The HOD Dichotomy, weak extender models, and universality

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October 25, 2015

Supercompactness

Definition

Suppose that κ is a regular cardinal and that $\kappa < \lambda$.

1. $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \mid |\sigma| < \kappa\}$.
2. Suppose that $\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ is a filter.
   - $\mathcal{F}$ is **fine** if for each $\alpha < \lambda$,
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in \mathcal{F}.$$
   - $\mathcal{F}$ is **normal** if for each function
     $$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
     such that
     $$\{\sigma \in \mathcal{F} \mid f(\sigma) \in \sigma\} \in \mathcal{F}$$
     there exists $\alpha < \lambda$ such that
     $$\{\sigma \in \mathcal{F} \mid f(\sigma) = \alpha\} \in \mathcal{F}$$

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is a **supercompact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

1. U is κ-complete,
2. U is a normal fine filter.

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2. U is a normal fine filter.

Lemma

Suppose κ is an uncountable regular cardinal. Then the following are equivalent.

(1) *κ is a supercompact cardinal.*

(2) *For each $\lambda > \kappa$, there exists an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, and such that $M^\lambda \subset M$.

- One can require that the transitive class M and the embedding j each be Σ_2-definable in V from parameters.

We shall need a specific variation of Solovay's Lemma on sets of measure one for normal fine κ-complete ultrafilters on $\mathcal{P}_\kappa(\lambda)$ where $\lambda > \kappa$ is a regular cardinal.

Lemma (Solovay's Lemma)

Suppose that $\kappa < \lambda$ are regular cardinals and $<$ is a wellordering of $H(\lambda^+)$. Then there exists a set $X \in \mathcal{P}_\kappa(\lambda)$ such that the following hold.

(1) *Suppose U is a κ-complete, normal, fine, ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Then $X \in U$.*

(2) *Suppose $\sigma, \tau \in X$ and $\sup(\sigma) = \sup(\tau)$. Then $\sigma = \tau$.*

(3) *X is uniformly definable in $(H(\lambda^+), <)$ from κ.*

Proof

Let $S = \{\alpha < \lambda \,|\, \mathrm{cof}(\alpha) = \omega\}$ and let

$$\langle S_\alpha : \alpha < \lambda \rangle$$

be the $<$-least partition of S into λ many stationary sets.

Proof

Let $S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}$ and let

$$\langle S_\alpha : \alpha < \lambda \rangle$$

be the $<$-least partition of S into λ many stationary sets.

Let X be the set of all $\sigma \in \mathcal{P}_\kappa(\lambda)$ such that

1. $\omega < \mathrm{cof}(\sup(\sigma)) < \kappa$,
2. σ is the set of $\alpha < \sup(\sigma)$ such that $S_\alpha \cap C \neq \emptyset$ for all closed cofinal subsets of $\sup(\sigma)$.

- Then X witnesses the lemma. $\square$

Weak Extender Models

Definition

A transitive class N model of ZFC is a **weak extender model for δ is supercompact** iff for every $\gamma > \delta$ there exists a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in U$,
2. $U \cap N \in N$.

Definition

Suppose N is a transitive class and that δ is a regular cardinal. Then N has the δ-**covering property** if for each $\sigma \subset N$ such that $|\sigma| < \delta$, there exists $\tau \in N$ such that

1. $\sigma \subset \tau$,
2. $|\tau| < \delta$.

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2. $|\tau| < \delta$.

Lemma (7)

Suppose that N is a weak extender model for δ is supercompact. Then N has the δ-covering property.

proof

Let $\sigma \subset N$ be a set with $|\sigma| < \delta$. Since

$$N \models \mathrm{ZFC}$$

we can reduce to the case that $\sigma \subset \mathrm{Ord}$.

- Let $\lambda > \delta$ be such that $\sigma \subset \lambda$.
- Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that
$$N \cap \mathcal{P}_\delta(\lambda) \in U.$$

Thus since U is fine and δ-complete,

$$\{\tau \in \mathcal{P}_\delta(\lambda) \mid \sigma \subset \tau\} \in U$$

and so there must exist

$$\tau \in \mathcal{P}_\delta(\lambda) \cap N$$

such that $\sigma \subset \tau$. $\square$

Lemma (8)

Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a regular cardinal in N.

- *Then* $(\mathrm{cof}(\gamma))^V = |\gamma|^V$.

Proof.

Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in U$,
2. $U \cap N \in N$.

Lemma (8)

Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a regular cardinal in N.

- *Then $(\mathrm{cof}(\gamma))^V = |\gamma|^V$.*

Proof.

Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in U$,
2. $U \cap N \in N$.

By Solovay's Lemma applied within N, there exists a set

$$X \in N \cap U$$

such that π is 1-to-1 on X where $\pi(\sigma) = \sup(\sigma)$.

- Let $C \subset \gamma$ be a closed cofinal set of ordertype $(\mathrm{cof}(\gamma))^V$.
- Let $j : V \to M$ be the ultrapower embedding given by U.

Thus $j[\gamma]$ is the unique element σ of $j(X)$ such that

$$\sup(\sigma) = \sup(j[\gamma]).$$

Proof continued

But C is closed cofinal in γ and so

$$\sup(j[\gamma]) \in j(C).$$

Therefore

$$\{\sigma \in X \mid \sup(\sigma) \in C\} \in U.$$

Finally since U is fine,

$$\cup\{\sigma \in X \mid \sup(\sigma) \in C\} = \gamma.$$

Proof continued

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Therefore

- $|\gamma|^V = |C|^V \cdot \delta = (\mathrm{cof}(\gamma))^V \cdot \gamma.$

Proof continued

But C is closed cofinal in γ and so

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Therefore

- $|\gamma|^V = |C|^V \cdot \delta = (\mathrm{cof}(\gamma))^V \cdot \gamma.$

Finally γ is a regular cardinal in N and N has the δ-covering property and so

$$(\mathrm{cof}(\gamma))^V \geq \delta.$$

Thus $|\gamma|^V = |C|^V \cdot \delta = (\mathrm{cof}(\gamma))^V \cdot \delta = (\mathrm{cof}(\gamma))^V$. □

Theorem (9)

Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and

$$(\gamma^+)^N = \gamma^+.$$

Proof.

If γ is a regular cardinal in N then by Lemma 8, $\mathrm{cof}(\gamma) = |\gamma|$ which contradicts that γ is singular.

Let $\lambda = (\gamma^+)^N$. Then

- λ is a regular cardinal in N,

and so by Lemma 8,

- $\mathrm{cof}(\lambda) = |\lambda| \geq \gamma$.

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- λ is a regular cardinal in N,

and so by Lemma 8,

- $\mathrm{cof}(\lambda) = |\lambda| \geq \gamma$.

But $\mathrm{cof}(\lambda)$ is a regular cardinal and so $\mathrm{cof}(\lambda) > \gamma$.

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But $\mathrm{cof}(\lambda)$ is a regular cardinal and so $\mathrm{cof}(\lambda) > \gamma$.

- This implies that $\lambda = \gamma^+$. □

Extendible cardinals and Magidor's Lemma

Definition

Suppose that δ is a cardinal. Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

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such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

Lemma (Magidor)

Suppose that δ is a regular cardinal. Then the following are equivalent.

