The coding obstruction

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If one can prove:

Conjecture

Suppose δ is an extendible cardinal. Then there exists a weak extender model N for δ is supercompact such that

$$N \subseteq \mathrm{HOD}.$$

Then one verifies the HOD Conjecture.

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Then one verifies the HOD Conjecture.

Theorem (By Kunen's construction and analysis of $L[U]$)

Suppose that δ is a measurable cardinal. Then there exists a weak extender model N for δ is measurable such that $N \subseteq \mathrm{HOD}$.

- So one just needs to generalize Kunen's construction of $L[U]$ to the level of supercompact cardinals.

Notation for extenders

Suppose E is an extender and let

$$j_E : V \to M_E \cong \mathrm{Ult}(V, E)$$

be the ultrapower embedding.

1. $\kappa_E = \mathrm{CRT}(j_E)$ *and* $\kappa^*_E = j_E(\kappa_E)$.
2. $\nu_E = \sup\{\xi + 1 \mid \xi \neq j_E(f)(s)$ *for all* $s \in [\xi]^{<\omega}$, $f \in V\}$.

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4. $\mathrm{SP}(E) = \sup\{\alpha \mid j_E(\alpha) < \nu_E\}$.

- $\mathrm{SP}(E)$ is a cardinal and $\rho(E) \geq \kappa_E + 1$.

Definition

An extender E is ω-**huge** if

$$\rho(E) \geq \lambda$$

where $\lambda > \mathrm{CRT}(E)$ is least such that $j_E(\lambda) = \lambda$.

Iteration trees and iteration hypotheses

Definition

A **coarse premouse** is a pair (M, δ) such that M is transitive, $\delta \in M$, and:

1. $M \models \mathrm{ZC} + \Sigma_2\text{-Replacement}.$
2. Suppose that $F : M_\delta \to M \cap \mathrm{Ord}$ is definable from parameters in M, then F is bounded in M.
3. δ is strongly inaccessible in M.

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Definition

Suppose that (M, δ) is a coarse premouse. An **iteration tree**, $\mathcal{T}$, on (M, δ) of length η is a tree order $<_{\mathcal{T}}$ on η with minimum element 0 and which is a suborder of the standard order, together with a sequence

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

such that the following conditions hold.

Iteration tree conditions

(1) $M_0 = M$ and M_α is transitive for all $\alpha < \eta$,

(2) $j_{\gamma,\alpha} : M_\gamma \to M_\alpha$ for all $\gamma <_{\mathcal{T}} \alpha < \eta$,

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(2) $j_{\gamma,\alpha} : M_\gamma \to M_\alpha$ for all $\gamma <_{\mathcal{T}} \alpha < \eta$,

(3) Suppose that $\alpha + 1 < \eta$. Then $\alpha + 1$ has an immediate predecessor, α^*, in the tree order $<_{\mathcal{T}}$ and:

  (a) $E_\alpha \in j_{0,\alpha}(M \cap V_\delta)$ and

    - $M_\alpha \models$ " E_α is an extender which is not ω-huge ";

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   (a) $E_\alpha \in j_{0,\alpha}(M \cap V_\delta)$ and
   - $M_\alpha \models$ " E_α is an extender which is not ω-huge ";

   (b) If $\alpha^* < \alpha$ then
   - $\mathrm{SP}(E_\alpha) + 1 \leq \min\{\rho(E_\beta) \,\big|\, \alpha^* \leq \beta < \alpha\}$
   - guarantees that E_α is an M_{α^*}-extender;

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   - $\mathrm{SP}(E_\alpha) + 1 \leq \min\{\rho(E_\beta) \mid \alpha^* \leq \beta < \alpha\}$
   - guarantees that E_α is an M_{α^*}-extender;

   (c) $M_{\alpha+1} = \mathrm{Ult}(M_{\alpha^*}, E_\alpha)$ and

$$j_{\alpha^*,\alpha+1} : M_{\alpha^*} \to M_{\alpha+1}$$

   is the associated embedding.

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  is the associated embedding.

(4) If $0 < \beta < \eta$ is a limit ordinal then the set of α such that $\alpha <_{\mathcal{T}} \beta$ is cofinal in β and M_β is the limit of the M_α where $\alpha <_{\mathcal{T}} \beta$ relative to the embeddings; $j_{\alpha,\beta}$.

Definition

Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Suppose that $\theta \in \mathrm{Ord}$.

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Suppose that $\theta \in \mathrm{Ord}$.

- Then the iteration tree, $\mathcal{T}$, is a $(+\theta)$-**iteration tree** if for all $\alpha + 1 < \eta$,

  $$\sup\{\mathrm{SP}(E_\beta) \mid \beta^* \leq \alpha < \beta\} + \theta \leq \rho(E_\alpha)$$

  where for each $\beta + 1 < \eta$, β^* is the $\mathcal{T}$ predecessor of $\beta + 1$.

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  where for each $\beta + 1 < \eta$, β^* is the $\mathcal{T}$ predecessor of $\beta + 1$.

- If $\beta^* \leq \alpha < \beta$ then $\mathrm{SP}(E_\beta) + 1 \leq \rho(E_\alpha)$ by the definition of an iteration tree. Thus:
  - Every iteration tree is a $(+0)$-iteration tree.
  - Every finite iteration tree is a $(+1)$-iteration tree.

Definition

Suppose that (M, δ) is a coarse premouse. An **iteration strategy of order** $\omega_1 + 1$ for (M, δ) is a function I such that the following hold.

1. Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of limit length such that $\mathrm{LTH}(\mathcal{T}) \leq \omega_1$.
   - Then $\mathcal{T} \in \mathrm{dom}(I)$ and $I(\mathcal{T})$ is a maximal wellfounded branch of $\mathcal{T}$ of limit length.

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   - Then $\mathcal{T} \in \mathrm{dom}(I)$ and $I(\mathcal{T})$ is a maximal wellfounded branch of $\mathcal{T}$ of limit length.
2. Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of limit length such that $\mathrm{LTH}(\mathcal{T}) \leq \omega_1$. Suppose that for all limit $\eta < \mathrm{LTH}(\mathcal{T})$, $I(\mathcal{T}|\eta) = \{\xi < \eta \,|\, \xi <_{\mathcal{T}} \eta\}$.
   - Then $I(\mathcal{T})$ is a cofinal wellfounded branch of $\mathcal{T}$.

Definition

Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

The iteration tree $\mathcal{T}$ is **strongly closed** if for all $\alpha + 1 < \eta$:

1. $\mathcal{T}$ is a $(+1)$-iteration tree; and
2. $\mathrm{LTH}(E_\alpha)$ is strongly inaccessible in M_α and E_α is $\mathrm{LTH}(E_\alpha)$-strong in M_α.

- A strongly closed iteration tree is a $(+\theta)$-iteration tree for all θ less than the least measurable cardinal of M.

Definition

Suppose that (M, δ) is a coarse premouse. A strongly closed iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

on (M, δ) is a 0-**strongly closed iteration tree** if:

- For all $\alpha + 1 < \eta$, $\mathrm{LTH}(E_\alpha) \leq j_{E_\alpha}(\mathrm{CRT}(E_\alpha))$;

where for each $\alpha + 1 < \eta$,

$$j_{E_\alpha} : M_\alpha \to \mathrm{Ult}(M_\alpha, E_\alpha)$$

is the ultrapower embedding (as computed in M_α).

Definition

Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is a 0-strongly closed iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha\rangle.$$

Then

- $\mathcal{T}$ is **weakly maximal** if $\kappa^*_{E_\alpha} \leq \kappa_{E_\beta}$ for all $\alpha + 1 \leq_{\mathcal{T}} \beta + 1 < \eta$.

Definition

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Then

- $\mathcal{T}$ is **weakly maximal** if $\kappa^*_{E_\alpha} \leq \kappa_{E_\beta}$ for all $\alpha + 1 \leq_{\mathcal{T}} \beta + 1 < \eta$.
- $\mathcal{T}$ is **maximal** if $j_{E_\beta}(\kappa_{E_\beta}) \leq \kappa_{E_\alpha}$ for all $\beta < \alpha^* < \alpha + 1 < \eta$.

Definition (**Weak $(\omega_1 + 1)$-Iteration Hypothesis**)

Suppose that (M, δ) is a countable coarse premouse and that

$$\pi : M \to V_\Theta$$

is an elementary embedding.

- Then (M, δ) has an iteration strategy of order $\omega_1 + 1$ for 0-strongly closed maximal iteration trees on (M, δ).

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Definition (**Weak Unique Branch Hypothesis**)

Suppose that (V_Θ, δ) is a coarse premouse that $\mathcal{T}$ is a countable 0-strongly closed maximal iteration tree on (V_Θ, δ) of limit length.

- Then $\mathcal{T}$ has at most one cofinal wellfounded branch.

A counterexample

Theorem

Suppose that there is a supercompact cardinal. Then there exist an extender E such that

$$\nu_E = \left(2^{2^\kappa}\right)^{M_E}$$

where $\kappa = \kappa_E$ and $M_E = \mathrm{Ult}(V, E)$, and a 0-strongly closed maximal iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega, \gamma <_{\mathcal{T}} \alpha \rangle$$

on M_E of length ω such that:

(1) $\kappa_{E_\alpha} > \kappa_E^*$ *for all* $\alpha < \omega$,

(2) $\mathcal{T}$ *has two wellfounded branches.*

Another counterexample

Theorem

Suppose that there is a supercompact cardinal. Then there exist an extender E such that

$$\nu_E = \left(2^{2^\kappa}\right)^{M_E}$$

where $\kappa = \kappa_E$ and $M_E = \mathrm{Ult}(V, E)$, and a 0-strongly closed maximal iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega, \gamma <_{\mathcal{T}} \alpha \rangle$$

on M_E of length ω^2 such that:

(1) $\kappa_{E_\alpha} > \kappa_E^*$ *for all* $\alpha < \omega$,

(2) $\mathcal{T}$ *has only one cofinal branch and that branch is not wellfounded.*

Generalizing $L[U]$

One extender is not enough

- Suppose $E = \langle E_a : a \in [\eta]^{<\omega} \rangle$ is an extender. Then
  - $L[E]$ denotes $L[P_E]$ where $P_E = \{(a, B) \mid B \in E_a\}$.

Lemma

Suppose that E is an extender such that $\mathrm{LTH}(E) \leq j_E(\kappa)$ where $j_E : V \to M_E \cong \mathrm{Ult}(V, E)$. is the ultrapower embedding. Let U be the normal ultrafilter on κ given by j_E.

- *Then $L[E] = L[U]$*

- $U = \{A \subset \kappa \mid \kappa \in j_E(A)\}$
  - This is equivalent to E_a where $a = \{\kappa\}$.

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- $U = \{A \subset \kappa \,|\, \kappa \in j_E(A)\}$
  - This is equivalent to E_a where $a = \{\kappa\}$.

Using longer extenders does not really help.

