ordering $<$ of V such that $d_{-}(x) < \mu$ $(x \in V)$,*

(c) *there is a set mapping $f : V \to [V]^{<\mu}$ such that if $\{x, y\} \in X$, then either $y \in f(x)$ or $x \in f(y)$,*

Proof. (a)$\rightarrow$(b) Clear from definition.

(b)$\rightarrow$(c) Direct all edges going down, i.e., $f(x) = \{y < x : \{y, x\} \in X\}$.

(c)$\to$(a) $V=\{v_\alpha:\alpha<\lambda\}$.

$$V_\alpha =$$

$$\{v_\beta:\beta<\alpha\}\cup\bigcup\{f(v_\beta):\beta<\alpha\}\cup\bigcup\{f^2(v_\beta):\beta<\alpha\}\cup\cdot$$

Clearly, $\{V_\alpha:\alpha<\lambda\}$ is increasing, continuous.
Set $W_\alpha=V_{\alpha+1}-V_\alpha$.
Then $W_\alpha\subseteq\{v_\alpha\}\cup f(v_\alpha)\cup f^2(v_\alpha)\cup\cdots$ so $|W_\alpha|\leq\mu$. Also, if $x\in W_\alpha$, $y\in V_\alpha$, $\{x,y\}\in X$, then $y\in f(x)$.
Well order W_α into $\leq\mu$ and make $W_\beta<W_\alpha$ $(\beta<\alpha)$.

Theorem. *If X is a graph on V with $|V| = \lambda$ and $\mathrm{Col}(X) \leq \mu$, then there is a well ordering of V witnessing $\mathrm{Col}(X) \leq \mu$ into ordinal λ.*

Proof. If $\lambda = \mu$, any well ordering into ordinal will do. If $\lambda > \mu$, the statement is given by the proof of (c)$\rightarrow$(a) of the previous Theorem.

When the coloring number is finite.
De Bruijn–Erdős is not true!

There is a countable graph X with $\mathrm{Col}(X) = 4$, $\mathrm{Col}(Y) \leq 3$ for each finite $Y \subseteq X$.

$\cdots T_{n+1} < T_n < \cdots < T_3 < T_2 < T_1 < T_0$
each T_n is a triangle.
Add 3 edges between T_n and T_{n+1} such that the downdegree of each nodes is 2, each of the 3 edges go into one node of T_{n+1}.
Each finite Y has $\mathrm{Col}(Y) \leq 3$.
Assume that X is well ordered. Let $\max(T_n)$ be minimal. Then $\max(T_n) < \max(T_{n+1}) = x$. From x, one edge goes to T_n, 2 edges to T_n, i.e., $d_{-}(x) \geq 3$.

Modifying this, Erdős and Hajnal constructed for $1 \leq n < \omega$, $2 \leq k < \omega$ a graph X such that $|X| = \aleph_n$, $\mathrm{Col}(X) = 2k$, and if $Y \subseteq X$ has $|Y| < \aleph_n$, then $\mathrm{Col}(Y) \leq k + 1$. This is sharp by the following Theorem.

Theorem. Let X be a graph on some vertex set V, $1 \leq k < \omega$. Then each of the following statements implies the next.

(1) If $Y \subseteq X$ is finite, then $\mathrm{Col}(Y) \leq k+1$.

(2) V has a (not necessarily well) ordering in which there at most k edges going down from any vertex.

(3) X is the union of k forests.

(4) $\mathrm{Col}(X) \leq 2k$.

Theorem. If $n < \omega$, X has $\mathrm{Col}(X) = n + 1$, then there is a subgraph $Y \subseteq X$, $\mathrm{Col}(Y) = n$.

Theorem. Assume that $\omega \leq \mu < \kappa = \mathrm{cf}(\kappa)$, X a graph on κ, all $Y \subseteq X$ with $|Y| < \kappa$ has $\mathrm{Col}(Y) \leq \mu$. Then $\mathrm{Col}(X) > \mu$ if and only if

$$S(X) = \{\alpha < \kappa : \exists \beta \geq \alpha, |N(\beta) \cap \alpha| \geq \mu\}$$

is stationary.

$\Longrightarrow$: If $S(X)$ is nonstationary, let C be a closed, unbounded set with $S(X) \cap C = \emptyset$. We can assume $0 \in C$. C splits κ into complementary intervals $[\gamma, \gamma')$
Each $X|[\gamma, \gamma')$ has coloring number $\leq \mu$, re-order it to a well order witnessing it, place them one after the other. This gives a well ordering witnessing $\mathrm{Col}(X) \leq \mu$.

$\Longleftarrow$: Assume that $f : \kappa \to [\kappa]^{<\mu}$ is such that if $\{\alpha, \beta\} \in X$, then $\alpha \in f(\beta)$ or $\beta \in f(\alpha)$.
Let $C \subseteq \kappa$ be a closed, unbounded set, s.t. it is closed under f, i.e., if $\beta < \alpha \in C$, then $f(\beta) \subseteq \alpha$.
Pick $\delta \in C \cap S(X)$. Now $f(\beta(\delta))$ contains $N(\beta(\delta)) \cap \delta$ which is of size μ, contradiction!