(1) *δ is supercompact.*

(2) *For each $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ and such that $\pi(\bar{\delta}) = \delta$.

The weak extender model version of Magidor's Lemma

Lemma (12)

Suppose that N is a weak extender model for δ is supercompact. Then for each $\lambda > \delta$ and for each $a \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{a} \in V_{\bar{\lambda}}$, and an elementary embedding

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that the following hold.

(1) $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, *and* $\pi(\bar{a}) = a$.

(2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Proof.

By increasing λ and replacing a by the pair (a, λ) if necessary, we can reduce to the case that

$$\lambda = |V_\lambda|$$

and that $\mathrm{cof}(\lambda) = \omega$. Thus $|N \cap V_\lambda|^N = \lambda$. Fix a bijection

$$\rho : \lambda \to N \cap V_\lambda$$

such that $\rho \in N$.

Proof.

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such that $\rho \in N$.

Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

1. $N \cap \mathcal{P}_\delta(\lambda) \in U$,
2. $U \cap N \in N$.

Proof continued

For each $\sigma \in \mathcal{P}_\delta(\lambda)$, let

$$X_\sigma = \{\rho(\alpha) \mid \alpha \in \sigma\}.$$

Let Z be the set of all $\sigma \in \mathcal{P}_\delta(\lambda)$ such that

$$X_\sigma \prec N \cap V_\sigma.$$

Thus by the normality and fineness of U:

- $Z \in U$.

Proof continued

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Let Z be the set of all $\sigma \in \mathcal{P}_\delta(\lambda)$ such that

$$X_\sigma \prec N \cap V_\sigma.$$

Thus by the normality and fineness of U:

- $Z \in U$.

For each $\sigma \in Z$, let M_σ be the transitive collapse of X_σ. The key claim is:

Claim (1)

$\{\sigma \in Z \mid M_\sigma = N \cap V_\alpha$ where α is the ordertype of $\sigma\} \in U$.

This follows easily by working in N.

Proof continued

Now let

$$j_U : V \to M_U \cong \mathrm{Ult}(V, U)$$

be the ultrapower embedding. Thus since $|V_\lambda| = \lambda$ and since $\mathrm{cof}(\lambda) = \omega$,

$$(M_U)^{V_{\lambda+1}} \subset M_U$$

and so $j_U|V_{\lambda+1} \in M_U$. Further by Claim (1),

$$j_U(N \cap V_\lambda) \cap V_\lambda = N \cap V_\lambda.$$

Thus the following hold.

1. $j_U|(N \cap V_\lambda) \in j_U(N)$,
2. $(\mathrm{cof}(\lambda))^N < \delta$,
3. $j_U|(N \cap V_\lambda) \in j_U(N)$ (since $j_U[\lambda] \in j_U(N)$),
4. $j_U(N) \cap V_\lambda = N \cap V_\lambda$.

Proof continued

Now let

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Thus the following hold.

1. $j_U | (N \cap V_\lambda) \in j_U(N)$,
2. $(\mathrm{cof}(\lambda))^N < \delta$,
3. $j_U | (N \cap V_\lambda) \in j_U(N)$ (since $j_U[\lambda] \in j_U(N)$),
4. $j_U(N) \cap V_\lambda = N \cap V_\lambda$.

- Note that 1–4 imply that the conclusion of the lemma holds for $(j_U(a), j_U(\lambda))$ in M_U for $j_U(N)$.

Therefore the conclusion of the lemma holds in V for (a, λ) relative to N. □

Elementary embeddings of weak extender models

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Theorem (13)

Suppose that N is a weak extender model for δ is supercompact and $\gamma > \delta$ is a cardinal in N. Suppose that

$$j : H(\gamma^{+})^{N} \to H(j(\gamma)^{+})^{N}$$

is an elementary embedding such that $\delta \leq \mathrm{CRT}(j)$.

- *Then $j \in N$.*

Proof

Fix $\lambda > j(\gamma)$ such that $\lambda = |V_\lambda|$. Letting $a = j$, by Lemma 12, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{a} \in V_{\bar{\lambda}}$, and an elementary embedding

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that the following hold.

1. $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, and $\pi(\bar{a}) = a$.
2. $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.
3. $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Thus $\bar{a} = \bar{j}$ where

$$\bar{j} : H(\bar{\gamma}^+)^N \to H(\bar{j}(\bar{\gamma})^+)^N.$$

Proof

Fix $\lambda > j(\gamma)$ such that $\lambda = |V_\lambda|$. Letting $a = j$, by Lemma 12, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{a} \in V_{\bar{\lambda}}$, and an elementary embedding

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such that the following hold.

1. $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, and $\pi(\bar{a}) = a$.
2. $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.
3. $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Thus $\bar{a} = \bar{j}$ where

$$\bar{j} : H(\bar{\gamma}^+)^N \to H(\bar{j}(\bar{\gamma})^+)^N.$$

It suffices to prove:

Claim

$\bar{j} \in N$.

- Since $\pi(\bar{j}) = j$ and since $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Proof continued

Let

$$E = \{(A, \xi) \mid A \in \mathcal{P}(\bar{\gamma}) \cap N, \xi < \bar{j}(\bar{\gamma}), \text{ and } \xi \in \bar{j}(A)\}$$

We prove that $E \in N$. This implies that

$$\bar{j} | (\mathcal{P}(\bar{\gamma}) \cap N) \in N$$

which implies that $\bar{j} \in N$.

Proof continued

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We prove that $E \in N$. This implies that

$$\bar{j} | (\mathcal{P}(\bar{\gamma}) \cap N) \in N$$

which implies that $\bar{j} \in N$.
The key point is:

$$\pi | (H(\bar{\gamma}^+))^N \in (H(\gamma^+))^N.$$

- This is because $\pi | (N \cap V_{\bar{\lambda}}) \in N$ noting that $(H(\gamma^+))^N$ is closed under γ-sequences in N.

Proof continued

Let

$$\pi^* = \pi | \left(H(\bar{\gamma}^+)\right)^N \in \left(H(\gamma^+)\right)^N .$$

Thus $\pi^* \in \left(H(\gamma^+)\right)^N$ and so $\pi^* \in \mathrm{dom}(j)$.

Proof continued

Let

$$\pi^* = \pi | \left(H(\bar{\gamma}^+)\right)^N \in \left(H(\gamma^+)\right)^N.$$

Thus $\pi^* \in (H(\gamma^+))^N$ and so $\pi^* \in \mathrm{dom}(j)$.

Now fix $A \in \mathcal{P}(\bar{\gamma}) \cap N$ and $\xi < \bar{j}(\bar{\gamma})$. Thus

$$\begin{aligned}
\xi \in \bar{j}(A) \quad &\Longleftrightarrow \quad \pi(\xi) \in \pi(\bar{j})(\pi(A)) \\
&\Longleftrightarrow \quad \pi(\xi) \in j(\pi(A)) \\
&\Longleftrightarrow \quad \pi(\xi) \in j(\pi^*(A)) \\
&\Longleftrightarrow \quad \pi(\xi) \in j(\pi^*)(j(A)) = j(\pi^*)(A).
\end{aligned}$$

Thus E can be computed from $\pi|\bar{j}(\bar{\gamma})$ and $j(\pi^*)$. Both these functions are in N and so $E \in N$. $\square$

Kunen's Theorem

Theorem (Kunen)

Suppose that λ is a cardinal. Then there is no non-trivial elementary embedding

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Proof.

