A failure of comparison

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Theorem

Assume that there is an extendible cardinal. Then there is a class-generic extension $V[G]$ of V in which the following hold.

(1) $V[G] = (\mathrm{HOD})^{V[G]}$.

(2) *There is an extendible cardinal.*

(3) *Suppose that $\tilde{E}$ is a generalized Martin-Steel extender sequence such that $\tilde{E}$ is Σ_2-definable and such that*

$$V[G] \neq L[\tilde{E}].$$

Then for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$:

- *If κ_{E_α} is an extendible cardinal in $V[G]$ then $\alpha \leq \kappa^*_{E_\alpha} + 1$.*

Thus

- There is really no direct generalization of Kunen's construction of $L[U]$ which can reach levels past superstrong cardinals.

Partial Extender Models

Rudimentary closure

Definition

A transitive set M is **rudimentarily closed** if

1. for all $a, b \in M$, $\{a\} \in M$, $a \times b \in M$, and $\cup a \in M$,
2. for all $a \in M$ if $b \subset a$ is Σ_0-definable with parameters from M then $b \in M$.

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Suppose P is a set. Then $J_\alpha[P]$ is defined by induction on α as follows.

1. $J_0[P] = \emptyset$,
2. $J_{\alpha+1}[P] = M$ where M is the smallest transitive rudimentarily closed set such that $J_\alpha[P] \in M$ and such that for each $b \in M$, $P \cap b \in M$.
3. $J_\alpha[P] = \cup\{J_\beta[P] \mid \beta < \alpha\}$ if $\alpha > 0$ and α is a limit ordinal.

- If $J_\alpha[P] \models \mathrm{ZF}\backslash\mathrm{Powerset}$ then $J_\alpha[P] \models \mathrm{Axiom\ of\ Choice}$.

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Suppose $\mathbb{E} = \langle E_\alpha : \alpha \in \mathrm{dom}(\mathbb{E}) \rangle$ is a sequence of partial extenders and that for all $\alpha \in \mathrm{dom}(\mathbb{E})$, $\mathrm{LTH}(E_\alpha) \leq \alpha$. Then for all $\eta \in \mathrm{Ord}$,

$$J_\eta^{\mathbb{E}} = J_\eta[P_{\mathbb{E}}]$$

where $P_{\mathbb{E}} = \{(\alpha, a, x) \mid \alpha \in \mathrm{dom}(\mathbb{E}), (a, x) \in E_\alpha\}$.

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$$J^{\mathbb{E}}_\eta = J_\eta[P_{\mathbb{E}}]$$

where $P_{\mathbb{E}} = \{(\alpha, a, x) \mid \alpha \in \mathrm{dom}(\mathbb{E}), (a, x) \in E_\alpha\}$.

Definition

Suppose that $\mathbb{E}$ is a partial extender sequence and $\alpha \in \mathrm{Ord}$. Then $J^{\mathbb{E}}_\alpha$ is **strongly acceptable** if for all $\beta < \alpha$ and for all $\kappa < \beta$, if

$$\mathcal{P}(\kappa) \cap J^{\mathbb{E}}_\beta \neq J^{\mathbb{E}}_{\beta+1}$$

then $|J^{\mathbb{E}}_\beta| \leq \kappa$ in $J^{\mathbb{E}}_{\beta+1}$.

Suppose that M is transitive, $M \models \mathrm{ZFC} \backslash \mathrm{Powerset}$ and that E is an M-extender. Let $j_E : M \to N \cong \mathrm{Ult}(M, E)$ be the ultrapower embedding.

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Definition

1. $\kappa_E = \mathrm{CRT}(E)$ and $\kappa_E^* = j_E(\kappa_E)$.
2. An ordinal $\xi < \mathrm{LTH}(E)$ is a **generator** of E if for all $f \in M$, for all $a \in [\xi]^{<\omega}$,
$$j_E(f)(a) \neq \xi.$$

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4. ι_E is the least cardinal γ of M such that $\nu_E \leq j_E(\gamma)$:
   - $\iota_E = \mathrm{SP}(E)$.

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4. ι_E is the least cardinal γ of M such that $\nu_E \leq j_E(\gamma)$:
    - $\iota_E = \mathrm{SP}(E)$.
5. F is the **Jensen completion** of $E|\nu_E$ if F is the M- extender of length η given by j_E where
$$\eta = ((j_E(\iota_E))^+)^N.$$

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- Then $\mathrm{SP}^*(E)$ is the set of M-cardinals γ such that
  1. $\sup(j_E[\gamma]) < j_E(\gamma)$,
  2. There is a generator ξ of E such that $\sup(J_E[\gamma]) < \xi < j_E(\gamma)$.

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- $\mathrm{SP}^*(E)$ is the set of M-cardinals γ such that for some

$$a \in [\mathrm{LTH}(E)]^{<\omega},$$

the M-ultrafilter given by E_a is a uniform M-ultrafilter on $[\gamma]^{|a|}$.

Definition

Suppose that $\mathbb{E}$ is a partial extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$. Then $\mathbb{E}$ is a **good partial extender sequence at** α if the following hold where E is the partial extender $\mathbb{E}_\alpha$.

1. $J_\alpha^{\mathbb{E}}$ is strongly acceptable and $J_\alpha^{\mathbb{E}} \models \mathrm{ZFC} \setminus \mathrm{Powerset}$.
2. E is an $J_\alpha^{\mathbb{E}}$-extender.

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3. (Indexing) E is the Jensen completion of $E|\nu_E$ and $\alpha = \mathrm{LTH}(E)$.
4. (Coherence) Let

$$j_E : J_\alpha^{\mathbb{E}} \to \mathrm{Ult}(J_\alpha^{\mathbb{E}}, E)$$

be the ultrapower embedding. Then

$$j_E(\mathbb{E}|\alpha)|(\alpha + 1) = \mathbb{E}|\alpha.$$

Comparison by least disagreement

Definition

Suppose $M \models \mathrm{ZFC}$, M is transitive, $\mathbb{E}$ is a sequence of partial extenders from M, and $\delta < \lambda < \mathrm{Ord}^M$.