Theorem. (Singular Cardinal Compactness, Shelah) Assume that λ is singular, $\mu < \lambda$, (V, X) is a graph with $|V| = \lambda$ such that for each $A \in [V]^{<\lambda}$, $\mathrm{Col}(X|A) \leq \mu$. Then $\mathrm{Col}(X) \leq \mu$.

Definition. $A \subseteq V$, $|A| < \lambda$ is *extendable*, if $|N(x) \cap A| < \mu$ for every $x \in V - A$ and so every μ-good well ordering of A can be endextended to any B with $|B| < \lambda$.

Lemma. (Erdős-Hajnal) *If $A \subseteq V$, $|A| \leq \kappa < \lambda$, then there is an extendable $A \subseteq A'$, $|A'| = \kappa$.*

Proof. Define the increasing continuous $\{A_\alpha : \alpha < \kappa^+\}$ such that $A_0 = A$, $|A_\alpha| = \kappa$, A_α can not be extended to $A_{\alpha+1}$. If $B = \bigcup\{A_\alpha : \alpha < \kappa^+\}$, $<$ is a μ-good well ordering of B into ordinal κ^+, then for some α, A_α is an initial seqment, contradiction.

Let $\{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ be a continuous, increasing sequence of cardinals, converging to λ, $\lambda_0 > \mathrm{cf}(\lambda), \mu$. Decompose V as $V = \bigcup\{V_\alpha : \alpha < \mathrm{cf}(\lambda)\}$, $|V_\alpha| = \lambda_\alpha$. The simplest way would be to construct a continuous, increasing sequence $\{A_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ of extandable sets such that $|A_\alpha| = \lambda_\alpha$, $A_\alpha \supseteq V_\alpha$.

But nothing guarantees that the increasing union of extandable sets is extendable, unless $\mathrm{cf}(\lambda) \leq \mathrm{cf}(\mu)$.

By mathematical induction on $n < \omega$ we define for all α simultaneously the extendable $A_{\alpha,n}$, $|A_{\alpha,n}| = \lambda_\alpha$ and a μ good well ordering $\prec_{\alpha,n}$ of $A_{\alpha,n}$ such that $\prec_{\alpha,n+1}$ end-extends $\prec_{\alpha,n}$. We will have $A_\alpha = \bigcup\{A_{\alpha,n} : n < \omega\}$ with the μ-good well ordering $\prec_\alpha = \bigcup\{\prec_{\alpha,n} : n < \omega\}$.

(1) $V_\alpha \subseteq A_{\alpha,0}$ $(\alpha < \mathrm{cf}(\lambda))$;

(2) $A_{\alpha,n+1} \supseteq A_{\beta,n}$ $(\beta \leq \alpha)$;

(3) $A_{\alpha,n}$ is split as $\bigcup\{B_\beta^{\alpha,n} : \beta < \alpha\}$ increasing, continuous, $|B_\beta^{\alpha,n}| = \lambda_\beta$ (α is limit);

(4) $B_\alpha^{\beta,n} \subseteq A_{\alpha,n+1}$ $(\beta > \alpha)$;

(5) if $x, y \in A_{\alpha,n}$, $\{x, y\} \in X$, $x \prec_{\alpha,n} y$, $y \in A_{\beta,n}$ $(\beta < \alpha)$, then $x \in A_{\beta,n+1}$.

Can be done. (3) immediate, the rest requires that some λ_α things be put into $A_{\alpha,n}$.

(1) $V_\alpha \subseteq A_{\alpha,0}$ $(\alpha < \mathrm{cf}(\lambda))$;

Guarantees $\bigcup\{A_\alpha : \alpha < \mathrm{cf}(\lambda)\} = V$.

(2) $A_{\alpha,n+1} \supseteq A_{\beta,n}$ $(\beta \leq \alpha)$;

Guarantees $A_{\alpha} \supseteq A_{\beta}$ $(\beta < \alpha)$.

(4) $B_\alpha^{\beta,n} \subseteq A_{\alpha,n+1}$ $(\beta > \alpha)$;

Guarantees that $\{A_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ is continuous: if α is limit, $x \in A_\alpha$, $x \in A_{\alpha,n}$ for some n, then $x \in B_\beta^{\alpha,n}$ for some $\beta < \alpha$, by (4) $x \in A_{\beta,n+1} \subseteq A_\beta$.

(5) if $x, y \in A_{\beta,n}$, $\{x, y\} \in X$, $x \prec_{\beta,n} y$, $y \in A_{\alpha,n}$ $(\alpha < \beta)$, then $x \in A_{\alpha,n+1}$.