Let j be given. Note that $V_{\lambda+2}$ is logically equivalent to $H(|V_\lambda|^{++})$ and so j yields an elementary embedding

$$\pi : H(\lambda^{++}) \to H(\lambda^{++}).$$

Necessarily $\pi(\lambda) = \lambda$ and $\pi(\lambda^+) = \lambda^+$.

Proof continued

Let $S = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$ and let $\langle S_\alpha : \alpha < \lambda^+\rangle$ be a partition of S into stationary sets. Let

- $\langle T_\alpha : \alpha < \lambda^+\rangle = \pi(\langle S_\alpha : \alpha < \lambda^+\rangle)$,
- $C = \{\alpha \in S \mid \pi(\alpha) = \alpha\}$.

Proof continued

Let $S = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$ and let $\langle S_\alpha : \alpha < \lambda^+ \rangle$ be a partition of S into stationary sets. Let

- $\langle T_\alpha : \alpha < \lambda^+ \rangle = \pi(\langle S_\alpha : \alpha < \lambda^+ \rangle)$,
- $C = \{\alpha \in S \mid \pi(\alpha) = \alpha\}$.

Thus C is ω-closed and cofinal in λ^+.
By the elementarity of π and since $\pi(S) = S$, for each $\alpha < \lambda^+$,

- T_α is a stationary subset of S and so for each $\alpha < \lambda^+$, $C \cap T_\alpha \neq \emptyset$.

Proof continued

Let $S = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$ and let $\langle S_\alpha : \alpha < \lambda^+ \rangle$ be a partition of S into stationary sets. Let

- $\langle T_\alpha : \alpha < \lambda^+ \rangle = \pi(\langle S_\alpha : \alpha < \lambda^+ \rangle)$,
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Thus C is ω-closed and cofinal in λ^+.
By the elementarity of π and since $\pi(S) = S$, for each $\alpha < \lambda^+$,

- T_α is a stationary subset of S and so for each $\alpha < \lambda^+$, $C \cap T_\alpha \neq \emptyset$.

Let $\kappa = \mathrm{CRT}(\pi)$ and choose

- $\xi \in C \cap T_\kappa$,
- $\beta < \lambda^+$ such that $\xi \in S_\beta$.

Proof continued

Let $S = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$ and let $\langle S_\alpha : \alpha < \lambda^+ \rangle$ be a partition of S into stationary sets. Let

- $\langle T_\alpha : \alpha < \lambda^+ \rangle = \pi(\langle S_\alpha : \alpha < \lambda^+ \rangle)$,
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By the elementarity of π and since $\pi(S) = S$, for each $\alpha < \lambda^+$,

- T_α is a stationary subset of S and so for each $\alpha < \lambda^+$, $C \cap T_\alpha \neq \emptyset$.

Let $\kappa = \mathrm{CRT}(\pi)$ and choose

- $\xi \in C \cap T_\kappa$,
- $\beta < \lambda^+$ such that $\xi \in S_\beta$.

Then

$$\xi = \pi(\xi) \in \pi(S_\beta) = T_{\pi(\beta)}.$$

This implies $\pi(\beta) = \kappa$ which contradicts that $\kappa = \mathrm{CRT}(\pi)$. $\square$

Theorem (15)

Let N be a weak extender model for δ is supercompact. Then there is no nontrivial elementary embedding $j : N \to N$ such that $\delta \leq \mathrm{CRT}(j)$.

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Let N be a weak extender model for δ is supercompact. Then there is no nontrivial elementary embedding $j : N \to N$ such that $\delta \leq \mathrm{CRT}(j)$.

Proof.

Assume toward a contradiction that $j : N \to N$ is a nontrivial elementary embedding such that $\delta \leq \mathrm{CRT}(j)$.

By Theorem 13, for each $\kappa > \delta$, $j|(N \cap V_{\kappa+1}) \in N$.

Thus j is amenable to N and in particular there must exist a cardinal λ of N such that $\mathrm{CRT}(j) < \lambda$, $j(\lambda) = \lambda$, and such that

$$j|(V_{\lambda+2} \cap N) \in N.$$

This contradicts Kunen's Theorem. □

Extenders

Definition

Suppose that M and N are transitive models of $\mathrm{ZFC}\backslash\mathrm{Powerset}$ and that $\pi : M \to N$ is an elementary embedding.
Then π is **cofinal** if

$$N = \cup\{\pi(a) \mid a \in M\}.$$

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Definition

Suppose that M and N are transitive models of $\mathrm{ZFC} \backslash \mathrm{Powerset}$ and that $\pi : M \to N$ is a cofinal elementary embedding which is not the identity.
Let $\kappa = \mathrm{CRT}(\pi)$ and suppose that $\eta \in \mathrm{Ord}^N$. For each $a \in [\eta]^{<\omega}$, let

$$E_a = \{A \in N \mid a \in \pi(A)\}.$$

Let $E = \langle E_a : a \in [\eta]^{<\omega} \rangle$. Then E is a (κ, η)-**extender over** M.

Definition

Suppose that M is a transitive model of $\mathrm{ZFC}\backslash\mathrm{Powerset}$ and that

$$E = \langle E_a : a \in [\eta]^{<\omega} \rangle$$

is an M-extender. Then

$$\mathrm{Ult}(M, E) = \lim_{a \in [\eta]^{<\omega}} \mathrm{Ult}(M, E_a).$$

Lemma

Suppose that M is a transitive model of $\mathrm{ZFC} \backslash \mathrm{Powerset}$ and that

$$E = \langle E_a : a \in [\eta]^{<\omega} \rangle$$

is an M-extender. Then:

(1) $\mathrm{Ult}(M, E)$ *is wellfounded.*

(2) *Let M_E be the transitive collapse of $\mathrm{Ult}(M, E)$ and let*

$$\pi_E : M \to M_E$$

be the ultrapower embedding. Then:

  (a) π_E *is an elementary embedding.*
  (b) $\mathrm{CRT}(\pi_E) < \eta < \mathrm{Ord}^{M_E}$.
  (c) *Let F be the (κ, η)-extender given by π_E where $\kappa = crt(\pi_E)$. Then $F = E$.*

Theorem (Universality Theorem)

Suppose that N is a weak extender model for δ is supercompact and E is an N-extender of length η with $\mathrm{CRT}(E) \geq \delta$. Let

$$\pi_E : N \to M_E \cong \mathrm{Ult}(N, E)$$

be the ultrapower embedding and suppose that for each $A \subset \eta$, $\pi_E(A) \cap \eta \in N$. Then

$$E \cap N \in N.$$

- The proof is essentially the same ase the proof of Theorem 13.

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- The proof is essentially the same ase the proof of Theorem 13.

Suppose that E is an L-extender of length η. Then

$$L \cong \mathrm{Ult}(L, E)$$

and so $\pi_E(A) \in L$ for all $A \in L$.

Definition

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Definition

An extender, E, of length η is λ-**complete** if

$$\eta^\lambda \subseteq M$$

where $M = \mathrm{Ult}(V, E)$.