- Then δ is witnessed by the partial extenders on the sequence $\mathbb{E}$ to be λ-supercompact in M if there exists $\alpha \in \mathrm{dom}(\mathbb{E})$ such that
  1. E is an M-extender,
  2. $\kappa_E = \delta$ and $\lambda \leq \iota_E$,
  3. $j_E[\lambda] \in M_E$

  where E is the partial extender $\mathbb{E}_\alpha$ and where

  $$j_E : M \to M_E \cong \mathrm{Ult}(M, E)$$

  is the ultrapower embedding.

We consider transitive structures of the form

$$(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$$

such that the following hold for all $\beta \in \mathrm{dom}(\mathbb{E})$ such that

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Let

1. $\iota_E = \mathrm{SP}(E)$,
2. $\gamma = (\iota_E^+)^{\mathcal{M}}$,
3. $j_E : \mathcal{M} \to \mathcal{M}_E \cong \mathrm{Ult}(\mathcal{M}, E)$ be the ultrapower embedding.

The conditions

(1) (**Suitability Condition**) $\iota_E < \kappa_E^*$ and no $\delta < \kappa_E$ is witnessed to be $(<\kappa_E)$-supercompact in $\mathcal{M}$ by $\mathbb{E}$.

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- $(\mathrm{cof}(\delta))^{\mathcal{M}} < \kappa_E$.
- Either $\iota_E = (\delta^+)^{\mathcal{M}}$ or $\mathbb{E}$ does not witness in $\mathcal{M}$ that κ_E is $(\delta^+)^{\mathcal{M}}$-supercompact.

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(3) (**Second Supercompactness Condition**) Suppose $\iota_E = \lambda^+$, $(\mathrm{cof}(\lambda))^{\mathcal{M}} \geq \kappa_E$, and $\mathbb{E}$ witnesses in $\mathcal{M}$ that κ_E is λ-supercompact.

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- Then $j_E[\iota_E] \in \mathcal{M}_E$ and for some $\xi \in \mathrm{Ord}^{\mathcal{M}}$:
  - (**Largest Generator Condition**) $\nu_E < j_E(\iota_E)$ and $\nu_E = \xi + 1$,

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    - (**First Initial Segment Condition**) $E|\eta \in \mathcal{M}_E$ for all $\eta < \xi$,
    - (**Second Initial Segment Condition**) if $E|\xi \notin \mathcal{M}_E$ then $(\mathrm{cof}(\xi))^{\mathcal{M}_E} < j_E(\kappa_E)$.

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Thus we are assuming that Jensen indexing is being used and that $\mathcal{M}|\beta$ makes sense.

- If $\mathcal{M}$ is of the form of $L[\mathbb{E}]$ then this is immediate, but we are not assuming that $\mathcal{M}$ has this form.

But with notation as above and by any reasonable notion of coherence

$$\mathcal{M}|\beta = \mathrm{Ult}(\mathcal{M}, E)|\beta.$$

Note β is a successor cardinal in $\mathrm{Ult}(\mathcal{M}, E)$. Thus

- $\mathrm{Ult}(\mathcal{M}, E)|\beta$ makes perfect sense by setting

$$\mathrm{Ult}(\mathcal{M}, E)|\beta = (H(\beta))^{\mathrm{Ult}(\mathcal{M}, E)}$$

  if $\mathcal{M}$ were simply a transitive set, and making the obvious adjustments otherwise.

Other conditions

- We assume there is a wellordering $<_{\mathcal{M}}$ of length $\mathrm{Ord}^{\mathcal{M}}$ such that for all uncountable regular cardinals γ of $\mathcal{M}$,

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    where $\mathcal{M}$ itself is a structure with additional predicates including the wellordering, $<_{\mathcal{M}}$.

Thus:

- Every element $a \in \mathcal{M}$ be definable in the structure

$$(\mathcal{M}, \mathbb{E})$$

  from ordinal parameters.

Finitely generated structures

Definition

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$.

1. $(\mathcal{M}, \mathbb{E})$ is finitely generated if for some $a \in \mathcal{M}$, every element $b \in \mathcal{M}$ is definable in $(\mathcal{M}, \mathbb{E})$ from a.
2. $X \prec (\mathcal{M}, \mathbb{E})$ is **finitely generated** if for some $a \in \mathcal{M}$, X is the set of all $b \in \mathcal{M}$ such that b is definable in $(\mathcal{M}, \mathbb{E})$ from a.

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Clearly,

- $X \prec (\mathcal{M}, \mathbb{E})$ is finitely generated if and only if $(\mathcal{M}_X, \mathbb{E}_X)$ is finitely generated where $(\mathcal{M}_X, \mathbb{E}_X)$ is the transitive collapse of X.
- Since every element $a \in \mathcal{M}$ is definable in the structure

$$(\mathcal{M}, \mathbb{E})$$

from ordinal parameters, every $a \in \mathcal{M}$ belongs to a $\subseteq$-least finitely generated elementary substructure of $(\mathcal{M}, \mathbb{E})$.

Backgrounding

Definition (12)

Suppose $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ and that $(\mathcal{M}, \mathbb{E})$ is transitive.

1. $(\mathcal{M}, \mathbb{E})$ is **weakly backgrounded at** κ if for all $\mathcal{M}$-extenders E given by $\mathbb{E}$ with $\kappa = \kappa_E$, if $\kappa_E < \gamma$,

$$j_E[\gamma] \in \mathcal{M}_E \cong \mathrm{Ult}(\mathcal{M}, E),$$

   and if U is the normal measure in $\mathcal{M}$ on $\mathcal{P}_\kappa(\gamma)$ given by E, then κ is a cardinal in V which is γ-supercompact in V and there is a normal fine κ-complete ultrafilter U^* on $\mathcal{P}_\kappa(\gamma)$ such that $U = U^* \cap \mathcal{M}$.

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2. $(\mathcal{M}, \mathbb{E})$ is **weakly backgrounded** if $(\mathcal{M}, \mathbb{E})$ is weakly backgrounded at κ for all $\kappa \in \mathrm{Ord}^{\mathcal{M}}$.