Guarantees that A_α is extendable: assume not and $y \notin A_\alpha$ yet it is joined to $\geq \mu$ elements of A_α. There is some $\beta > \alpha$, $y \in A_\beta$. As $\prec_\beta$ is a μ-good well order, these μ elements of A_α cannot all precede y by $\prec_\beta$. There are, therefore, $n < \omega$, $x \in A_{\alpha,n}$, $y \in A_{\beta,n}$, $y \prec_{\beta,n} x$, $\{x, y\} \in X$. But then by (5), $y \in A_{\alpha,n+1} \subseteq A_\alpha$, contradiction.

Given μ, we call a graph X *of the first kind*, and of type (λ, μ), if it is a bipartite graph on the bipartition classes A and B, $|A| = \lambda$, $|B| = \lambda^+$ for some cardinal $\lambda \geq \mu$ and $d(x) = \mu$ for each $x \in B$.

A graph is a *graph of the second kind* of type (κ, μ), if it is isomorphic to a (κ, X) where κ is a regular cardinal $\kappa > \omega$ and there is a stationary set $S \subseteq \kappa$ such that $N^-(\alpha)$ is a cofinal subset of α of type μ.

Theorem. (a) If X is a graph of first or second kind, of type (λ, μ), then $\mathrm{Col}(X) > \mu$.
(b) If X is a graph with $\mathrm{Col}(X) > \mu$, then X contains a subgraph of the first or second kind of type (λ, μ) for some λ.

Proof. (a) Assume that X is a graph of first kind on $A \cup B$, $|A| = \lambda$, $|B| = \lambda^+$, each $|N(x)| = \mu$ $(x \in B)$. Assume $f : A \cup B \to [A \cup B]^{<\mu}$ is such that if $\{x, y\} \in X$, then either $x \in f(y)$ or $y \in f(x)$. If $y \in B - \bigcup\{f(x) : x \in A\}$, then $|f(y)| \geq |N(y)| = \mu$, contradiction.
If X is of the second kind, use Fodor.

(b) If X is a graph with $\mathrm{Col}(X) > \mu$, $|X| = \kappa$, minimal cardinality, so $S(X)$ is stationary in κ. For each $\alpha \in S(X)$, set $f(\alpha) =$ sup of the first *mu* elements of $N^{-}(\alpha)$. If $f(\alpha) < \alpha$ stat often, find a subgraph of first kind. If $f(\alpha) = \alpha$ stat often, find a subgraph of the second kind.

Corollary.

(a) (Erdős–Hajnal) If $\mathrm{Col}(X) > \omega$, then X contains C_4, even every complete bipartite graph $K_{n,m}$ $(n, m < \omega)$.

(b) (Halin) If $\mathrm{Col}(X) > \kappa$, then X contains a topological K_κ.

(c) (Thomassen) If $\mathrm{Col}(X) > \kappa$, then X contains a κ-edge-connected Y such that $\mathrm{Col}(Y) > \kappa$.

Proof. (a) Let X be a graph of the first kind, on $V = A \cup B$, $|A| = \lambda$, $|B| = \lambda^+$, $|N(x)| = \omega$ $(x \in B)$.
For $\{a, b\} \in [A]^2$ set $N(\{a, b\}) = N(a) \cap N(b)$.
Clearly, $\bigcup\{N(s) : s \in [A]^2, |N(s)| \leq 1\}$ has size $\leq \lambda$.
There are, therefore, $a, b \in A$, with $c, d \in N(\{a, b\})$, but $\{a, c, b, d\}$ is a C_4.
If X is a graph of the second kind, use Fodor's lemma.

Incompactness

Theorem. If $\kappa > \mu \geq \omega$ are regular and there is a nonreflecting stationary set $S \subseteq S^\kappa_\mu$, then there is a graph X on κ with $\mathrm{Col}(X) = \mu^+$ such that $\mathrm{Col}(X|\alpha) \leq \mu$ for $\alpha < \kappa$.

Proof. Join each $\alpha \in S$ into a μ-sequence converging to it.

Compactness

Lemma. *If X is a graph with $\mathrm{Col}(X) > \omega$, then $\mathrm{Col}(X) > \omega$ will stay after forcing with an ω_1-closed (or just proper) forcing.*

Proof. Let $(P, \leq)$ be the forcing.
(a) Let X be a graph of the first kind on $A \cup B$, $|A| = \lambda$, $|B| = \lambda^+$, $|N(x)| = \omega$ $(x \in B)$.

Assume that $p \in P$ forces that $f : A \cup B \to [A \cup B]^{<\omega}$ is such that if $\{x, y\} \in X$, then either $x \in f(y)$ or $y \in f(x)$. Let M be an elementary submodel (of some $\mathcal{H}(\theta)$) such that $p, P, X, \underline{f} \in M$, $|M| = \lambda$. Pick $y \in B - M$, $N(y) = \{x_0, x_1, \ldots\}$, then select $p \geq p_0 \geq p_1 \geq \cdots$ such that p_i forces $f(x_i) = s_i$ for some finite $s_i \subseteq B$. If $p' \leq p_i$ ($i < \omega$) then p' forces that $\{x_0, x_1, \ldots\} \subseteq f(y)$.
(b) X second kind, proof similar, have M such that $\delta = \sup(M \cap \lambda) \in S$.