Suppose that E is a (κ, η)-extender, $\mathbb{P} \in V_\kappa$, and $G \subsetneq \mathbb{P}$ is V-generic. Then

- E naturally defines a (κ, η)-extender in $V[G]$ and

$$(j_E)^{V[G]}|V = (j_E)^V.$$

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Lemma (23)

Suppose that $\delta < \kappa$, E is an extender which is δ-complete with critical point κ, and that

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) $M = \{j(f)(\alpha) \mid \alpha < \delta\}$,

(ii) *G is V-generic for some partial order $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \delta$ in V.*

- *Then $(j_E)^{V[G]}|M = (j_F)^M$ where $F = j(E)$.*

Proof.

By (i), $M = \mathrm{Ult}(V, H)$ where H is a V-extender of length δ.
Let $\eta = \mathrm{LTH}(E)$ and for each $a \in [\eta]^\delta$ let E_a be the ultrafilter,

$$E_a = \{A \subseteq [\hat{\eta}]^\delta \mid a \in j_E(A)\},$$

where $\hat{\eta} = \min\{\gamma \mid \eta \leq j_E(\gamma)\}$.

- Since E is δ-complete for each $a \in [\eta]^\delta$, $a \in \mathrm{Ult}(V, E)$ and so E_a is defined.

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- Since E is δ-complete for each $a \in [\eta]^\delta$, $a \in \mathrm{Ult}(V, E)$ and so E_a is defined.

Suppose that $a \subseteq b$ and $b \in [\eta]^\delta$. Then there is a natural elementary embedding,

$$j_{a,b} : \mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, E_b).$$

- This defines a directed system indexed by the directed set, $([\eta]^\delta, \subseteq)$ with limit, $\mathrm{Ult}(V, E)$.

This is just the usual analysis of $\mathrm{Ult}(V, E)$ as the limit of a directed system of ultrapowers except here the underlying directed set is $([\eta]^\delta, \subseteq)$ instead of the directed set, $([\eta]^{<\omega}, \subseteq)$.

Proof continued.

Let $X = [\hat{\eta}]^\delta$. For each $a \in [\eta]^\delta$,

- $E_a \subseteq \mathcal{P}(X)$ and E_a is an ultrafilter on X.

Proof continued.

Let $X = [\hat{\eta}]^\delta$. For each $a \in [\eta]^\delta$,

- $E_a \subseteq \mathcal{P}(X)$ and E_a is an ultrafilter on X.

Fix $a \in [\eta]^\delta$. We prove the following.

Claim (1)

Suppose that $f : X \to M$ is a function in $V[G]$.

- *Then there exists a function*

$$f^* : j(X) \to M$$

such that $f^* \in M$ and such that

$$\{y \in X \mid f(y) = f^*(j(y))\} \in (E_a)_G$$

where $(E_a)_G$ is the ultrafilter in $V[G]$ generated by E_a.

Proof continued.

Fix f and work in $V[G]$. For each $y \in X$ there exists a pair (g_y, α_y) such that

1. $\alpha_y < \delta$,
2. $g_y \in V$,
3. $f(y) = j(g_y)(\alpha_y)$.

This defines a function

$$F : X \to V$$

where for all $y \in X$, $F(y) = (g_y, \alpha_y)$.

Proof continued.

Fix f and work in $V[G]$. For each $y \in X$ there exists a pair (g_y, α_y) such that

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This defines a function

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where for all $y \in X$, $F(y) = (g_y, \alpha_y)$.
Since E_a is κ-complete and since $|\mathbb{P}|^V \leq \delta < \kappa$, it follows that there exists $Z \in E_a$ and there exists $\alpha < \delta$ such that

1. $F|Z \in V$,
2. $\alpha_y = \alpha$ for all $y \in Z$.

Proof continued.

Define

$$f^* : j(X) \to M$$

by $f^*(t) = 0$ if $t \notin j(Z)$ and if $t \in j(Z)$ then

$$f^*(t) = j(F)_t(\alpha)$$

where for each $y \in X$, $F_y = g_y$.
Thus for each $y \in Z$,

$$f^*(j(y)) = j(F)_{j(y)}(\alpha) = (j(F_y))(\alpha) = (j(g_y))(\alpha) = (j(g_y))(\alpha_y) = f(y),$$

and so f^* is as required.

Proof continued.

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and so f^* is as required.

- What we have done is show that for each $a \in [\eta]^\delta$ the lemma holds with E replaced by E_a. This special case is due to Steel.

Proof continued.

Now we use the hypothesis that E is δ-complete. The key claim is:

Claim (2)

The set $\{j(a) \mid a \in [\eta]^\delta\}$ *is cofinal in the directed set,*

$$\{a \mid a \in j([\eta]^\delta)\}.$$

Proof continued.

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To see this suppose $b \in j([\eta]^\delta)$.

- Then there exists $\alpha < \delta$ and a function

  $$g : \delta \to [\eta]^\delta$$

  such that $j(g)(\alpha) = b$ noting that $\delta \leq j(\delta)$.

Proof continued.

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Let $a = \cup\{g(\beta) \mid \beta < \delta\}$. Thus $a \in [\eta]^\delta$, $a \in V$ and $b \subseteq j(a)$.

Proof continued.

Claim (3)

$\mathrm{Ult}(M, j(E))$ *is the limit of* $\mathrm{Ult}(M, j(E_a))$ *over the directed set*

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Proof continued.

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Proof continued.

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$$([\eta]^\delta, \subseteq)^V.$$

- This is an immediate consequence of Claim 2.

The lemma follows by the correspondence of functions established above. □

There is a useful corollary of Lemma 23.

Lemma (24)

Suppose that $\delta < \kappa$, κ is supercompact, and that

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) $M = \{j(f)(\alpha) \mid \alpha < \delta\}$,

(ii) *G is V-generic for some partial order $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \delta$ in V.*

- *Then in $V[G]$, M is a weak extender model for κ is supercompact.*

Proof.

By Lemma 23, for each extender $E \in V$, if (in V),

1. $\mathbb{P} \in V_{\mathrm{CRT}(E)}$,
2. $\rho(E) = \mathrm{LTH}(E)$,
3. $\mathrm{cof}(\mathrm{LTH}(E)) > \delta$,

then in $V[G]$, $E_G \cap M \in M$ where E_G is the extender in $V[G]$ generated by E.

- The point is that by Lemma 23,

$$j(E) = E_G \cap M.$$

Proof.

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then in $V[G]$, $E_G \cap M \in M$ where E_G is the extender in $V[G]$ generated by E.

- The point is that by Lemma 23,

$$j(E) = E_G \cap M.$$

Since κ is supercompact in V, the class of all such extenders, E_G, witnesses the Magidor characterization that κ is supercompact in $V[G]$.
The corollary follows. $\square$

Lemma

Suppose that δ is a supercompact cardinal. Then there is N a weak extender model for δ is supercompact and a nontrivial $j : N \to N$ with $\mathrm{CRT}(j) < \delta$.

Proof.

Let $\kappa < \delta$ be a measurable cardinal and let U a normal κ-complete uniform ultrafilter on κ.
Let $\langle (M_n, U_n, j_{n,n+1}) : n < \omega \rangle$ be the iteration of (V, U) of length ω.