Semi-iterations

Definition (13)

Suppose $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ and $(\mathcal{M}, \mathbb{E})$ is transitive.

- A **semi-iteration** of $(\mathcal{M}, \mathbb{E})$ is a continuous (linearly) directed system

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

such that the following hold for all $\alpha < \beta < \eta$.

Semi-iteration conditions where $\alpha < \beta \leq \eta$

(1) E_α is an $\mathcal{N}_\alpha$-extender, $\mathcal{N}_{\alpha+1} = \mathrm{Ult}(\mathcal{N}_\alpha, E_\alpha)$, and $\pi_{\alpha,\alpha+1}$ is the ultrapower embedding.

(2) $(\mathcal{N}_0, \mathbb{F}_0) = (\mathcal{M}, \mathbb{E})$ and $\mathcal{N}_\beta$ is transitive.

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(5) (**First Supercompactness Condition**) Suppose that $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \notin \mathcal{N}_{\alpha+1}$ and let $\delta \leq \iota_{E_\alpha}$ be least such that $\pi_{\alpha,\alpha+1}[\delta] \notin \mathcal{N}_{\alpha+1}$.

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(5) (**First Supercompactness Condition**) Suppose that $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \notin \mathcal{N}_{\alpha+1}$ and let $\delta \leq \iota_{E_\alpha}$ be least such that $\pi_{\alpha,\alpha+1}[\delta] \notin \mathcal{N}_{\alpha+1}$.

Suppose that $\delta < \iota_{E_\alpha}$. Then:

- $(\mathrm{cof}(\delta))^{\mathcal{N}_\alpha} < \kappa_{E_\alpha}$.
- Either $\iota_{E_\alpha} = (\delta^+)^{\mathcal{N}_\alpha}$ or the sequence $\mathbb{F}_\alpha$ does not witness in $\mathcal{N}_\alpha$ that κ_{E_α} is $(\delta^+)^{\mathcal{N}_\alpha}$-supercompact.

Semi-iteration conditions continued where $\alpha < \beta \leq \eta$

(5) (**Second Supercompactness Condition**) Suppose that $\iota_{E_\alpha} = (\lambda^+)^{\mathcal{N}_\alpha}$, $(\mathrm{cof}(\lambda))^{\mathcal{N}_\alpha} \geq \kappa_{E_\alpha}$, and that $\mathbb{F}_\alpha$ witnesses in $\mathcal{N}_\alpha$ that κ_{E_α} is λ-supercompact.

Semi-iteration conditions continued where $\alpha < \beta \leq \eta$

(5) (**Second Supercompactness Condition**) Suppose that $\iota_{E_\alpha} = (\lambda^+)^{\mathcal{N}_\alpha}$, $(\mathrm{cof}(\lambda))^{\mathcal{N}_\alpha} \geq \kappa_{E_\alpha}$, and that $\mathbb{F}_\alpha$ witnesses in $\mathcal{N}_\alpha$ that κ_{E_α} is λ-supercompact.

Then $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$ and for some $\xi < \mathrm{Ord}^{\mathcal{N}_{\alpha+1}}$:

- (**Largest Generator Condition**)$\nu_{E_\alpha} < \pi_{\alpha,\alpha+1}(\iota_{E_\alpha})$ and $\nu_{E_\alpha} = \xi + 1$,

Semi-iteration conditions continued where $\alpha < \beta \leq \eta$

(5) (**Second Supercompactness Condition**) Suppose that $\iota_{E_\alpha} = (\lambda^+)^{\mathcal{N}_\alpha}$, $(\mathrm{cof}(\lambda))^{\mathcal{N}_\alpha} \geq \kappa_{E_\alpha}$, and that $\mathbb{F}_\alpha$ witnesses in $\mathcal{N}_\alpha$ that κ_{E_α} is λ-supercompact.

Then $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$ and for some $\xi < \mathrm{Ord}^{\mathcal{N}_{\alpha+1}}$:

- (**Largest Generator Condition**)$\nu_{E_\alpha} < \pi_{\alpha,\alpha+1}(\iota_{E_\alpha})$ and $\nu_{E_\alpha} = \xi + 1$,
- (**First Initial Segment Condition**) $E_\alpha | \eta \in \mathcal{N}_{\alpha+1}$

Semi-iteration conditions continued where $\alpha < \beta \leq \eta$

(5) (**Second Supercompactness Condition**) Suppose that $\iota_{E_\alpha} = (\lambda^+)^{\mathcal{N}_\alpha}$, $(\mathrm{cof}(\lambda))^{\mathcal{N}_\alpha} \geq \kappa_{E_\alpha}$, and that $\mathbb{F}_\alpha$ witnesses in $\mathcal{N}_\alpha$ that κ_{E_α} is λ-supercompact.

Then $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$ and for some $\xi < \mathrm{Ord}^{\mathcal{N}_{\alpha+1}}$:

- (**Largest Generator Condition**)$\nu_{E_\alpha} < \pi_{\alpha,\alpha+1}(\iota_{E_\alpha})$ and $\nu_{E_\alpha} = \xi + 1$,
- (**First Initial Segment Condition**) $E_\alpha | \eta \in \mathcal{N}_{\alpha+1}$ for all $\eta < \xi$,
- (**Second Initial Segment Condition**) if $E_\alpha | \xi \notin \mathcal{N}_{\alpha+1}$ then $(\mathrm{cof}(\xi))^{\mathcal{N}_{\alpha+1}} < \pi_{\alpha,\alpha+1}(\kappa_{E_\alpha})$.

Semi-iteration conditions continued where $\alpha < \beta \leq \eta$

(5) (**Second Supercompactness Condition**) Suppose that $\iota_{E_\alpha} = (\lambda^+)^{\mathcal{N}_\alpha}$, $(\mathrm{cof}(\lambda))^{\mathcal{N}_\alpha} \geq \kappa_{E_\alpha}$, and that $\mathbb{F}_\alpha$ witnesses in $\mathcal{N}_\alpha$ that κ_{E_α} is λ-supercompact.