Theorem. (GCH) *If* κ *is supercompact, then after forcing with* $\mathrm{Coll}(\omega_1, < \kappa)$ *each* X *with* $\mathrm{Col}(X) > \omega$ *contains a* Y *with* $|Y| = \mathrm{Col}(Y) = \omega_1$*.*

Proof. Set $P = \mathrm{Coll}(\omega_1, < \kappa)$. Let G be $V - P$-generic.
Let $\lambda = |X|$.
Pick an elementary embedding $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $[M]^\lambda \subseteq M$.
$j(P) = P \oplus Q$ where $Q = \mathrm{Coll}(\omega_1, [\kappa, j(\kappa)0)$.
Pick a $V[G] - Q$-generic H.

As $P \subseteq V_\kappa$ we can elevate j to $j : V[G] \to M[G, H]$ by $j(\tau^G) = j(\tau)^{G,H}$.
By the Lemma, $\mathrm{Col}(X) > \omega$ holds in $V[G, H]$.
As $V[G, H] \models [M[G, H]]^\kappa \subseteq M[G, H]$,
we have $M[G, H] \models \mathrm{Col}(X) > \omega$.

Set $Y = j[X] \in M[G, H]$ (as $j|\lambda \in M$).
In $M[G, H]$, there is a subgraph Y of $j(X)$,
$|Y| < j(\kappa)$, $\mathrm{Col}(Y) > \omega$.
Apply j backwards: in $V[G]$, there is $Y \subseteq X$,
$\mathrm{Col}(Y) > \omega$, $|Y| < \kappa = \aleph_2$.

Theorem. (Shelah) If the existence of a proper class of supercompact cardinals is consistent, then it is consistent that if $\mathrm{Col}(X) > \mu$, then X contains a graph Y with $|Y| = \mathrm{Col}(Y) = \mu^+$.

Proof. Let $\{\kappa_\alpha : \alpha \in \mathrm{ORD}\}$ be an increasing, continuous equence of cardinals, $\kappa_0 = \omega$, if κ_α is regular, then $\kappa_{\alpha+1}$ is supercompact, if κ_α is singular, then $\kappa_{\alpha+1} = \kappa_\alpha^+$. Iterate such that if κ_α is regular then $Q_\alpha = \mathrm{Coll}(\kappa_\alpha, < \kappa_{\alpha+1})$, if κ_α is singular, then Q_α shoots a club through the approachable ordinals in $\kappa_{\alpha+1}$.

The shift graph $\mathrm{Sh}_2(\lambda)$: the vertex set $V = [\lambda]^2$, if $x < y < z$, $\{x, y\}$ is joined to $\{y, z\}$.

Theorem. $\mathrm{Chr}(\mathrm{Sh}_2(\lambda)) \leq \kappa$ iff $\lambda \leq 2^\kappa$.

Proof. If $\lambda > 2^\kappa$, $f : [\lambda]^2 \to \kappa$, then there are $x < y < z$, $f(x, y) = f(y, z)$ by the Erdős-Rado theorem.
Enumerate ${}^\kappa 2 = \{r_\xi : \xi < 2^\kappa\}$. Define $F(\xi, \eta) = \langle \alpha, i \rangle$ where α is the first difference of r_ξ and r_η, and $i = 0$ if $r_\xi(\alpha) = 0 < r_\eta(\alpha) = 1$, $i = 1$, OW. If $\xi < \eta < \zeta$, $F(\xi, \eta) = F(\eta, \zeta)$, then $r_\xi(\alpha) < r_\eta(\alpha) < r_\zeta(\alpha)$, imp.

General shift graph $\mathrm{Sh}_n(\lambda)$: vertex set $V = [\lambda]^n$, if $x_0 < x_1 < \cdots < x_n$, then $x_0, \ldots, x_{n-1}$ is joined with $\{x_1, \ldots, x_n\}$.

Theorem. $\mathrm{Chr}(\mathrm{Sh}_n(\lambda)) \leq \kappa$ iff $\lambda \leq \exp_{n-1}(\kappa)$.

Theorem. (Erdős–Hajnal) If $\mathrm{Chr}(X) > \aleph_0$, then each finite bipartite graph occurs in X and each finite nonbipartite graph can be omitted in graphs of arbitrarily large chromatic number.

What are the obligatory families of finite graphs?

Theorem. (Erdős–Hajnal–Shelah, Thomassen) If $\mathrm{Chr}(X) > \aleph_0$, then X contains for some n all circuits $C_{2n+1}, C_{2n+3}, \ldots$.

If $\mathrm{Chr}(X) > \aleph_0$ let $f_X : \omega \to \omega$ be defined as $f_X(n)$ is the number of vertices of the least n-chromatic subgraph of X. Clearly, $f_X(n) \geq n$ and so $f_X \to \infty$.

Question. (Erdős–Hajnal) Can f_X tend to ∞ arbitrarily fast?