Lemma

Suppose that F is an extender and $E = F|j_F(\xi)$ for some $\xi < j_F(\kappa_F)$ such that $j_F(\xi) + \omega < \rho(F)$.

- *Then in $L[E]$ there is no Woodin cardinal.*

Martin-Steel extender sequences

Martin-Steel extender models are of the form $L[\tilde{E}]$ where

$$\tilde{E} \subseteq (\mathrm{Ord} \times \mathrm{Ord}) \times V$$

is a predicate defining a sequence of extenders.

- The predicate $\tilde{E}$ is defined such that for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, the set,
$$\{(a, B) \in V \mid ((\alpha, \beta), (a, B)) \in \tilde{E}\},$$
defines an extender of length α which we denote by E^{α}_{β}.

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$$\{(a, B) \in V \mid ((\alpha, \beta), (a, B)) \in \tilde{E}\},$$
defines an extender of length α which we denote by E^{α}_{β}.
- For $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, $\tilde{E}|(\alpha, \beta)$ is the extender sequence given by restricting $\tilde{E}$ to the set of all (η, γ) such that $(\eta, \gamma) <_{\mathcal{L}} (\alpha, \beta)$ in the lexicographical ordering of pairs of ordinals:
$$\tilde{E}|(\alpha, \beta) = \{((\eta, \gamma), (a, B)) \in \tilde{E} \mid (\eta, \gamma) <_{\mathcal{L}} (\alpha, \beta)\}.$$

Martin-Steel extender sequences

Definition

A cardinal κ is **superstrong** if there is an elementary embedding
$$j : V \to M$$
such that $\mathrm{CRT}(j) = \kappa$ and such that $V_{j(\kappa)} \subset M$.

- Assuming the Weak $(\omega_1 + 1)$-Iteration Hypothesis and the existence of a supercompact cardinal, Martin-Steel extender sequences yield the generalization of $L[U]$ to the level of superstrong cardinals.

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Definition (Martin-Steel Extender Sequences)

An extender sequence,

$$\tilde{E} = \langle E^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{E}) \rangle$$

is a **Martin-Steel extender sequence** if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ the following conditions hold.

Coherence and Novelty

1. (Coherence) There exists an extender F such that
   - $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible and Mahlo.
   - $E_\beta^\alpha = F|\alpha$,
   - (shortness) $\alpha \leq j_F(\mathrm{CRT}(F))$,
   - $j_F(\tilde{E})|(\alpha+1,0) = \tilde{E}|(\alpha,\beta)$.

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   - $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.

2. (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^{\alpha}_{\beta^*} \cap L[\tilde{E}|(\alpha, \beta)] \neq E^{\alpha}_{\beta} \cap L[\tilde{E}|(\alpha, \beta)]$$

Initial Segment Condition

3. (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ is the critical point associated to E^{α}_{β}.

- Then there exists β^* such that $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^{\alpha}_{\beta}|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$

Doddages

Definition

A **Doddage** is a sequence $\tilde{\mathcal{E}}$ such that

$$\mathrm{dom}(\tilde{\mathcal{E}}) \subseteq \mathrm{Ord} \times \mathrm{Ord}$$

and such that for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, $\tilde{\mathcal{E}}(\alpha, \beta)$ is a set of extenders of length α.

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Definition

Suppose that $\tilde{\mathcal{E}}$ is a Doddage. Then $L[\tilde{\mathcal{E}}]$ denotes $L[P_{\tilde{\mathcal{E}}}]$ where $P_{\tilde{\mathcal{E}}}$ is the set of all (α, β, a, B) such that

1. $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$,
2. $a \in [\alpha]^{<\omega}$,
3. $B \in E_a$ for all $E \in \tilde{\mathcal{E}}(\alpha, \beta)$.

Suppose $\tilde{\mathcal{E}}$ is a Doddage. For each $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ we denote $\tilde{\mathcal{E}}(\alpha, \beta)$ by $\mathcal{E}^{\alpha}_{\beta}$.

Definition

A Doddage,

$$\tilde{\mathcal{E}} = \langle \mathcal{E}^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle$$

is a **Martin-Steel Doddage** if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and for each extender $E \in \mathcal{E}^{\alpha}_{\beta}$ the following conditions hold.

1. (Coherence) There exists an extender F such that
   - $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible and Mahlo,
   - $E = F|\alpha$,
   - (shortness) $\alpha \leq j_F(\mathrm{CRT}(F))$,
   - $j_F(\tilde{\mathcal{E}})|(\alpha+1, 0) = \tilde{\mathcal{E}}|(\alpha, \beta)$.

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   - $j_F(\tilde{\mathcal{E}})|(\alpha+1, 0) = \tilde{\mathcal{E}}|(\alpha, \beta)$.
2. (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\mathcal{E})$ and for all $E^* \in \mathcal{E}^{\alpha}_{\beta^*}$,

$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)] \neq E \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)]$$

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$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha,\beta)] \neq E \cap L[\tilde{\mathcal{E}}|(\alpha,\beta)]$$

3. (Initial Segment Condition) Suppose that

$$\mathrm{CRT}(E) < \alpha^* < \alpha,$$

Then there exists $(\alpha^*,\beta^*) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and there exists $E^* \in \mathcal{E}^{\alpha^*}_{\beta^*}$ such that

$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha^*+1,0)] = (E|\alpha^*) \cap L[\tilde{\mathcal{E}}|(\alpha^*+1,0)].$$

Definition

Suppose $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage.