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Let $\kappa < \delta$ be a measurable cardinal and let U a normal κ-complete uniform ultrafilter on κ.
Let $\langle (M_n, U_n, j_{n,n+1}) : n < \omega \rangle$ be the iteration of (V, U) of length ω.

Thus

1. $(M_0, U_0) = (V, U)$,
2. $M_{n+1} = \mathrm{Ult}(M_n, U_n)$ and $j_{n,n+1} : M_n \to M_{n+1}$ is the ultrapower embedding.
3. $U_{n+1} = j_{n,n+1}(U_n)$.

Proof continued

Let

$$M_\omega = \lim_{n<\omega} M_n$$

be the direct limit under the composition of the elementary embeddings,

$$\langle j_{n,n+1} : n < \omega \rangle.$$

Thus M_ω is wellfounded. Define $N = M_\omega$ and let

$$j_{0,\omega} : V \to N$$

be the associated elementary embedding.

Proof continued

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Thus M_ω is wellfounded. Define $N = M_\omega$ and let

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be the associated elementary embedding.
Let $\eta = j_{0,\omega}(\kappa)$. Then $\eta < (2^\kappa)^+ < \delta$ and

$$N = \mathrm{Ult}(V, E)$$

where E is the (κ, η)-extender given by $j_{0,\omega}$.

Proof continued

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- Thus N is a weak extender model for δ is supercompact by Lemma 24.

Proof continued

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- Thus N is a weak extender model for δ is supercompact by Lemma 24.

Finally, $j_{0,1}(N) = N$ and so $j_{0,1}|N$ yields an elementary embedding

$$j : N \to N \qquad \square$$

The HOD Dichotomy Theorem

Theorem (Jensen)

Exactly one of the following holds.

(1) *L is correct about singular cardinals and computes their successors correctly.*

(2) *Every uncountable cardinal is inaccessible in L.*

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Exactly one of the following holds.

(1) *L is correct about singular cardinals and computes their successors correctly.*

(2) *Every uncountable cardinal is inaccessible in L.*

Theorem (HOD Dichotomy Theorem)

Assume that δ is an extendible cardinal. Then exactly one of the following holds.

(1) *For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.*

(2) *Every regular cardinal greater than δ is measurable in HOD.*

Definition

Let λ be an uncountable regular cardinal. Then λ is ω-**strongly measurable in** HOD iff there is $\kappa < \lambda$ such that

1. $(2^{\kappa})^{\mathrm{HOD}} < \lambda$ and
2. there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of $\mathrm{cof}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa \rangle \in \mathrm{HOD}$.

Lemma (29)

Suppose that λ is an uncountable regular cardinal and that $\mathcal{F}$ is a λ-complete uniform filter on λ. Let

$$\mathbb{B} = \mathcal{P}(\lambda)/I$$

where I is the ideal dual to $\mathcal{F}$. Suppose that $\mathbb{B}$ is γ-cc for some γ such that $2^\gamma < \lambda$.

- *Then $|\mathbb{B}| \leq 2^\gamma$ and $\mathbb{B}$ is atomic.*

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- *Then $|\mathbb{B}| \leq 2^\gamma$ and $\mathbb{B}$ is atomic.*

Proof.

It suffices to prove that $\mathbb{B}$ is atomic. Equivalently, it suffices to show that if $A \subseteq \kappa$ and $A \notin I$ then there exists $B \subseteq A$ such that $B \notin I$ and such that B cannot be split into 2 sets each of which is I-positive.
This in turn reduces to simply proving that $\mathbb{B}$ has an atom since if $\mathbb{B}$ is not atomic then we can replace I by the ideal generated by $I \cup \{A\}$ where A/I is the join in $\mathbb{B}$ of all the atoms of $\mathbb{B}$.

Proof continued

Therefore we assume toward a contradiction that $\mathbb{B}$ has no atoms. Let

$$\langle (P_\alpha, Z_\alpha) : \alpha < \Theta \rangle$$

be a maximal sequence such that $\Theta \leq \gamma + 1$ and such that for all $\alpha < \beta$,

1. $2^{|\beta|} < \kappa$,
2. $Z_\beta \in \mathcal{F}$ and $Z_\beta \subseteq Z_\alpha$,
3. P_α is a partition of Z_β into I-positive sets,
4. P_β refines P_α,
5. for each $B \in P_\alpha$, there exist distinct $X, Y \in P_\beta$ such that $X \cup Y \subset B$.

Proof continued

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4. P_β refines P_α,
5. for each $B \in P_\alpha$, there exist distinct $X, Y \in P_\beta$ such that $X \cup Y \subset B$.

For each $\alpha < \Theta$, $|P_\alpha| < \gamma$ since $\mathbb{B}$ is γ-cc. We prove

- $2^{|\Theta|} \geq \kappa$.

Proof continued

Assume toward a contradiction that $2^{|\Theta|} < \kappa$. Thus $\Theta < \kappa$ and so $Z \in \mathcal{F}$ where

$$Z = \cap\{Z_\alpha \mid \alpha < \Theta\}.$$

Define an equivalence relation $\sim$ on Z by $\xi_1 \sim \xi_2$ if for all $\alpha < \Theta$, for all $A \in P_\alpha$, $\xi_1 \in A$ if and only if $x_2 \in A$.

We have:

- $2^{|\Theta|} < \kappa$ and $2^\gamma < \kappa$,
- for each $\alpha < \kappa$, $|P_\alpha| < \gamma$.

Proof continued

Assume toward a contradiction that $2^{|\Theta|} < \kappa$. Thus $\Theta < \kappa$ and so $Z \in \mathcal{F}$ where

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We have:

- $2^{|\Theta|} < \kappa$ and $2^\gamma < \kappa$,
- for each $\alpha < \kappa$, $|P_\alpha| < \gamma$.

Therefore $|Z/\!\sim| < \kappa$. But then

$$\cup\{[\xi]_\sim \,|\, \xi \in Z \text{ and } [\xi]_\sim \notin I\} \in \mathcal{F}$$

where for each $\xi \in Z$, $[\xi]_\sim$ is the $\sim$-equivalence class of ξ.

Proof continued

Assume toward a contradiction that $2^{|\Theta|} < \kappa$. Thus $\Theta < \kappa$ and so $Z \in \mathcal{F}$ where

$$Z = \cap\{Z_\alpha \mid \alpha < \Theta\}.$$

Define an equivalence relation $\sim$ on Z by $\xi_1 \sim \xi_2$ if for all $\alpha < \Theta$, for all $A \in P_\alpha$, $\xi_1 \in A$ if and only if $x_2 \in A$.

We have:

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where for each $\xi \in Z$, $[\xi]_\sim$ is the $\sim$-equivalence class of ξ.

Define $Z_\Theta = Z$ and $P_\Theta = \{[\xi]_\sim \mid \xi \in Z \text{ and } [\xi]_\sim \notin I\}$. This contradicts the maximality of the sequence

$$\langle (P_\alpha, Z_\alpha) : \alpha < \Theta \rangle.$$

Proof continued

This proves that $2^{|\Theta|} \geq \kappa$.

- But this implies that $\Theta > \gamma$.