Then $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$ and for some $\xi < \mathrm{Ord}^{\mathcal{N}_{\alpha+1}}$:

- (**Largest Generator Condition**)$\nu_{E_\alpha} < \pi_{\alpha,\alpha+1}(\iota_{E_\alpha})$ and $\nu_{E_\alpha} = \xi + 1$,
- (**First Initial Segment Condition**) $E_\alpha | \eta \in \mathcal{N}_{\alpha+1}$ for all $\eta < \xi$,
- (**Second Initial Segment Condition**) if $E_\alpha | \xi \notin \mathcal{N}_{\alpha+1}$ then $(\mathrm{cof}(\xi))^{\mathcal{N}_{\alpha+1}} < \pi_{\alpha,\alpha+1}(\kappa_{E_\alpha})$.

(6) (**Closeness Condition**) For all $a \in [\mathrm{LTH}(E_\alpha)]^{<\omega}$, $(E_\alpha)_a \in \mathcal{N}_\alpha$.

An abstract form of comparison

Definition (14)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ and $(\mathcal{M}, \mathbb{E})$ is transitive.

- Then $(\mathcal{M}, \mathbb{E})$ satisfies **comparison** if for all
$$X \prec (\mathcal{M}, \mathbb{E})$$
and
$$Y \prec (\mathcal{M}, \mathbb{E}),$$
the following hold where $(\mathcal{M}_X, \mathbb{E}_X)$ is the transitive collapse of X and $(\mathcal{M}_Y, \mathbb{E}_Y)$ is the transitive collapse of X.

Suppose that

- X and Y are finitely generated,
- $X \neq Y$,
- $X \cap \mathbb{R} = Y \cap \mathbb{R}$.

Definition continued

Then there exists semi-iterations,

$$((\mathcal{N}_\alpha^X, \mathbb{F}_\alpha^X), \pi_{\alpha,\beta}^X, E_\alpha^X : \alpha < \beta \leq \eta_X)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}_\alpha^Y, \mathbb{F}_\alpha^Y), \pi_{\alpha,\beta}^Y, E_\alpha^Y : \alpha < \beta \leq \eta_Y)$$

of $(\mathcal{M}_Y, \mathbb{E}_Y)$ such that:

Definition continued

Then there exists semi-iterations,

$$((\mathcal{N}^X_\alpha, \mathbb{F}^X_\alpha), \pi^X_{\alpha,\beta}, E^X_\alpha : \alpha < \beta \leq \eta_X)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}^Y_\alpha, \mathbb{F}^Y_\alpha), \pi^Y_{\alpha,\beta}, E^Y_\alpha : \alpha < \beta \leq \eta_Y)$$

of $(\mathcal{M}_Y, \mathbb{E}_Y)$ such that:

1. $(\mathcal{N}^X_{\eta_X}, \mathbb{F}^X_{\eta_X}) = (\mathcal{N}^Y_{\eta_Y}, \mathbb{F}^Y_{\eta_Y})$.
2. (**First Disagreement Condition** $E^X_0 \neq E^Y_0$.

Definition continued

Then there exists semi-iterations,

$$((\mathcal{N}^X_\alpha, \mathbb{F}^X_\alpha), \pi^X_{\alpha,\beta}, E^X_\alpha : \alpha < \beta \leq \eta_X)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}^Y_\alpha, \mathbb{F}^Y_\alpha), \pi^Y_{\alpha,\beta}, E^Y_\alpha : \alpha < \beta \leq \eta_Y)$$

of $(\mathcal{M}_Y, \mathbb{E}_Y)$ such that:

1. $(\mathcal{N}^X_{\eta_X}, \mathbb{F}^X_{\eta_X}) = (\mathcal{N}^Y_{\eta_Y}, \mathbb{F}^Y_{\eta_Y})$.
2. (**First Disagreement Condition** $E^X_0 \neq E^Y_0$.
3. (**Second Disagreement Condition**) Suppose that $\iota_{E^X_0} < \lambda$, $\iota_{E^Y_0} < \lambda$, and that

$$\mathcal{P}(\lambda) \cap \mathcal{M}_X = \mathcal{P}(\lambda) \cap \mathcal{M}_Y.$$

Then

$$\pi^X_{0,\eta_X} | \mathcal{P}(\lambda) \neq \pi^Y_{0,\eta_Y} | \mathcal{P}(\lambda).$$

At the finite levels of supercompact

Theorem (Weak $(\omega_1 + 1)$ Iteration Hypothesis)

Assume there is an extendible cardinal. Then there exists a partial extender sequence $\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \mathrm{dom}(\mathbb{E}) \rangle$ such that the following hold.

1. *$L[\mathbb{E}]$ is weakly backgrounded and $\mathbb{E}$ is weakly Σ_2-definable.*
2. *$L[\mathbb{E}]$ satisfies comparison.*
3. *For each ξ there exists $\alpha \in \mathrm{dom}(\mathbb{E})$ such that*
   - *$\alpha > \xi$,*
   - *$\mathbb{E}_\alpha$ is an $L[\mathbb{E}]$-extender which witnesses that κ is ω-extendible in $L[\mathbb{E}]$ where $\kappa = \mathrm{CRT}(\mathbb{E}_\alpha)$.*

- κ is ω-extendible if there is an elementary embedding

$$j : V_{\kappa+\omega} \to V_{j(\kappa)+\omega}$$

such that $\mathrm{CRT}(j) = \kappa$.

Coherent pairs

Definition

Suppose that $(\mathcal{M}_0, \mathbb{E}_0) \models \mathrm{ZFC}$ and that $(\mathcal{M}_1, \mathbb{E}_1) \models \mathrm{ZFC}$. Suppose each structure is transitive and κ is a regular cardinal of both structures.

- Then the pair
$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$
is a **coherent pair at κ** if
$$(\kappa^+)^{\mathcal{M}_0} = (\kappa^+)^{\mathcal{M}_1}$$
and
$$(\mathcal{M}_0, \mathbb{E}_0)|(\kappa^+)^{\mathcal{M}_0} = (\mathcal{M}_1, \mathbb{E}_1)|(\kappa^+)^{\mathcal{M}_1}.$$

Definition (17)

Suppose that

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a coherent pair at κ. A **semi-iteration** at κ of the (ordered) pair,

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a continuous (linearly) directed system

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

such that the following hold for all $\alpha < \beta < \eta$ where for all $\alpha < \eta$, E_α is the $\mathcal{N}_\alpha$-extender defined by $\pi_{\alpha,\alpha+1}$.