- Then $\tilde{\mathcal{E}}$ is **good** if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E_0, E_1 \in \mathcal{E}^{\alpha}_{\beta}$, $E_0 \cap L[\tilde{\mathcal{E}}] = E_1 \cap L[\tilde{\mathcal{E}}]$.

Definition

Suppose $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage.

- Then $\tilde{\mathcal{E}}$ is **good** if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E_0, E_1 \in \mathcal{E}^\alpha_\beta$, $E_0 \cap L[\tilde{\mathcal{E}}] = E_1 \cap L[\tilde{\mathcal{E}}]$.

Theorem (Martin-Steel)

Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds and that $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage. Then $\tilde{\mathcal{E}}$ is good.

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Theorem (Martin-Steel)

Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds and that there is a supercompact cardinal. Then there exists a Martin-Steel Doddage $\tilde{\mathcal{E}}$ such that there is a superstrong cardinal in $L[\tilde{\mathcal{E}}]$.

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Theorem (Martin-Steel)

Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds and that there is a supercompact cardinal. Then there exists a Martin-Steel Doddage $\tilde{\mathcal{E}}$ such that there is a superstrong cardinal in $L[\tilde{\mathcal{E}}]$.

- This gives a generalization of Kunen's Theorem for $L[U]$ up to the level of superstrong cardinals
  - assuming that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds.

A stronger generalization of Kunen's Theorem

Theorem (24)

Suppose that the Weak (ω_1+1)*-Iteration Hypothesis holds. Suppose that* $\tilde{\mathcal{E}}_0$ *and* $\tilde{\mathcal{E}}_1$ *are Martin-Steel Doddages such that*

$$\mathrm{dom}(\tilde{\mathcal{E}}_0) = \mathrm{dom}(\tilde{\mathcal{E}}_1).$$

Then

$$L[\tilde{\mathcal{E}}_0] = L[\tilde{\mathcal{E}}_1],$$

and moreover for all $(\alpha,\beta) \in \mathrm{dom}(\tilde{\mathcal{E}}_0)$*, for all* $E_0 \in \tilde{\mathcal{E}}_0(\alpha,\beta)$*, for all* $E_1 \in \tilde{\mathcal{E}}_1(\alpha,\beta)$*,*

$$E_0 \cap L[\tilde{\mathcal{E}}_0] = E_1 \cap L[\tilde{\mathcal{E}}_1].$$

Martin-Steel extender sequences with long extenders

Definition

An extender sequence,

$$\tilde{E} = \langle E^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{E}) \rangle$$

is a **generalized Martin-Steel extender sequence** if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ the following hold.

1. (Coherence) There exists an extender F such that
   - $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible and Mahlo,
   - $E^{\alpha}_{\beta} = F|\alpha$,
   - $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.

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   - $E^\alpha_\beta = F|\alpha$,
   - $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.
2. (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^\alpha_{\beta^*} \cap L[\tilde{E}|(\alpha, \beta)] \neq E^\alpha_\beta \cap L[\tilde{E}|(\alpha, \beta)]$$

1. (Coherence) There exists an extender F such that
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$$E^\alpha_{\beta^*} \cap L[\tilde{E}|(\alpha,\beta)] \neq E^\alpha_\beta \cap L[\tilde{E}|(\alpha,\beta)]$$

3. (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

   where κ is the critical point associated to E^α_β.
   Then there exists β^* such that $(\alpha^*,\beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^*+1,0)] = (E^\alpha_\beta|\alpha^*) \cap L[\tilde{E}|(\alpha^*+1,0)].$$

Fast club forcing

For each strongly inaccessible cardinal δ, let $\mathbb{Q}_\delta$ be the following partial order (which adds a fast club at δ).

- Conditions are pairs (c, X) where c is a bounded closed subset of δ and X is a set of closed cofinal subsets of δ with $|X| < \delta$.

Suppose $(d, Y), (c, X) \in \mathbb{Q}_\delta$. Then $(d, Y) \leq (c, X)$ if the following hold.

1. $c = d \cap (\sup(c) + 1)$ and $d \setminus c \subseteq \cap X$,
2. $X \subseteq Y$.

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Suppose $(d, Y), (c, X) \in \mathbb{Q}_\delta$. Then $(d, Y) \leq (c, X)$ if the following hold.

1. $c = d \cap (\sup(c) + 1)$ and $d \backslash c \subseteq \cap X$,
2. $X \subseteq Y$.

- $\mathbb{Q}_\delta$ is $(<\delta)$-closed.
- Suppose $G \subsetneq \mathbb{Q}_\delta$ is V-generic and let

$$C_G = \cup\{c \mid (c, X) \in G\}.$$

  Then C_G is a closed cofinal subset of δ such that for all closed cofinal sets $D \subseteq \delta$ with $D \in V$, $C_G \backslash D$ is bounded in δ
  - C_G is a fast club in δ.

A useful lemma

Lemma (26)

Suppose κ is strongly inaccessible and $A \subseteq \kappa$. Suppose $G \subset \mathbb{Q}_\kappa$ is V-generic and in $V[G]$ there is a club $D \subseteq C_G$ such that

$$D \cap \gamma \in L[A]$$

for all $\gamma < \kappa$. Then $V_\kappa \subset L[A]$.

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Proof.