Proof continued

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- But this implies that $\Theta > \gamma$.

Fix $\xi \in Z_\gamma$. For each $\alpha < \gamma$, let $X_\alpha \in P_\alpha$ be such that $\xi \in X_\alpha$. Thus

$$\langle X_\alpha : \alpha < \gamma \rangle$$

is a decreasing sequence of I-positive sets and for each $\alpha < \gamma$, $A_{\alpha+1} \backslash A_\alpha$ is I-positive.

- This yields an antichain in $\mathbb{B}$ is cardinality γ which contradicts that $\mathbb{B}$ is γ-cc.

□

Lemma (30)

Assume λ is ω-strongly measurable in HOD. Then

$$\mathrm{HOD} \models \lambda \textit{ is a measurable cardinal.}$$

Proof.

Let $S = \{\alpha < \lambda \mid (\mathrm{cof}(\alpha))^V = \omega\}$ and let

$\mathcal{F} = \{A \in \mathcal{P}(\kappa) \cap \mathrm{HOD} \mid S \backslash A$ is not a stationary subset of λ in $V\}$.

Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ.

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Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ.

- Since λ is ω-strongly measurable in HOD, there exists $\gamma < \lambda$ such that in HOD:
  - $2^\gamma < \lambda$,
  - $\mathcal{P}(\lambda)/I$ is γ-cc where I is the ideal dual to $\mathcal{F}$.

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Assume λ is ω-strongly measurable in HOD. Then

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Let $S = \{\alpha < \lambda \,|\, (\mathrm{cof}(\alpha))^V = \omega\}$ and let

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Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ.

- Since λ is ω-strongly measurable in HOD, there exists $\gamma < \lambda$ such that in HOD:
  - $2^\gamma < \lambda$,
  - $\mathcal{P}(\lambda)/I$ is γ-cc where I is the ideal dual to $\mathcal{F}$.

Therefore by Lemma 29, the Boolean algebra

$$\langle \mathcal{P}(\lambda) \cap \mathrm{HOD} \rangle / I$$

is atomic. □

Theorem (31)

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) HOD *is a weak extender model for δ is supercompact.*

(2) *There exists a regular cardinal $\lambda > \delta$ which is not ω-strongly measurable in* HOD.

Theorem (31)

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) HOD *is a weak extender model for δ is supercompact.*

(2) *There exists a regular cardinal $\lambda > \delta$ which is not ω-strongly measurable in* HOD.

Proof.

By Theorem 9, (1) implies that for every singular cardinal $\gamma > \delta$,

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$$

and by Lemma 30, this implies (2).

Theorem (31)

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) HOD *is a weak extender model for δ is supercompact.*

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Proof.

By Theorem 9, (1) implies that for every singular cardinal $\gamma > \delta$,

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$$

and by Lemma 30, this implies (2).
Thus it suffices to show that (2) implies (1). We first prove:

Claim (1.1)

For each $\alpha > \delta$ there exists a regular cardinal $\lambda > \alpha$ such that λ is not ω-strongly measurable in HOD.

Proof continued

Fix a regular cardinal $\lambda_0 > \delta$ such that λ_0 is not ω-strongly measurable in HOD. Let $\kappa > \lambda_0$ be such that $\kappa > \alpha$ and

$$V_\kappa \prec_{\Sigma_2} V.$$

Thus

$$V_\kappa \models \text{“}\lambda_0 \text{ is not } \omega\text{-strongly measurable in } \mathrm{HOD}\text{”}.$$

Proof continued

Fix a regular cardinal $\lambda_0 > \delta$ such that λ_0 is not ω-strongly measurable in HOD. Let $\kappa > \lambda_0$ be such that $\kappa > \alpha$ and

$$V_\kappa \prec_{\Sigma_2} V.$$

Thus

$$V_\kappa \models \text{“}\lambda_0 \text{ is not } \omega\text{-strongly measurable in } \mathrm{HOD}\text{”}.$$

Since δ is extendible, there exists and elementary embedding

$$\pi : V_{\kappa+1} \to V_{\pi(\kappa)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \kappa > \alpha$. Thus

$$V_{\pi(\kappa)} \models \text{“}\pi(\lambda_0) \text{ is not } \omega\text{-strongly measurable in } \mathrm{HOD}\text{”}.$$

Proof continued

Fix a regular cardinal $\lambda_0 > \delta$ such that λ_0 is not ω-strongly measurable in HOD. Let $\kappa > \lambda_0$ be such that $\kappa > \alpha$ and

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Since δ is extendible, there exists and elementary embedding

$$\pi : V_{\kappa+1} \to V_{\pi(\kappa)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \kappa > \alpha$. Thus

$$V_{\pi(\kappa)} \models \text{“}\pi(\lambda_0) \text{ is not } \omega\text{-strongly measurable in } \mathrm{HOD}\text{”}.$$

But

$$(\mathrm{HOD})^{V_{\pi(\kappa)}} \subset \mathrm{HOD}$$

and so $\pi(\lambda_0)$ is not ω-strongly measurable in HOD. This proves Claim 1.1.

Proof continued

Fix $\kappa_0 > \delta$ and fix:

- $\kappa > \kappa_0$ such that $|V_\kappa| = \kappa$.
- A regular cardinal $\lambda_0 > 2^\kappa$ which is ω-strongly measurable in HOD.
- $\lambda > \lambda_0$ such that $V_\lambda \prec_{\Sigma_2} V$.

Thus $\lambda = |V_\lambda|$ and $\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$.

Proof continued

Fix $\kappa_0 > \delta$ and fix:

- $\kappa > \kappa_0$ such that $|V_\kappa| = \kappa$.
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Thus $\lambda = |V_\lambda|$ and $\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$.

Let $S = \{\alpha < \lambda_0 \mid \mathrm{cof}(\alpha) = \omega\}$. Thus:

- There exists a partition $\langle S_\alpha : \alpha < \kappa\rangle \in \mathrm{HOD}$ of S into stationary subsets of S.

Proof continued

Fix $\kappa_0 > \delta$ and fix:

- $\kappa > \kappa_0$ such that $|V_\kappa| = \kappa$.
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Thus $\lambda = |V_\lambda|$ and $\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$.

Let $S = \{\alpha < \lambda_0 \mid \mathrm{cof}(\alpha) = \omega\}$. Thus:

- There exists a partition $\langle S_\alpha : \alpha < \kappa \rangle \in \mathrm{HOD}$ of S into stationary subsets of S.

Let

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

be an elementary embedding such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.
Let

- $T = \pi(S)$
- $\langle T_\alpha : \alpha < \pi(\kappa) \rangle = \pi(\langle S_\alpha : \alpha < \kappa \rangle)$.

Proof continued

Thus:

- $\pi(\lambda_0)$ is a regular cardinal.
- $T = \{\alpha < \pi(\lambda_0) \mid \operatorname{cof}(\alpha) = \omega\}$,
- $\langle T_\alpha : \alpha < \pi(\kappa)\rangle$ is a partition of T into stationary sets.
- $\langle T_\alpha : \alpha < \pi(\kappa)\rangle \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$.