Theorem. (Shelah) Consistently for every function $f : \omega \to \omega$ there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ and $f_X(n) \geq f(n)$ $(n \geq 3)$.

Taylor conjecture (Taylor, Erdős–Hajnal–Shelah)
If X is a graph, $\mathrm{Chr}(X) > \aleph_0$, then for every cardinal λ there is a graph Y with the same finite subgraphs as X and $\mathrm{Chr}(Y) > \lambda$.

Notice:

if X is a graph, Y is a graph with the same finite subgraphs and $\mathrm{Chr}(Y) > \lambda$, then Y embeds into the ultraproduct of its finite subgraphs, which embeds into the ultrapower of X. There is, therefore, an ultrapower Z of X with $\mathrm{Chr}(Z) > \lambda$.

Theorem. It is consistent that there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ and if Y is a graph all whose finite subgraphs occur in X, then $\mathrm{Chr}(Y) \leq \aleph_2$.

Proof. P adds a Cohen real, a function $f:\omega\longrightarrow\omega$ undominated by any function in V. Q adds a graph X on ω_1, $\mathrm{Chr}(X)=\omega_1$ with $f_X\geq f$. $|P*Q|=\aleph_1$. Ass. $Y\in V^{P,Q}$ is a graph with the same finite graphs as X. Y splits into $|P*Q|=\aleph_1$ graphs in V: $Z_p=\{e:p\Vdash e\in Y\}$. If $Z\subseteq Y$, then $f_Y\leq f_Z$, if $Z\in V$, then $f_Z\in V$, so $\mathrm{Chr}(Z)$ is finite. So Y is the union of $\aleph_1$ finite chromatic graphs, and so $\mathrm{Chr}(Y)\leq 2^{\aleph_1}=\aleph_2$.

Theorem. It is consistent that if $\mathrm{Chr}(X) \geq \aleph_2$, then there are arbitrarily large chromatic graphs with the same finite subgraphs.

Proof. Let κ be such that if $\mathrm{Chr}(X) \geq \kappa$, then there are arbitrarily large chromatic graphs with the same finite subgraphs. Force with $P = \mathrm{Coll}(\omega, \kappa)$. If X is a graph in V^P with $\mathrm{Chr}(X) \geq \aleph_2$, then X is the union of $|P| = \omega$ graphs $Y \in V$, one has $\mathrm{Chr}(Y) \geq \kappa$. Then there are arbitrarily large chromatic graphs Z in V with the same finite subgraphs as Y. But if $\lambda \geq \kappa$, and $V \models \mathrm{Chr}(Z) = \lambda$, then $V^P \models \mathrm{Chr}(Z) = \lambda$.

Theorem (Shelah, Rinot) ($2^\lambda = \lambda^+$, $\Box_\lambda$) There is a graph X with $|X| = \mathrm{Chr}(X) = \lambda^+$, $\mathrm{Chr}(Y) \leq \omega$ for $Y \subseteq X$, $|Y| < \lambda^+$.

Theorem. (Shelah) (GCH) If $\lambda > \mathrm{cf}(\lambda) = \mu^+$ is a singular cardinal, then there is a cardinal, cofinality, and GCH preserving forcing extension in which there is a graph X with $|X| = \lambda$, $\mathrm{Chr}(X) = \mu^+$ on λ, such that $\mathrm{Chr}(Y) \leq \mu$ holds for every subgraph Y of X with $|Y| < \lambda$.

Theorem. (Shelah) (V=L) If κ is regular, not weakly compact, $\omega \leq \theta < \kappa$, $\lambda > \mathrm{cf}(\lambda) = \kappa$, then there is a graph X on λ with $\mathrm{Chr}(X) = \theta^+$, such that if Y is a subgraph of X with $|Y| < \lambda$ then $\mathrm{Chr}(Y) \leq \theta$.

Theorem. (Foreman–Laver) *Relative to the existence of a huge cardinal, it is consistent that if* $|X| = \mathrm{Chr}(X) = \aleph_2$, *then* X *contains a subgraph* Y *with* $|Y| = \mathrm{Chr}(Y) = \aleph_1$.

Proof. (Foreman) There is a model of GCH in which there is an $\aleph_1$-dense, ω_1-complete ideal I on ω_2. Let $\{A_\xi : \xi < \omega_1\}$ be dense in $\mathcal{P}(\omega_1)/I$. Assume that $f_\alpha : \alpha \to \omega$ is a good coloring of $X|\alpha$ $(\alpha < \omega_2)$. For each $\beta < \omega_2$ there are $\xi < \omega_1$, $i < \omega$, such that for almost all $\alpha \in A_\xi$, $f_\alpha(\beta) = i$. $F(\beta) = \langle \xi, i \rangle$ is a good coloring.

Conjecture. It is consistent that each graph X with $|X| = \aleph_2$, $\mathrm{Chr}(X) \geq \aleph_1$ contains a subgraph Y with $|Y| = \mathrm{Chr}(Y) = \aleph_1$.

Implies non-CH.