Definition (17)

Suppose that

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a coherent pair at κ. A **semi-iteration** at κ of the (ordered) pair,

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is a continuous (linearly) directed system

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

such that the following hold for all $\alpha < \beta < \eta$ where for all $\alpha < \eta$, E_α is the $\mathcal{N}_\alpha$-extender defined by $\pi_{\alpha,\alpha+1}$.

1. $(\mathcal{N}_0, \mathbb{F}_0) \in \{(\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1)\}$ and

   $$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

   is a semi-iteration of $(\mathcal{N}_0, \mathbb{F}_0)$.
2. If $\mathcal{N}_0 = \mathcal{M}_1$ then $\kappa < \iota$ for some $\iota \in \mathrm{SP}^*(E_0)$.

Definition (18)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$, $(\mathcal{M}, \mathbb{E})$ is transitive, κ is a measurable cardinal in V, U is a normal measure on κ, and $U \cap \mathcal{M} \in \mathcal{M}$. Let

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}\left((\mathcal{M}, \mathbb{E}), U\right)$$

and suppose that

$$((\mathcal{M}, \mathbb{E}), (\mathcal{M}_U, \mathbb{E}_U))$$

is a coherent pair at κ.

- Then $(\mathcal{M}_U, \mathbb{E}_U)$ satisfies **comparison backed up by** $(\mathcal{M}, \mathbb{E})$ at κ if the following hold.

Definition (18)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$, $(\mathcal{M}, \mathbb{E})$ is transitive, κ is a measurable cardinal in V, U is a normal measure on κ, and $U \cap \mathcal{M} \in \mathcal{M}$. Let

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}\,((\mathcal{M}, \mathbb{E}), U)$$

and suppose that

$$((\mathcal{M}, \mathbb{E}), (\mathcal{M}_U, \mathbb{E}_U))$$

is a coherent pair at κ.

- Then $(\mathcal{M}_U, \mathbb{E}_U)$ satisfies **comparison backed up by** $(\mathcal{M}, \mathbb{E})$ at κ if the following hold.

- Suppose $X \prec (\mathcal{M}, \mathbb{E})$, X is finitely generated, $U \cap \mathcal{M} \in X$,

$$(\mathcal{M}_X, \mathbb{E}_X)$$

is the transitive collapse of X, κ_X is the image of κ under the transitive collapse, and $(\mathcal{M}_U^X, \mathbb{E}_U^X)$ is the image of $(X \cap \mathcal{M}_U, X \cap \mathbb{E}_U)$ under the transitive collapse.

Definition continued

Then there exist semi-iterations,

$$((\mathcal{N}^0_\alpha, \mathbb{F}^0_\alpha), \pi^0_{\alpha,\beta}, E^0_\alpha : \alpha < \beta \leq \eta_0)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}^1_\alpha, \mathbb{F}^1_\alpha), \pi^1_{\alpha,\beta}, E^1_\alpha : \alpha < \beta \leq \eta_1)$$

of the pair $\big((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}^X_U, \mathbb{E}^X_U)\big)$ at κ_X such that:

Definition continued

Then there exist semi-iterations,

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of the pair $\left((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}^X_U, \mathbb{E}^X_U)\right)$ at κ_X such that:

1. $\left(\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0}\right) = \left(\mathcal{N}^1_{\eta_1}, \mathbb{F}^1_{\eta_1}\right)$.
2. (**First Disagreement Condition**) $E^0_0 \neq E^1_0$.

Definition continued

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of the pair $\big((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}^X_U, \mathbb{E}^X_U)\big)$ at κ_X such that:

1. $\big(\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0}\big) = \big(\mathcal{N}^1_{\eta_1}, \mathbb{F}^1_{\eta_1}\big)$.
2. (**First Disagreement Condition**) $E^0_0 \neq E^1_0$.
3. (**Second Disagreement Condition**) Suppose that $\iota_{E^X_0} < \lambda$, $\iota_{E^Y_0} < \lambda$, and that

   $$\mathcal{P}(\lambda) \cap \mathcal{N}^0_0 = \mathcal{P}(\lambda) \cap \mathcal{N}^1_0.$$

   Then

   $$\pi^0_{0,\eta_0} | \mathcal{P}(\lambda) \neq \pi^1_{0,\eta_1} | \mathcal{P}(\lambda).$$

Lemma (19)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$*,* $(\mathcal{M}, \mathbb{E})$ *is weakly backgrounded,* $\delta < \mathrm{Ord}^{\mathcal{M}}$*, and that* δ *is witnessed by the partial extenders on the sequence* $\mathbb{E}$ *to be supercompact in* $\mathcal{M}$*.*
Suppose $\delta < \kappa < \mathrm{Ord}^{\mathcal{M}}$ *and that* U *is a* δ*-complete ultrafilter on* κ*.*

- *Then* $U \cap \mathcal{M} \in \mathcal{M}$*.*

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- *Then* $U \cap \mathcal{M} \in \mathcal{M}$*.*

Proof.

Let $\lambda = |V_{\kappa+\omega} \cap \mathcal{M}|^{\mathcal{M}}$ and let μ be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

1. $\mathcal{M} \cap \mathcal{P}_\delta(\lambda) \in \mu$,
2. $\mu \cap \mathcal{M} \in \mathcal{M}$.

Lemma (19)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$, $(\mathcal{M}, \mathbb{E})$ *is weakly backgrounded,* $\delta < \mathrm{Ord}^{\mathcal{M}}$, *and that* δ *is witnessed by the partial extenders on the sequence* $\mathbb{E}$ *to be supercompact in* $\mathcal{M}$.
Suppose $\delta < \kappa < \mathrm{Ord}^{\mathcal{M}}$ *and that* U *is a* δ*-complete ultrafilter on* κ.

- *Then* $U \cap \mathcal{M} \in \mathcal{M}$.

Proof.