Proof continued

Thus:

- $\pi(\lambda_0)$ is a regular cardinal.
- $T = \{\alpha < \pi(\lambda_0) \mid \mathrm{cof}(\alpha) = \omega\}$,
- $\langle T_\alpha : \alpha < \pi(\kappa)\rangle$ is a partition of T into stationary sets.
- $\langle T_\alpha : \alpha < \pi(\kappa)\rangle \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$.

Let

$$\Theta = \sup\{\pi(\xi) \mid \xi < \lambda_0\}.$$

Thus $\Theta < \pi(\lambda_0)$. Let σ be the set of all $\alpha < \pi(\kappa)$ such that

$$T_\alpha \cap C \neq \emptyset$$

for all closed cofinal subsets of Θ. Therefore

$$\sigma = \{\pi(\alpha) \mid \alpha < \kappa\}.$$

But $\sigma \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$ since

$$\langle T_\alpha : \alpha < \pi(\kappa)\rangle \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

Proof continued

This proves that

$$\pi|\kappa \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

But there is a bijection

$$\rho : \kappa \to \mathrm{HOD} \cap V_\kappa$$

such that $\rho \in (\mathrm{HOD})^{V_\lambda}$ and so

$$\pi|(\mathrm{HOD} \cap V_\kappa) \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

Proof continued

This proves that

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But there is a bijection

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such that $\rho \in (\mathrm{HOD})^{V_\lambda}$ and so

$$\pi|(\mathrm{HOD} \cap V_\kappa) \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

Let U_0 be the normal fine ultrafilter on $\mathcal{P}_\delta(\kappa_0)$ given by π. Thus

- $\mathcal{P}_\delta(\kappa_0) \cap \mathrm{HOD} \in U_0$,
- $U_0 \cap \mathrm{HOD} \in (\mathrm{HOD})^{V_{\pi(\lambda)}} \subset \mathrm{HOD}$.

This proves that HOD is a weak extender model for δ is supercompact. $\square$

Theorem (HOD Dichotomy Theorem)

Suppose that δ is an extendible cardinal. Then one of the following hold.

(1) *Every regular cardinal above δ is ω-strongly measurable in HOD.*

(2) *No regular cardinal above δ is ω-strongly measurable in HOD.*

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Suppose that δ is an extendible cardinal. Then one of the following hold.

(1) *Every regular cardinal above δ is ω-strongly measurable in HOD.*

(2) *No regular cardinal above δ is ω-strongly measurable in HOD.*

Proof.

Assume toward a contradiction that $\kappa < \gamma$ are regular cardinals above δ such that κ is not ω-strongly measurable in HOD and that γ is ω-strongly measurable in HOD.

- Then there exists a stationary set $S \subset \{\alpha < \gamma \mid \mathrm{cof}(\alpha) = \omega\}$ such that
  - $S \in \mathrm{HOD}$,
  - $\mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma))$ is an ultrafilter

  where $\mathcal{F}$ is the club filter (of V) restricted to S.

Proof continued

Let

$$U = \mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma)).$$

Thus in HOD:

- U is a γ-complete, normal, uniform ultrafilter on γ.
- $S \in U$.

Proof continued

Let

$$U = \mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma)).$$

Thus in HOD:

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By Theorem 31

- HOD is a weak extender model for δ is supercompact.

Proof continued

Let

$$U = \mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma)).$$

Thus in HOD:

- U is a γ-complete, normal, uniform ultrafilter on γ.
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By Theorem 31

- HOD is a weak extender model for δ is supercompact.

Therefore, for each $\xi \in S$,

$$(\mathrm{cof}(\xi))^{\mathrm{HOD}} < \delta.$$

Proof continued

Let

$$U = \mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma)).$$

Thus in HOD:

- U is a γ-complete, normal, uniform ultrafilter on γ.
- $S \in U$.

By Theorem 31

- HOD is a weak extender model for δ is supercompact.

Therefore, for each $\xi \in S$,

$$(\mathrm{cof}(\xi))^{\mathrm{HOD}} < \delta.$$

But then

$$\{\xi < \gamma \mid (\mathrm{cof}(\xi))^{\mathrm{HOD}} < \xi\} \in U$$

and this contradicts that in HOD, U is a γ-complete, normal, uniform ultrafilter on γ. □

Theorem

Suppose there exists an extendible cardinal. Then there is a measurable cardinal in HOD.

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Proof.

Let δ be an extendible cardinal. By Lemma 30, we can reduce to the case that there is a regular cardinal above δ which is not ω-strongly measurable in HOD. But then by Theorem 31, HOD is a weak extender model for δ is supercompact and so there is a supercompact cardinal in HOD. $\square$

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Theorem

Suppose there exists an elementary embedding

$$j : V_{\kappa+\omega} \to V_{j(\kappa)+\omega}$$

with $\mathrm{CRT}(j) = \kappa$. *Then there is a measurable cardinal in* HOD.

The HOD Hypothesis and the HOD Conjecture

Definition (The HOD Hypothesis)

There exists a proper class of regular cardinals λ which are not ω-strongly measurable in HOD.

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1. *It is not known if there can exist 4 regular cardinals which are ω-strongly measurable in HOD.*
2. *Suppose γ is a singular strong limit cardinal of uncountable cofinality. It is not known if γ^+ can ever be ω-strongly measurable in HOD.*

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2. *Suppose γ is a singular strong limit cardinal of uncountable cofinality. It is not known if γ^+ can ever be ω-strongly measurable in HOD.*

Theorem (36)

Assume the HOD Hypothesis. Suppose that δ is an extendible cardinal.

- *Then HOD is a weak extender model for δ is supercompact.*

Theorem (37)

Assume the HOD *Hypothesis. Suppose that there is an extendible cardinal. Then there is an ordinal* λ *such that for all* $\gamma > \lambda$, *if*

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with $j(\lambda) = \lambda$ *then* $j \in \mathrm{HOD}$.

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is an elementary embedding with $j(\lambda) = \lambda$ *then* $j \in \mathrm{HOD}$.

Proof.

Let δ be an extendible cardinal and let $\lambda_0 = \delta^{+\omega}$ be the ω-th cardinal above δ. Clearly $(\mathrm{cof}(\lambda_0))^{\mathrm{HOD}} = \omega$. Further by Theorem 9 and Theorem 36,

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+.$$

Theorem (37)

Assume the HOD *Hypothesis. Suppose that there is an extendible cardinal. Then there is an ordinal* λ *such that for all* $\gamma > \lambda$*, if*

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Proof.

Let δ be an extendible cardinal and let $\lambda_0 = \delta^{+\omega}$ be the ω-th cardinal above δ. Clearly $(\mathrm{cof}(\lambda_0))^{\mathrm{HOD}} = \omega$. Further by Theorem 9 and Theorem 36,

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+.$$

Therefore if $\eta < \lambda_0^+$ then $(\mathrm{cof}(\eta))^{\mathrm{HOD}} < \lambda_0$. Let κ_0 be least such that

$$\{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\}$$

is stationary in λ_0^+.

Define $\lambda = \lambda_0 + \kappa_0$. We show that λ is as required.

Proof continued

Suppose $\gamma > \lambda$ and

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding such that $j(\lambda) = \lambda$.

- By Theorem 36, if $j|\delta$ is the identity then $j \in \mathrm{HOD}$.