Theorem. (Shelah) Modulo the consistency of a supercompact cardinal it is consistent that (GCH and) each graph $|X| = \aleph_{\omega+1}$, $\mathrm{Chr}(X) \geq \aleph_1$ contains a subgraph Y with $|Y| < \aleph_\omega$, $\mathrm{Chr}(Y) = \aleph_1$.

Galvin's question: does the chromatic number have the Darboux-property: if $\mathrm{Chr}(X) = \lambda$ and $\kappa < \lambda$, then there is a subgraph $Y \subseteq X$ with $\mathrm{Chr}(Y) = \kappa$? W. l. o. g. $\omega < \kappa$.

Theorem. (Galvin) If $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2}$, then there is a graph X, $\mathrm{Chr}(X) > \aleph_1$, but it has no induced subgraph Y with $\mathrm{Chr}(Y) = \aleph_1$.

Proof. $X = \mathrm{Sh}_2(2^{\aleph_2})$.

Theorem. *It is consistent that there is a graph* X *with* $|X| = \mathrm{Chr}(X) = \aleph_2$ *such that there is no subgraph* $Y \subseteq X$ *with* $\mathrm{Chr}(Y) = \aleph_1$*.*

Proof. Finite support iteration of length ω_3. Q_0 adds a graph X on ω_2 with finite conditions. For $0 < \alpha < \omega_3$ let Y_α be a subgraph of X with $\mathrm{Chr}(Y_\alpha) = \aleph_1$. Q_α forces a good coloring of Y_α with elements of ω, with finite approximations.

If X is a graph, let

$$I(X) =$$

$$\{\mathrm{Chr}(Y) : Y \text{ induced subgr. in } X\} - \{0, 1, \ldots, \aleph_0\}.$$

Then $I(X)$ is closed, and if $\lambda \in I(X)$ is singular, then $\lambda \in I(X)'$.

Further, if $A \neq \emptyset$ is a set consisting of uncountable cardinals, then there is a cardinal, cofinality preserving forcing that adds a graph X such that $I(X) = A$.

If X is a graph, set

$$S(X) = \{\mathrm{Chr}(Y) : Y \subseteq X\} - \{0, 1, \ldots, \aleph_0\}.$$

Then, if $\lambda \in S(X)$ is singular, then $\lambda \in S(X)'$ and if $\lambda \in S(X)'$ is singular, then $\lambda \in S(X)$.

But $S(X)$ is not necessarily closed at regular cardinals:

Theorem. *If the existence of a measurable cardinal is consistent, then it is consistent that $S(X)$ is not closed at a regular cardinal.*

Chromatic number of graph products

If (V, X), (W, Y) are graphs, define their product $(V \times W, X \times Y)$ as

$$X \times Y = \{\{\langle x, x' \rangle, \langle y, y' \rangle\} : \{x, y\} \in X, \{x', y'\} \in Y\}$$

Hajnal: $\mathrm{Chr}(X \times Y) = \min(\mathrm{Chr}(X), \mathrm{Chr}(Y))$?

Theorem. (Hajnal) If $\mathrm{Chr}(X) < \omega \leq \mathrm{Chr}(X \times Y)$, then $\mathrm{Chr}(X \times Y) = \mathrm{Chr}(X)$.
Proof. Ass. $\mathrm{Chr}(X) = k + 1$, $\mathrm{Chr}(X \times Y) \leq k$. $f : V \times W \to k$ good coloring. U ultrafilter on W extending co-finite-chromatic sets. For $v \in V$ there is unique $g(v) < k$ s.t. $\{w : f(v, w) = g(v)\} \in U$. There are v, v', $\{v, v'\} \in X$, $g(v) = g(v')$.

$$\{w \in W : f(v, w) = f(v', w) = g(v)\} \in U$$

so it contains an edge: $\{w, w'\} \in Y$. Now $\{\langle v, w\rangle, \langle v', w'\rangle\} \in X \times Y$ and $f(v, w) = f(v', w')$.

Theorem. (Hajnal) If $\mathrm{Chr}(X), \mathrm{Chr}(Y) > \kappa$, $\mathrm{Chr}(X \times Y) < \kappa$, then $\mathrm{Chr}(X') < \kappa$ for $X' \subseteq X$, $|X'| = \kappa$.

Theorem. (Hajnal) If $\kappa \geq \omega$, there are $\{X_i : i < \kappa^+\}$ on 2^κ s.t. $\mathrm{Chr}(X_i) = \kappa^+$, $\mathrm{Chr}(X_i \times X_j) = \kappa$ $(i \neq j)$.

Proof.

Set $V = \{f : \alpha \to \kappa, \mathrm{inj}, \alpha < \kappa^+\}$, $\{f, g\} \in X$ iff $f \subsetneqq g$.

Claim 1. $\mathrm{Chr}(X) = \kappa^+$.