Fix a term τ for D. By the homogeneity of $\mathbb{Q}_\kappa$, we can suppose

$$1 \Vdash \text{“}\tau \cap \gamma \in L[A] \text{ for all } \gamma < \kappa\text{”}$$

and that

$$1 \Vdash \text{“}\tau \text{ is closed, cofinal in } C_G\text{”}.$$

Proof continued

For each $\gamma < \kappa$, let D_γ be the set of $(c, X) \in \mathbb{Q}_\kappa$ such that

- $\gamma < \max(c)$,
- for all $\alpha < \max(c)$, either $(c, X) \Vdash$ "$\alpha \in \tau$" or $(c, X) \Vdash$ "$\alpha \notin \tau$",
- $\{\alpha < \max(c) \mid (c, X) \Vdash \text{"}\alpha \in \tau\text{"}\}$ is cofinal in $\max(c)$.

Proof continued

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- $\{\alpha < \max(c) \,|\, (c, X) \Vdash \text{“}\alpha \in \tau\text{”}\}$ is cofinal in $\max(c)$.

Thus for each $\gamma < \kappa$, D_γ is dense in $\mathbb{Q}_\kappa$. Further

- if

  $$\langle (c_\alpha, X_\alpha) : \alpha < \eta \rangle$$

  is a decreasing sequence in D_γ where $\eta < \kappa$, then

  $$(c, X) \in D_\gamma$$

  where

  - $c = \left(\cup\{c_\alpha \,\middle|\, \alpha < \eta\}\right) \cup \{\sup\left(\cup\{c_\alpha \,\middle|\, \alpha < \eta\}\right)\}$,
  - $X = \cup\{X_\alpha \,\middle|\, \alpha < \eta\}$.

Proof continued

Let $\mathbb{D} = \{D_\gamma \mid \gamma < \kappa\}$. Thus:

- A filter $\mathcal{F} \subsetneq \mathbb{Q}_\kappa$ is $\mathbb{D}$-generic if and only if for each $\gamma < \kappa$ there exists $(c, X) \in D_0 \cap \mathcal{F}$ such that $\gamma < \sup(c)$.

If $\mathcal{F}$ is a $\mathbb{D}$-generic filter let $D_\mathcal{F}$ be the interpretation of τ by $\mathcal{F}$.

- $D_\mathcal{F}$ is closed cofinal in κ and for all $\gamma < \kappa$, $D_\mathcal{F} \cap \gamma \in L[A]$.

Proof continued

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If $\mathcal{F}$ is a $\mathbb{D}$-generic filter let $D_{\mathcal{F}}$ be the interpretation of τ by $\mathcal{F}$.

- $D_{\mathcal{F}}$ is closed cofinal in κ and for all $\gamma < \kappa$, $D_{\mathcal{F}} \cap \gamma \in L[A]$.

The key claim is the following.

Claim

For each $B \subseteq \kappa$, there exists a pair $(\mathcal{F}_0, \mathcal{F}_1)$ of $\mathbb{D}$-generic filters such that if

$$\langle \eta_\alpha : \alpha < \kappa \rangle$$

is the increasing enumeration of $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1}$ then for all $\alpha < \kappa$, $\alpha \in B$ if and only if

$$\min\{\eta \in D_{\mathcal{F}_0} \mid \eta_\alpha < \eta\} < \min\{\eta \in D_{\mathcal{F}_1} \mid \eta_\alpha < \eta\}.$$

Proof continued

Since:

- For all $\gamma < \kappa$, $(D_{\mathcal{F}_0} \cap \gamma, D_{\mathcal{F}_1} \cap \gamma) \in L[A]$,

the claim implies that for all $\gamma < \kappa$, $B \cap \gamma \in L[A]$.

Proof continued

Since:

- For all $\gamma < \kappa$, $(D_{\mathcal{F}_0} \cap \gamma, D_{\mathcal{F}_1} \cap \gamma) \in L[A]$,

the claim implies that for all $\gamma < \kappa$, $B \cap \gamma \in L[A]$.

The proof of the claim involves noting the following.

Suppose $(c_0, X_0) \in \mathbb{Q}_\kappa$ *and that either* $(c_0, X_0) \in \mathbb{D}$ *or* $c_0 = \emptyset$.

- *Then for each* $\eta < \kappa$ *such that* $\max(c_0) < \eta$*, there exists* $(c_1, X_1) \in \mathbb{D}$ *such that*
  - $(c_1, X_1) < (c_0, X_0)$,
  - $\eta < \max(c_1)$,
  - $c_1 \cap \eta = c_0$.

Proof continued

One uses this to construct decreasing sequences

$$\langle (c^0_\alpha, X^0_\alpha) : \alpha < \kappa \rangle$$

and

$$\langle (c^1_\alpha, X^1_\alpha) : \alpha < \kappa \rangle$$

of conditions in D_0 by induction on α such that for all α the following hold.

Proof continued

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of conditions in D_0 by induction on α such that for all α the following hold.

- $c^0_0 \cap c^1_0 = \emptyset$.
- $c^0_{\alpha+1} \cap c^1_{\alpha+1} = c^0_\alpha \cap c^1_\alpha$.

Proof continued

One uses this to construct decreasing sequences

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of conditions in D_0 by induction on α such that for all α the following hold.

- $c^0_0 \cap c^1_0 = \emptyset$.
- $c^0_{\alpha+1} \cap c^1_{\alpha+1} = c^0_\alpha \cap c^1_\alpha$.
- If $\alpha > 0$ and α is a limit then
  - $c^0_\alpha = \cup\{c^0_\beta \mid \beta < \alpha\} \cup \sup\left(\cup\{c^0_\beta \mid \beta < \alpha\}\right)$,
  - $c^1_\alpha = \cup\{c^1_\beta \mid \beta < \alpha\} \cup \sup\left(\cup\{c^1_\beta \mid \beta < \alpha\}\right)$,
  - $\max(c^0_\alpha) = \max(c^1_\alpha)$,
  - $\eta \in B$ if and only if

    $$\min(c^0_{\alpha+1} \backslash c^0_\alpha) < \min(c^1_{\alpha+1} \backslash c^1_\alpha)$$

    where $\alpha = \omega \cdot \eta$.