Proof continued

Suppose $\gamma > \lambda$ and

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding such that $j(\lambda) = \lambda$.

- By Theorem 36, if $j|\delta$ is the identity then $j \in \mathrm{HOD}$.

Therefore we have only to prove that $j|\delta$ is the identity.
Since $\kappa_0 < \lambda_0$ and since $j(\lambda) = \lambda$,

- $j(\lambda_0) = \lambda_0$ and $j(\kappa_0) = \kappa_0$.

Clearly j induces canonically an elementary embedding

$$j^* : \langle H(\lambda_0^{++}) \rangle^{\mathrm{HOD}} \to \langle H(\lambda_0^{++}) \rangle^{\mathrm{HOD}}$$

with the property that $j|\lambda_0 = j^*|\lambda_0$.

Proof continued

Let

$$S = \{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\}.$$

Thus since S is stationary in λ_0^+ and since

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+,$$

There is a partition $\langle S_\alpha : \alpha < \lambda_0^+ \rangle \in \mathrm{HOD}$ of S into stationary sets.

Proof continued

Let

$$S = \{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\}.$$

Thus since S is stationary in λ_0^+ and since

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+,$$

There is a partition $\langle S_\alpha : \alpha < \lambda_0^+\rangle \in \mathrm{HOD}$ of S into stationary sets.

Let

$$\langle T_\beta : \beta < \lambda_0^+\rangle = j^*(\langle S_\alpha : \alpha < \lambda_0^+\rangle).$$

Note that if $\eta \in S$ and if η is closed under j^* then $j^*(\eta) = \eta$.

- This is because $(\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0$ and because $j^*(\kappa_0) = \kappa_0$.

Proof continued

Therefore for all $\beta < \lambda_0^+$:

Claim

$T_\beta \cap S$ is stationary in λ_0^+ if and only if $\beta = j^*(\alpha)$ for some $\alpha < \lambda_0^+$.

Proof continued

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Therefore:

Claim

$\{j^*(\alpha) \mid \alpha < \lambda_0^+\} \in \mathrm{HOD}$

- since $\{\beta < \lambda_0^+ \mid T_\beta \cap S \text{ is stationary in } \lambda_0^+\} \in \mathrm{HOD}$.

Proof continued

Therefore for all $\beta < \lambda_0^+$:

Claim

$T_\beta \cap S$ is stationary in λ_0^+ if and only if $\beta = j^*(\alpha)$ for some $\alpha < \lambda_0^+$.

Therefore:

Claim

$\{j^*(\alpha) \mid \alpha < \lambda_0^+\} \in \mathrm{HOD}$

- since $\{\beta < \lambda_0^+ \mid T_\beta \cap S \text{ is stationary in } \lambda_0^+\} \in \mathrm{HOD}$.

But by the elementarity of j^* and since $j^*(S) = S$, for all $\beta < \lambda_0^+$,

$$\mathrm{HOD} \models \text{“ } T_\beta \cap S \text{ is stationary in } \lambda_0^+ \text{ ”},$$

which implies (since $\{j^*(\alpha) \mid \alpha < \lambda_0^+\} \in \mathrm{HOD}$) that $j^*|\lambda_0^+$ is the identity. Thus $\mathrm{CRT}(j) > \delta$ and so by Theorem 13 and Theorem 36, $j \in \mathrm{HOD}$. □

Theorem (HOD Hypothesis)

Suppose that there exists an extendible cardinal. Then there is no sequence of non-trivial elementary embeddings,

$$j_i : \mathrm{HOD} \to \mathrm{HOD}$$

such that the direct limit,

$$\lim_{i<\omega} j_i \circ \cdots \circ j_0(\mathrm{HOD}),$$

is wellfounded.

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such that the direct limit,

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is wellfounded.

Proof.

Assume toward a contradiction that the direct limit is wellfounded. Then for every ordinal λ,

$$j_i(\lambda) = \lambda$$

for all sufficiently large $i < \omega$. Therefore by Theorem 37 and Kunen' Theorem, j_i must be the identity for all sufficiently large $i < \omega$. □

Theorem (HOD Hypothesis)

Suppose that there is exists an extendible cardinal. Let T be the Σ_2-theory of V with ordinal parameters. Then there is no non-trivial elementary embedding,

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T).$$

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Suppose that there is exists an extendible cardinal. Let T be the Σ_2-theory of V with ordinal parameters. Then there is no non-trivial elementary embedding,

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Proof.

By Theorem 37, there exists $\lambda \in \mathrm{Ord}$ such that for all $\gamma > \lambda$, if

$$k : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{k(\gamma)+1}$$

is an elementary embedding with $k(\lambda) = \lambda$, then $k \in \mathrm{HOD}$. Let λ_0 be the least such λ. Clearly λ_0 is definable in V and so λ_0 is definable in (HOD, T).

Proof continued

Suppose toward a contradiction that

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T)$$

is a non-trivial elementary embedding. Therefore $j(\lambda_0) = \lambda_0$ and so for all $\gamma > \lambda_0$,

$$j| \mathrm{HOD} \cap V_{\gamma+1} \in \mathrm{HOD},$$

which is a contradiction. $\square$

The HOD Conjecture and its consequences

Definition (HOD Conjecture)

The theory

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}$$

proves the HOD Hypothesis.

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Theorem (ZF)

Assume the HOD Conjecture. Suppose δ is an extendible cardinal. Then there is a transitive class $M \subseteq V$ such that:

(1) $M \models \mathrm{ZFC}$

(2) *M is $\Sigma_2(a)$-definable for some $a \in V_\delta$*

(3) *Every set of ordinals is $< \delta$-generic over M*

(4) $M \models$ *“δ is an extendible cardinal”*

Theorem (ZF)

Assume the HOD *Conjecture. Suppose* δ *is an extendible cardinal. Then for all* $\lambda > \delta$ *there is no non-trivial elementary embedding* $j : V_{\lambda+2} \to V_{\lambda+2}$.

Theorem (ZF)

Assume the HOD *Conjecture. Suppose* δ *is an extendible cardinal. Then for all* $\lambda > \delta$ *there is no non-trivial elementary embedding* $j : V_{\lambda+2} \to V_{\lambda+2}$.

Definition (Axiom of Choice Conjecture (ZF))

Suppose that δ is an extendible cardinal and that $G \subset \mathrm{Coll}(\omega, V_\delta)$ is V-generic. Then the Axiom of Choice holds in $V[G]$.

Theorem (ZF)

Assume the HOD *Conjecture. Suppose* δ *is an extendible cardinal. Then for all* $\lambda > \delta$ *there is no non-trivial elementary embedding* $j : V_{\lambda+2} \to V_{\lambda+2}$.

Definition (Axiom of Choice Conjecture (ZF))

Suppose that δ is an extendible cardinal and that $G \subset \mathrm{Coll}(\omega, V_\delta)$ is V-generic. Then the Axiom of Choice holds in $V[G]$.

Theorem (ZF)

Assume the HOD *Conjecture. Suppose that* δ *is an extendible cardinal. Then in* $L(\mathcal{P}(\mathrm{Ord}))$*:*

(1) δ *is an extendible cardinal.*

(2) *The Axiom of Choice Conjecture holds.*