Proof. Let $F \to \kappa$ be a good coloring. Define f_α ($\alpha < \kappa^+$): $f_0 = 0$, $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$ (α limit), $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, F(f_\alpha)\rangle\}$. Induction gives $f_\beta \subseteq f_\alpha$ ($\beta < \alpha$) and $f_\alpha \in V$. Then $\bigcup\{f_\alpha : \alpha < \kappa^+\}$ is an injective $\kappa^+ \to \kappa$ function, contradiction!

Define $\Delta = \{\langle f, g\rangle \in V \times V, \mathrm{Dom}(f) = \mathrm{Dom}(g)\}$.

Claim 2. $\mathrm{Chr}(X \times X | (V \times V - \Delta)) \leq \kappa$.

Proof. Define for $\mathrm{Dom}(f) = \alpha$, $\mathrm{Dom}(g) = \beta$, $\alpha < \beta$, $F(\langle f, g\rangle) = g(\alpha)$ and similarly for $\beta < \alpha$.

If $A \subseteq \kappa^+$, set $V|A = \{f \in V : \mathrm{Dom}(f) \in A\}$,
$X|A = X|(V|A) \times (V|A)$.
$I = \{A \subseteq \kappa^+ : \mathrm{Chr}(X|A) \leq \kappa\}$

Claim 2. *I is a κ^+-complete, normal ideal on κ^+.*

Ulam matrix implies there are κ^+ disjoint sets in I^+, which gives the κ^+ graphs with κ-chromatic product.

Theorem. (Rinot) ($2^\lambda = \lambda^+$, $\diamondsuit_\lambda$) *There are graphs* $\{X_i : i < \lambda^+\}$ *with* $|X_i| = \mathrm{Chr}(X_i) = \lambda^+$, $\mathrm{Chr}(X_i \times X_j) = \omega$.

The *list chromatic number* $\mathrm{List}(X)$ of a graph (V, X) is the least cardinal μ such that: if $F(v)$ is arbitrary with $|F(v)| = \mu$ $(v \in V)$ then there is a good coloring f of X such that $f(v) \in F(v)$ $(v \in V)$.

Lemma. For every graph X $\mathrm{Chr}(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X)$ holds.

Theorem. *If X is a bipartite graph on the bipartition classes A,B, with $|A| = \kappa$, $|B| = 2^\kappa$, $|N(x)| = \kappa$ $(x \in B)$, then $\mathrm{List}(X) > \kappa$.*

Proof. First, let $\{F(a) : a \in A\}$ be disjoint sets of size κ. For each choice function $g \in \prod\{F(a) : a \in A\}$ select an element $b_g \in B$ and set $F(b_g) = \{g(a) : \{a, b_g\} \in X\}$. $|F(b_g)| = \kappa$ by condition. If $f(x) \in F(x)$ $(x \in A \cup B)$, then let $g = f|A$, now $f(b_g)$ cannot be any element of $F(b_g)$.

Theorem. Consistently, for graphs of size $\aleph_1$ $\mathrm{List}(X) = \aleph_1 \iff \mathrm{Chr}(X) = \aleph_1$.

Proof. MA_{ω_1} proves this.
Let X be a graph on ω_1, $\mathrm{Chr}(X) \leq \omega$, we want to prove that $\mathrm{List}(X) \leq \omega$. Let $h : \omega \to \omega$ be a good coloring of X, and let $F(x)$ be given for each $x \in \omega_1$, $|F(x)| = \omega$.
Set $p \in P$ if p is a function, $\mathrm{Dom}(p) \in [\omega_1]^{<\omega}$, $p(x) \in F(x)$ $(x \in \mathrm{Dom}(p))$, and $h(x) \neq h(y)$ implies $p(x) \neq p(y)$. $p' \leq p$ iff $p' \supseteq p$.
The hard part is to show that $(P, \leq)$ is ccc.

Theorem. *Consistently, GCH holds and* $\mathrm{List}(X) = \mathrm{Col}(X)$ *whenever the latter is infinite.*

Theorem. (GCH) If $\mathrm{List}(X)$ is infinite, then $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$.

Theorem. (Kojman) $\mathrm{Col}(X) \leq \exp_\omega(\mathrm{List}(X))^+$.

Ramsey-theory

Definition. A *topological* K_κ is a set of κ vetices plus a collection of paths between any two, disjoint except at ends.

Theorem. (E–H, 1964) If κ is infinite, n finite, then $\kappa \to (\mathrm{Top}K_\kappa)^2_n$.

The proof

Assume X is the complete graph on V, $\kappa = |V|$. Let U be an ultrafilter on V such that if $W \subseteq V$, $|W| < |V|$, then $V - W \in U$.
Let $f : X \to \{0, 1, \ldots, n-1\}$ be a coloring of X.
Each $v \in V$ has a principal color $i(v)$ s. t.

$$A(v) = \{w : f(v, w) = i(v)\} \in U.$$

Principal color: $B = \{v : i(v) = i\} \in U$.

Select the distinct vertices $\{v(\alpha) : \alpha < \kappa\}$ and $u(\alpha, \beta)$ such that

(1) $i(v(\alpha)) \in B$,
(2) $u(\alpha, \beta) \in A(v(\alpha)) \cap A(v(\beta))$ $(\beta < \alpha)$.