Proof continued

The filters

- $\mathcal{F}_0$ generated by $\{(c^0_\alpha, X^0_\alpha) : \alpha < \kappa\}$,
- $\mathcal{F}_1$ generated by $\{(c^1_\alpha, X^1_\alpha) : \alpha < \kappa\}$,

witness the claim since:

- $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1} = \{\max(c^0_\alpha) \mid \alpha \text{ is a nonzero limit ordinal}\}$.
- $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1} = \{\max(c^1_\alpha) \mid \alpha \text{ is a nonzero limit ordinal}\}$. $\square$

Weakly Σ_2-definable sequences

Definition

A sequence $N = \langle N_\alpha : \alpha \in \mathrm{Ord} \rangle$ is **weakly Σ_2-definable** if there is a formula $\varphi(x)$ such that:

1. for all $\beta < \eta_1 < \eta_2 < \eta_3$, if $(N_\varphi)^{V_{\eta_1}} | \beta = (N_\varphi)^{V_{\eta_3}} | \beta$ then
$$(N_\varphi)^{V_{\eta_1}} | \beta = (N_\varphi)^{V_{\eta_2}} | \beta = (N_\varphi)^{V_{\eta_3}} | \beta;$$
2. for all $\beta \in \mathrm{Ord}$, $N | \beta = (N_\varphi)^{V_\eta} | \beta$ for all sufficiently large η,

where for all γ,

- $(N_\varphi)^{V_\gamma} = \{a \in V_\gamma \mid V_\gamma \models \varphi[a]\}$.

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where for all γ,

- $(N_\varphi)^{V_\gamma} = \{a \in V_\gamma \mid V_\gamma \models \varphi[a]\}$.

Definition

Suppose that $N \subset V$ is an inner model and $N \models \mathrm{ZFC}$.

- Then N is **weakly Σ_2-definable** if the sequence $\langle N \cap V_\alpha : \alpha \in \mathrm{Ord} \rangle$ is weakly Σ_2-definable.

If $N \subset V$ is a class which is Σ_2-definable then the sequence

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For each $\alpha \in \mathrm{Ord}$, let T_α be the Σ_2-theory of V with parameters from V_α. Then the sequence

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The increasing enumeration $\langle \delta_\alpha : \alpha \in \mathrm{Ord} \rangle$ of all supercompact cardinals is weakly Σ_2-definable.

Definition

Suppose that N is a transitive inner model of ZFC which is is weakly Σ_2-definable and $V_\delta \prec_{\Sigma_2} V$. Then $(N)^{V_\delta}$ denotes the union of the sequence

$$\langle N^*_\alpha : \alpha < \delta \rangle = (N_\varphi)^{V_\delta}$$

where φ is a formula which witnesses that

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Claim

This is well-defined in the sense that it does not depend on the choice of the formula φ which witnesses that $\langle N \cap V_\alpha : \alpha \in \mathrm{Ord} \rangle$ is weakly Σ_2-definable.

Definition

A cardinal κ is a **strong cardinal** if for every λ there is an elementary embedding $j : V \to M$ such that $\mathrm{CRT}(j) = \kappa$ such that $j(\kappa) > \lambda$ and such that $V_\lambda \subset M$.

- If κ is a supercompact cardinal then κ is a strong cardinal.

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- If κ is a supercompact cardinal then κ is a strong cardinal.

Lemma (31)

Suppose that

$$N = \langle N_\alpha : \alpha \in \mathrm{Ord} \rangle$$

is weakly Σ_2-definable and δ is a strong cardinal.

- *Then $N \cap V_\delta = (N)^{V_\delta}$.*

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- *Then $N \cap V_\delta = (N)^{V_\delta}$.*

Proof.

Let $\varphi(x)$ be a formula which witnesses that N is weakly Σ_2-definable.

Proof continued

Assume toward a contradiction that $N|\delta \neq (N)^{V_\delta}$.

- Then there exists $\eta > \delta$ and $\beta < \delta$ such that

$$N|\beta = (N_\varphi)^{V_\eta}|\beta \neq (N_\varphi)^{V_\delta}|\beta.$$

Since δ is a strong cardinal, $V_\delta \prec_{\Sigma_2} V$ and so there exists $\beta < \eta_0 < \delta$ such that

$$N|\beta = (N_\varphi)^{V_{\eta_0}}|\beta.$$

But then

1. $\beta < \eta_0 < \delta < \eta$,
2. $(N_\varphi)^{V_{\eta_0}}|\beta = (N_\varphi)^{V_\eta}$,
3. $(N_\varphi)^{V_{\eta_0}}|\beta \neq (N_\varphi)^{V_\delta}|\beta$,

which is a contradiction. □

Lemma (32)

Suppose that N is a transitive inner model of ZFC, N is weakly Σ_2-definable, δ is an extendible cardinal, and that

$$V_\delta \subset N.$$

Then $V = N$.

Lemma (32)

Suppose that N is a transitive inner model of ZFC, N is weakly Σ_2-definable, δ is an extendible cardinal, and that

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Then $V = N$.

Proof.