Let $\lambda = |V_{\kappa+\omega} \cap \mathcal{M}|^{\mathcal{M}}$ and let μ be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

1. $\mathcal{M} \cap \mathcal{P}_\delta(\lambda) \in \mu$,
2. $\mu \cap \mathcal{M} \in \mathcal{M}$.

- The ultrafilter μ must exist since $\mathcal{M}$ is weakly backgrounded and since δ is witnessed by the partial extenders on the sequence $\mathbb{E}$ to be supercompact in $\mathcal{M}$.

Proof continued

Fix a bijection

$$\Pi : \Lambda \to V_{\kappa+\omega} \cap \mathcal{M}$$

with $\pi \in \mathcal{M}$ and let I be the set of all $\sigma \in \mathcal{P}_\delta(\lambda) \cap M$ such that for each $\xi < \kappa$ there exists $\eta < \lambda$ such that

1. $\eta \in \sigma$,
2. $\pi(\eta)$ is a δ-compete ultrafilter in κ in $\mathcal{M}$,
3. for all $A \in \mathcal{P}(\kappa) \cap \pi[\sigma]$, $A \in \pi(\eta)$ if and only if $\xi \in A$.

Proof continued

Fix a bijection

$$\Pi : \Lambda \to V_{\kappa+\omega} \cap \mathcal{M}$$

with $\pi \in \mathcal{M}$ and let I be the set of all $\sigma \in \mathcal{P}_\delta(\lambda) \cap M$ such that for each $\xi < \kappa$ there exists $\eta < \lambda$ such that

1. $\eta \in \sigma$,
2. $\pi(\eta)$ is a δ-compete ultrafilter in κ in $\mathcal{M}$,
3. for all $A \in \mathcal{P}(\kappa) \cap \pi[\sigma]$, $A \in \pi(\eta)$ if and only if $\xi \in A$.

The key point is that $I \in \mu$. This is easily verified by working in $\mathcal{M}$ and using that in $\mathcal{M}$, $\mu \cap \mathcal{M}$ is a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$.

Proof continued

Define

$$f : I \to \lambda$$

by $f(\sigma) = \eta$ such that

1. $\pi(\eta)$ is a δ-complete ultrafilter on κ in $\mathcal{M}$,
2. $\eta \in \sigma$,
3. $\pi(\eta) \cap \pi[\sigma] = U \cap \pi[\sigma]$.

Proof continued

Define

$$f : I \to \lambda$$

by $f(\sigma) = \eta$ such that

1. $\pi(\eta)$ is a δ-complete ultrafilter on κ in $\mathcal{M}$,
2. $\eta \in \sigma$,
3. $\pi(\eta) \cap \pi[\sigma] = U \cap \pi[\sigma]$.

Since $I \in \mu$, there must exist $\eta_0 < \lambda$ such that

$$\{\sigma \in I \mid f(\sigma) = \eta_0\} \in \mu.$$

Thus $\pi(\eta_0) = U \cap \mathcal{M}$ and this proves the lemma. $\square$

Close embeddings

Definition (20)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ and that

$$\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

is an elementary embedding.

- Then π is **close** to $(\mathcal{M}, \mathbb{E})$ if for each $X \in \mathcal{M}$ and each $a \in \pi(X)$,

$$\{Z \in \mathcal{P}(X) \cap \mathcal{M} \mid a \in \pi(Z)\} \in \mathcal{M}.$$

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- Then π is **close** to $(\mathcal{M}, \mathbb{E})$ if for each $X \in \mathcal{M}$ and each $a \in \pi(X)$,

$$\{Z \in \mathcal{P}(X) \cap \mathcal{M} \mid a \in \pi(Z)\} \in \mathcal{M}.$$

Lemma (21)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ and that $\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$ is an elementary embedding which is given by a semi-iteration of $(\mathcal{M}, \mathbb{E})$.

- *Then π is close to $(\mathcal{M}, \mathbb{E})$.*

Proof.

The key point is that the composition of close embeddings is close. Suppose that

$$\pi_0 : (\mathcal{M}_0, \mathbb{E}_0) \to (\mathcal{M}_1, \mathbb{E}_1)$$

and

$$\pi_1 : (\mathcal{M}_1, \mathbb{E}_1) \to (\mathcal{M}_2, \mathbb{E}_2)$$

are each close embeddings. Fix $Y \in \mathcal{M}_0$ and $a \in \pi_1 \circ \pi_0(Y)$.

Proof.

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$$\pi_0 : (\mathcal{M}_0, \mathbb{E}_0) \to (\mathcal{M}_1, \mathbb{E}_1)$$

and

$$\pi_1 : (\mathcal{M}_1, \mathbb{E}_1) \to (\mathcal{M}_2, \mathbb{E}_2)$$

are each close embeddings. Fix $Y \in \mathcal{M}_0$ and $a \in \pi_1 \circ \pi_0(Y)$.
We must show that

- $\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0$.

Proof.

The key point is that the composition of close embeddings is close. Suppose that

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and

$$\pi_1 : (\mathcal{M}_1, \mathbb{E}_1) \to (\mathcal{M}_2, \mathbb{E}_2)$$

are each close embeddings. Fix $Y \in \mathcal{M}_0$ and $a \in \pi_1 \circ \pi_0(Y)$.
We must show that

- $\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0$.

Let

- $W = \{Z \in \mathcal{P}(\pi_0(Y)) \cap \mathcal{M}_1 \mid a \in \pi_1(Z)\}$.
- $W^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid W \in \pi_0(U)\}$.

Proof.

The key point is that the composition of close embeddings is close. Suppose that

$$\pi_0 : (\mathcal{M}_0, \mathbb{E}_0) \to (\mathcal{M}_1, \mathbb{E}_1)$$

and

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are each close embeddings. Fix $Y \in \mathcal{M}_0$ and $a \in \pi_1 \circ \pi_0(Y)$.
We must show that

- $\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0$.

Let

- $W = \{Z \in \mathcal{P}(\pi_0(Y)) \cap \mathcal{M}_1 \mid a \in \pi_1(Z)\}$.
- $W^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid W \in \pi_0(U)\}$.