A topological K_κ is given by $\{v(\alpha) : \alpha < \kappa\}$ and the paths $\{v(\alpha), u(\alpha, \beta), v(\beta)\}$.

Question (E–H) $\kappa \to (\kappa, \mathrm{Top}K_\kappa)^2$?

Stronger than $\kappa \to (\mathrm{Top}K_\kappa)^2_n$ (n finite)

Theorem. $\kappa \to (\kappa, \mathrm{Top}K_\kappa)^2$ if and only if κ is regular and there is no κ-Suslin tree.

If κ is singular, let X be the disjoint union of $\mathrm{cf}(\kappa)$ complete graphs. In X, there is no independent set of size κ (or even $\mathrm{cf}(\kappa)^+$), neither any connected subgraph of size κ, in particular, no topological K_κ.

If $(T, \leq)$ is a tree (or any partially ordered set) then the *comparison graph* of $(T, \leq)$ is the graph (T, X) where $\{t, t'\} \in X$ iff $t < t'$ or $t' < t$.

The comparison graph of a κ-Suslin tree does not contain an independent set of size κ—it would be an antichain of size κ.

Assume that $\{v(\alpha) : \alpha < \kappa\}$ gives a topological K_κ with the connectings paths $\{p(\alpha, \beta) : \alpha < \beta < \kappa\}$. For any $\alpha < \kappa$, there are only $< \kappa$ nodes $\alpha < \beta < \kappa$ such that $p(\alpha, \beta) \cup \{v(\beta)\}$ has any elements $< v(\alpha)$ (as there are $< \kappa$ points $< \kappa$).
There is a subsequence $\{\alpha_\xi : \xi < \kappa\}$ such that if $\xi < \eta$, then all elements of $p(\alpha_\xi, \alpha_\eta) \cup v(\alpha_\eta)$ are above $v(\alpha_\xi)$ and so $\{v(\alpha_\xi) : \xi < \kappa\}$ is a κ-branch.

Can we have $\kappa \to (\mathrm{Top}K_\kappa)^2_{\aleph_0}$?

Not for $\kappa = \aleph_1$ as $[\omega_1]^2$ is the union of countably many (graph theoretical) trees (Erdős–Kakutani).

So set $\kappa = \aleph_2$.

Theorem. If the existence of a huge cardinal is consistent then $\omega_2 \to (\mathrm{Top} K_{\omega_2})^2_{\aleph_0}$ is consistent.

(∗) If $f : [\omega_2]^2 \to \omega$, then there are $i < \omega$ and $A \in [\omega_2]^{\aleph_2}$ such that if $\alpha < \beta$ are in A, then

$$|\{\beta < \gamma : f(\alpha, \gamma) = f(\beta, \gamma) = i\}| = \aleph_2.$$

$(*)$ implies $\omega_2 \to (\mathrm{Top} K_{\aleph_2})^2_{\aleph_0}$: given A as above, select $a(\xi) \in A$, $b(\xi, \eta)$ $(\xi < \eta < \omega_2)$ such that $a(\xi) < a(\eta) < b(\xi, \eta)$, $f(a(\xi), b(\xi, \eta)) = f(a(\eta), b(\xi, \eta)) = i$ and all a's and b's are distinct.

Theorem. (Foreman) Relative to the existence of a huge cardinal, it is consistent that there exists an ω_1-complete, uniform, $\aleph_1$-dense ideal I on ω_2.

Implies $(*)$ by the E–H principal color argument.

Theorem. (Shelah–K) *It is consistent with GCH that* $[\omega_2]^2$ *is the union of countably many Suslin-trees. That is,* $K_{\aleph_2}$ *is the union of countably many graphs, each the comparison graph of an* $\aleph_2$*-Suslin tree.*

$X \Longrightarrow (Y)^e_\mu$ denotes that if the edges of the graph X are colored with μ colors, then there is an induced copy of Y, all whose edges get the same color.

Two simple examples:

1. $K_{(2^\kappa)^+} \Longrightarrow (K_{\kappa^+})^e_\kappa$
2. $K_{\aleph_1,\aleph_2} \Longrightarrow (C_4)^e_{\aleph_0}$

Theorem. (Hajnal–K) It is consistent that there is a graph X on ω_1 such that $Y \not\Longrightarrow (X)^e_2$ holds for every graph Y.

Theorem. (Shelah) It is consistent that for every graph X and cardinal μ, there is a graph Y such that $Y \Longrightarrow (X)^e_\mu$.

Theorem. (Hajnal) For every finite graph X and cardinal μ, there is a graph Y such that $Y \Longrightarrow (X)^e_\mu$.

What if X is countable?

Theorem. (Shelah) If μ is a cardinal, then there is a forcing extension in which there is a graph X with no K_4, such that if the edges of X are colored with μ colors, then there is a monocolored K_3.

Erdős: Does this hold in ZFC?

Implies the existence of a finite graph X, no K_4, when 2-coloring, there is a monochromatic triangle.