Let φ be a formula which witnesses that

$$\langle N \cap V_\alpha : \alpha \in \mathrm{Ord} \rangle$$

is weakly Σ_2-definable.

Proof continued

Since δ is a strong cardinal, by Lemma 31,

$$\langle V_\alpha : \alpha < \delta \rangle = (N_\varphi)^{V_\delta}.$$

Since δ is an extendible cardinal, for a proper class of κ,

$$V_\delta \prec V_\kappa$$

and so for a proper class of κ,

$$\langle V_\alpha : \alpha < \kappa \rangle = (N_\varphi)^{V_\kappa}.$$

Therefore

$$\langle V_\alpha : \alpha \in \mathrm{Ord} \rangle = \langle N \cap V_\alpha : \alpha \in \mathrm{Ord} \rangle$$

and so $V = N$. □

Theorem (33)

Suppose that there is an extendible cardinal. Then there is a class-generic extension $V[G]$ of V in which the following hold.

(1) $V[G] = (\mathrm{HOD})^{V[G]}$ *and* $\mathbb{R}^{V[G]} = \mathbb{R}^V$.

(2) *There is an extendible cardinal.*

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(3) *Suppose* $\mathbb{E} \subset \mathrm{Ord}$ *and* δ *are such that the following hold.*

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(3) *Suppose* $\mathbb{E} \subset \mathrm{Ord}$ *and* δ *are such that the following hold.*
   - (a) $L[\mathbb{E}]$ *is weakly* Σ_2*-definable,*
   - (b) δ *is an extendible cardinal,*
   - (c) *Let* $X \subset \delta$ *be the set of all* $\kappa < \delta$ *such that there is an elementary embedding,*

$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$

with $\mathrm{CRT}(j) = \kappa$ *and* $j(\kappa) = \delta$, *where* λ *is the least strongly inaccessible cardinal above* κ.

- *Then there exists* $Y \subset X$ *such that* $Y \cap \xi \in L[\mathbb{E}]$ *for all* $\xi < \delta$ *and such that*

$$\sup(Y) = \sup(X) = \delta.$$

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   - *Then there exists* $Y \subset X$ *such that* $Y \cap \xi \in L[\mathbb{E}]$ *for all* $\xi < \delta$ *and such that*

$$\sup(Y) = \sup(X) = \delta.$$

Then $L[\mathbb{E}] = V[G]$.

Proof.

Let $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle$ be the backward Easton iteration:

1. If α is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\alpha}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{B} * \mathbb{Q}$$

   where $\mathbb{B}$ adds a Cohen generic subset to α^+ and $\mathbb{Q}$ is the fast-club forcing $\mathbb{Q}_\gamma$ defined in $V^{\mathbb{P}_\alpha * \mathbb{B}}$ with $\gamma = \alpha$.

Proof.

Let $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle$ be the backward Easton iteration:

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   where $\mathbb{B}$ adds a Cohen generic subset to α^+ and $\mathbb{Q}$ is the fast-club forcing $\mathbb{Q}_\gamma$ defined in $V^{\mathbb{P}_\alpha * \mathbb{B}}$ with $\gamma = \alpha$.

2. If $\alpha = \beta + 1$ and β is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\beta}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{H}$$

   where $\mathbb{Q}$ codes $(G_\alpha, V_{\alpha+1}, \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle)$ into the powerset function before the next strongly inaccessible cardinal above β and where $\mathbb{H}$ is (β^+)-closed in $V^{\mathbb{P}_\alpha}$.

   - The set being coded is naturally a set of ordinals by the definition of of $\mathbb{P}_{\beta+1}$ as the iteration $\mathbb{P}_\beta * \mathbb{B} * \mathbb{Q}$ and so $\mathbb{H}$ can be chosen canonically.

Proof.

Let $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle$ be the backward Easton iteration:

1. If α is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\alpha}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{B} * \mathbb{Q}$$

   where $\mathbb{B}$ adds a Cohen generic subset to α^+ and $\mathbb{Q}$ is the fast-club forcing $\mathbb{Q}_\gamma$ defined in $V^{\mathbb{P}_\alpha * \mathbb{B}}$ with $\gamma = \alpha$.

2. If $\alpha = \beta + 1$ and β is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\beta}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{H}$$

   where $\mathbb{Q}$ codes $(G_\alpha, V_{\alpha+1}, \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle)$ into the powerset function before the next strongly inaccessible cardinal above β and where $\mathbb{H}$ is (β^+)-closed in $V^{\mathbb{P}_\alpha}$.
   - The set being coded is naturally a set of ordinals by the definition of of $\mathbb{P}_{\beta+1}$ as the iteration $\mathbb{P}_\beta * \mathbb{B} * \mathbb{Q}$ and so $\mathbb{H}$ can be chosen canonically.

3. Otherwise $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$.

Proof continued

Let G be V-generic for the backward Easton iteration

$$\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle.$$

By standard lifting arguments:

- Every extendible cardinal of V remains extendible in $V[G]$.

Proof continued

Let G be V-generic for the backward Easton iteration

$$\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle.$$

By standard lifting arguments:

- Every extendible cardinal of V remains extendible in $V[G]$.

We note that the following must hold in $V[G]$ where for each strongly inaccessible Mahlo cardinal γ of $V[G]$, C_γ is the fast club added by G at stage $\gamma + 1$.

Claim (Club Coherence)

Suppose that

$$\pi : V[G]_{\kappa+1} \to V[G]_{\pi(\kappa)+1}$$

is an elementary embedding such that $\mathrm{CRT}(\pi) < \kappa$ *and such that* κ *is strongly inaccessible in* $V[G]$. *Let* $\gamma = \mathrm{CRT}(\pi)$. *Then*

- $\pi(C_\gamma) = C_{\pi(\gamma)}$ *and* $C_{\pi(\gamma)} \cap \gamma = C_\gamma$.