Then

- $W \in \mathcal{M}_1$ by the closeness of π_1 to $\mathcal{M}_1$.
- $W^* \in \mathcal{M}_0$ by the closeness of π_0 to $\mathcal{M}_0$.

Proof continued

Fix $Z \in \mathcal{P}(Y) \cap \mathcal{M}_0$. Then

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $\pi_0(Z) \in W$.

Proof continued

Fix $Z \in \mathcal{P}(Y) \cap \mathcal{M}_0$. Then

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $\pi_0(Z) \in W$.

Let

$$Z^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid Z \in U\}.$$

Thus

- $\pi_0(Z) \in W$ if and only if $W \in \pi_0(Z^*)$.

Proof continued

Fix $Z \in \mathcal{P}(Y) \cap \mathcal{M}_0$. Then

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $\pi_0(Z) \in W$.

Let

$$Z^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid Z \in U\}.$$

Thus

- $\pi_0(Z) \in W$ if and only if $W \in \pi_0(Z^*)$.

But

- $W \in \pi_0(Z^*)$ if and only if $Z^* \in W^*$.

Therefore

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $Z^* \in W^*$,

and this implies

$$\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0.$$

Proof continued

Fix $Z \in \mathcal{P}(Y) \cap \mathcal{M}_0$. Then

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $\pi_0(Z) \in W$.

Let

$$Z^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid Z \in U\}.$$

Thus

- $\pi_0(Z) \in W$ if and only if $W \in \pi_0(Z^*)$.

But

- $W \in \pi_0(Z^*)$ if and only if $Z^* \in W^*$.

Therefore

- $a \in \pi_1 \circ \pi_0(Z)$ if and only if $Z^* \in W^*$,

and this implies

$$\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0.$$

This proves

- $\pi_1 \circ \pi_0$ is close to $(\mathcal{M}_0, \mathbb{E}_0)$.

The lemma now follows easily by induction of the length of semi-iterations. $\square$

A uniqueness lemma for close embeddings

Lemma (22)

Suppose that $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ *and is finitely generated. Suppose that*

$$\pi_0 : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

and

$$\pi_1 : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

are elementary embeddings each of which is close to $(\mathcal{M}, \mathbb{E})$*.*

- *Then* $\pi_0 = \pi_1$*.*

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- *Then* $\pi_0 = \pi_1$*.*

Proof.

Let $\xi \in \mathcal{M} \cap \mathrm{Ord}$ be least such that every element of $\mathcal{M}$ is definable in $(\mathcal{M}, \mathbb{E})$ from ξ. It suffices to show that

$$\pi_0(\xi) = \pi_1(\xi).$$

Let $\xi_0 = \pi_0(\xi)$ and let $\xi_1 = \pi_1(\xi)$.

- Assume toward a contradiction that $\xi_0 < \xi_1$.

Proof continued

Let $U = \{Z \subset \xi \mid \xi_0 \in \pi_1(Z)\}$. Thus $U \in \mathcal{M}$. Let

- $j_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$ be the ultrapower embedding,
- $k_U : (\mathcal{M}_U, \mathbb{E}_U) \to (\mathcal{N}, \mathbb{F})$ be the factor embedding such that
  - $\pi_1 = k_U \circ j_U$.

Proof continued

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Let ξ_0^U be the element of $\mathcal{M}_U$ represented by the identity function. Thus

- $k_U(\xi_0^U) = \xi_0$.

Proof continued

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Let ξ_0^U be the element of $\mathcal{M}_U$ represented by the identity function. Thus

- $k_U(\xi_0^U) = \xi_0$.

Let $(\mathcal{N}_X, \mathbb{F}_X)$ be the transitive collapse of X where X is the set of all $a \in \mathcal{N}$ such that a is definable in $(\mathcal{N}, \mathbb{F})$ from ξ_0. Thus:

Claim

$(\mathcal{N}_X, \mathbb{F}_X) = (\mathcal{M}, \mathbb{E})$.

But $X \subset k_U[\mathcal{M}_U]$ since $\xi_0 = k_U(\xi_0^U)$ and since $\pi_1 = k_U \circ j_U$. Further

- $k_U(\xi_0^U) = \xi_0 < \pi_1(\xi) = k_U \circ j_U(\xi)$.

Proof continued

Therefore

1. $\xi_0^U < j_U(\xi)$,
2. Let X_U be the set of all $a \in \mathcal{M}_U$ such that a is definable in $(\mathcal{M}_U, \mathbb{E}_U)$ from ξ_0^U, and let $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U})$ be the transitive collapse of X. Then
   - $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U}) = (\mathcal{M}, \mathbb{E})$.

Proof continued

Therefore

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2. Let X_U be the set of all $a \in \mathcal{M}_U$ such that a is definable in $(\mathcal{M}_U, \mathbb{E}_U)$ from ξ_0^U, and let $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U})$ be the transitive collapse of X. Then
    - $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U}) = (\mathcal{M}, \mathbb{E})$.

Let

$$\pi_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

invert the transitive collapse of X_U. Thus $\pi_U(\xi) = \xi_0^U < j_U(\xi)$.

- The key point is:
    - There is a canonical elementary embedding

$$j : \mathrm{Ult}(\mathcal{M}, U) \to \mathrm{Ult}(\mathcal{M}_U, \pi_U(U)).$$

Proof continued

Therefore

1. $\xi_0^U < j_U(\xi)$,
2. Let X_U be the set of all $a \in \mathcal{M}_U$ such that a is definable in $(\mathcal{M}_U, \mathbb{E}_U)$ from ξ_0^U, and let $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U})$ be the transitive collapse of X. Then
   - $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U}) = (\mathcal{M}, \mathbb{E})$.

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Now one can generate an illfounded iteration of $\mathcal{M}$ of length ω and this is a contradiction.

Claim

Thus one can generate an illfounded iteration of $\mathcal{M}$ of length ω and this is a contradiction. □

Theorem (23)

Suppose that δ is supercompact and that $\Omega > \delta$ is a a strongly inaccessible cardinal. Then there is no weakly backgrounded structure $(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$ such that the following hold.

(1) $\Omega = \mathrm{Ord}^{\mathcal{M}}$ *and δ is witnessed by the partial extenders on the sequence $\mathbb{E}$ to be supercompact in $\mathcal{M}$.*

(2) *There exists a measurable cardinal $\delta < \kappa < \Omega$ and a normal measure U on κ such that the following hold where*

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}\left((\mathcal{M}, \mathbb{E}), U\right).$$

  (a) $((\mathcal{M}, \mathbb{E}), (\mathcal{M}_U, \mathbb{E}_U))$ *is a coherent pair at κ.*

  (b) $U \cap \mathcal{M} \in \mathcal{M}$.

  (c) $(\mathcal{M}_U, \mathbb{E}_U)$ *satisfies comparison backed up by $(\mathcal{M}, \mathbb{E})$ at κ.*

Proof.

Assume toward a contradiction that $(\mathcal{M}, \mathbb{E})$ is weakly backgrounded and that $(\mathcal{M}, \mathbb{E})$, U, and κ satisfy (1) and (2). Let

$$e_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

be the ultrapower embedding as defined in $(\mathcal{M}, \mathbb{E})$ using $U \cap \mathcal{M}$. Let

$$X \prec (\mathcal{M}, \mathbb{E})$$

be the elementary substructure given by the set of all $a \in \mathcal{M}$ such that a is definable in $(\mathcal{M}, \mathbb{E})$ from $\{U \cap \mathcal{M}\}$.

Proof continued

Notation

- *Let* $(\mathcal{M}_X, \mathbb{E}_X)$ *be the transitive collapse of* X.
- *Let* κ_X *be the image of* κ *under the transitive collapse of* X.
- *Let* δ_X *be the image of* δ *under the transitive collapse of* X.
- *Let* U_X *be the image of* $U \cap \mathcal{M}$ *under the transitive collapse of* X.
- *Let* $(\mathcal{M}_U^X, \mathbb{E}_U^X)$ *be the transitive collapse of* $(X \cap \mathcal{M}_U, X \cap \mathbb{E}_U)$.
- *Let*
$$e_U^X : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{M}_U^X, \mathbb{E}_U^X)$$
be the image of e_U *under the transitive collapse of* X.

Proof continued

Since $(\mathcal{M}_U, \mathbb{E}_U)$ satisfies comparison backed up by $(\mathcal{M}, \mathbb{E})$ at κ, there exist semi-iterations

$$((\mathcal{N}^0_\alpha, \mathbb{F}^0_\alpha), \pi^0_{\alpha,\beta}, E^0_\alpha : \alpha < \beta \leq \eta_0)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}^1_\alpha, \mathbb{F}^1_\alpha), \pi^1_{\alpha,\beta}, E^1_\alpha : \alpha < \beta \leq \eta_1)$$

of the pair $\big((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}^X_U, \mathbb{E}^X_U)\big)$ at κ_X such that:

1. $\big(\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0}\big) = \big(\mathcal{N}^1_{\eta_1}, \mathbb{F}^1_{\eta_1}\big)$.
2. $E^0_0 \neq E^1_0$.
3. $\iota_{E^0_0} < \theta$, $\iota_{E^1_0} < \theta$, and that and that

$$\mathcal{N}^0_0 \cap \mathcal{P}(\theta) = \mathcal{N}^1_0 \cap \mathcal{P}(\theta).$$

Then $\pi^0_{0,\eta_0} | \mathcal{P}(\theta) \neq \pi^1_{0,\eta_1} | \mathcal{P}(\theta)$.

The key claim

Claim (1)

(1.1) $(\mathcal{N}_0^1, \mathbb{F}_0^1) = \left(\mathcal{M}_U^X, \mathbb{E}_U^X\right)$.

(1.2) $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e_U^X$.

The key claim

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(1.2) $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e_U^X$.

Assume toward a contradiction that $(\mathcal{N}_0^1, \mathbb{F}_0^1) = (\mathcal{M}_X, \mathbb{E}_X)$. Then

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

and

$$\pi^1_{0,\eta_1} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^1_{\eta_0}, \mathbb{F}^1_{\eta_1})$$

are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure.

- By Lemma 21, each embedding is close to $(\mathcal{M}_X, \mathbb{E}_X)$.

The key claim

Claim (1)

(1.1) $(\mathcal{N}_0^1, \mathbb{F}_0^1) = \big(\mathcal{M}_U^X, \mathbb{E}_U^X\big)$.
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Assume toward a contradiction that $(\mathcal{N}_0^1, \mathbb{F}_0^1) = (\mathcal{M}_X, \mathbb{E}_X)$. Then

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

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are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure.

- By Lemma 21, each embedding is close to $(\mathcal{M}_X, \mathbb{E}_X)$.

Therefore by Lemma 22,

- $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1}$

and this is a contradiction. This proves (1.1).

The proof that $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e^X_U$

Thus $(\mathcal{N}^1_0, \mathbb{F}^1_0) = \left(\mathcal{M}^X_U, \mathbb{E}^X_U\right)$ and so

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

and

$$\pi^1_{0,\eta_1} \circ e^X_U : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^1_{\eta_0}, \mathbb{F}^1_{\eta_1})$$

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are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure.

By Lemma 21,

- π^0_{0,η_0} is close to $(\mathcal{M}_X, \mathbb{E}_X)$ and π^1_{0,η_1} is close to $(\mathcal{M}^X_U, \mathbb{E}^X_U)$.

Further e_U is trivially close to $(\mathcal{M}_X, \mathbb{E}_X)$. Therefore

- $\pi^1_{0,\eta_1} \circ e^X_U$ is close to $(\mathcal{M}_X, \mathbb{E}_X)$

since close embeddings are closed under composition.

The proof that $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e^X_U$

Thus $(\mathcal{N}^1_0, \mathbb{F}^1_0) = \left(\mathcal{M}^X_U, \mathbb{E}^X_U\right)$ and so

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

and

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are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure.

By Lemma 21,

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since close embeddings are closed under composition.

- Therefore $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e^X_U$ by Lemma 22 on the uniqueness of close embeddings.

This proves (1.1) and (1.2).

The remainder of the proof is a routine technical calculation using the properties of semi-iterations to obtain a contradiction.