Proof continued

We have:

- $X \subset \delta$ is the set of all $\kappa < \delta$ such that there is an elementary embedding,
$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$
with $\mathrm{CRT}(j) = \kappa$ and $j(\kappa) = \delta$ where λ is the least strongly inaccessible cardinal above κ.

Proof continued

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- $X \subset \delta$ is the set of all $\kappa < \delta$ such that there is an elementary embedding,
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Therefore by the Club Coherence Claim,

- $X \subset C_\delta$, where C is the fast-club added by G at stage δ.

Proof continued

We have:

- $X \subset \delta$ is the set of all $\kappa < \delta$ such that there is an elementary embedding,
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Therefore by the Club Coherence Claim,

- $X \subset C_\delta$, where C is the fast-club added by G at stage δ.

Thus:

Claim (1)

Y is a cofinal subset of C_δ such that $Y \cap \xi \in L[\mathbb{E}]$ for all $\xi < \delta$.

Proof continued

Since $\mathbb{E}$ is weakly Σ_2-definable in $V[G]$ and since δ is a strong cardinal in $V[G]$, by Lemma 31:

- $L[\mathbb{E}] \cap V[G]_\delta = (L[\mathbb{E}])^{V[G]_\delta}$.

Further since δ is strongly inaccessible and Mahlo in $V[G]$,

- $V[G]_\delta \subset V[G|\delta]$.

Proof continued

Since $\mathbb{E}$ is weakly Σ_2-definable in $V[G]$ and since δ is a strong cardinal in $V[G]$, by Lemma 31:

- $L[\mathbb{E}] \cap V[G]_\delta = (L[\mathbb{E}])^{V[G]_\delta}$.

Further since δ is strongly inaccessible and Mahlo in $V[G]$,

- $V[G]_\delta \subset V[G|\delta]$.

Therefore by Lemma 26 and Claim (1),

$$V[G]_\delta \subset L[\mathbb{E}].$$

But then by Lemma 32, $V[G] = L[\mathbb{E}]$. $\square$

Applications

Theorem

Suppose that $V = \mathrm{HOD}$ and that there is an extendible cardinal.

- *Then there is a generalized Martin-Steel extender sequence $\tilde{E}$ such that:*
  - *$\tilde{E}$ is Σ_2-definable.*
  - *For each $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, $\alpha \leq \kappa^*_{E^\alpha_\beta} + 1$.*
  - *$V = L[\tilde{E}]$.*

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  - *$V = L[\tilde{E}]$.*

A natural conjecture

We just need more conditions beyond Novelty and the Initial Segment Condition.

Theorem

Assume that there is an extendible cardinal. Then there is a class-generic extension $V[G]$ of V in which the following hold.

(1) $V[G] = (\mathrm{HOD})^{V[G]}$.

(2) *There is an extendible cardinal.*

(3) *Suppose that $\tilde{E}$ is a generalized Martin-Steel extender sequence such that $\tilde{E}$ is Σ_2-definable and such that*

$$V[G] \neq L[\tilde{E}].$$

Then for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$:

- *If κ_{E_α} is an extendible cardinal in $V[G]$ then $\alpha \leq \kappa^*_{E_\alpha} + 1$.*

Proof.

Let $V[G]$ be the generic extension given by Theorem 33.
Suppose $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$, $\kappa_{E^{\alpha}_{\beta}}$ is a strong cardinal of $V[G]$, and that

$$\alpha > \kappa^{*}_{E^{\alpha},\beta} + 1.$$

Let $\delta = \kappa_{E^{\alpha}_{\beta}}$ and $X \subset \delta$ be the set of all $\kappa < \delta$ such that there is an elementary embedding,

$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$

with $\mathrm{CRT}(j) = \kappa$ and $j(\kappa) = \delta$, where λ is the least strongly inaccessible cardinal above κ.
Let

$$Y = \{\kappa_{E^{\delta+1}_{\eta}} \mid (\delta+1, \eta) \in \mathrm{dom}(\tilde{E}) \text{ and } \delta = \kappa^{*}_{E^{\delta+1}_{\eta}}\}.$$

By the Novelty and Initial Segment Condition

- $\sup(Y) = \delta$.

By the Coherence Condition, $Y \subset X$. Therefore by Theorem 33,

- $V = L[\tilde{E}]$. □

Lemma

Suppose that $\tilde{E}$ is a generalized Martin-Steel extender sequence,

$$(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$$

and $\alpha \leq \kappa^*_E + 1$. Then

$$(\rho(E))^{L[\tilde{E}]} \leq (\nu_E)^{L[\tilde{E}]} \leq (\kappa^*_E)^{L[\tilde{E}]}.$$

where E is the $L[\tilde{E}]$-extender given by E^α_β.

Lemma

Suppose that $\tilde{E}$ is a generalized Martin-Steel extender sequence,

$$(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$$

and $\alpha \leq \kappa_E^* + 1$. Then

$$(\rho(E))^{L[\tilde{E}]} \leq (\nu_E)^{L[\tilde{E}]} \leq (\kappa_E^*)^{L[\tilde{E}]}.$$

where E is the $L[\tilde{E}]$-extender given by E_β^α.

Summary

There is no direct generalization of Kunen's construction to $L[U]$ which can reach beyond the level of superstrong cardinals.