The higher sharp

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- $A \subseteq \mathbb{R}$ is $\boldsymbol{\Pi}^1_n$ iff $\mathbb{R} \setminus A$ is $\boldsymbol{\Sigma}^1_n$.
- $A \subseteq \mathbb{R}$ is $\boldsymbol{\Sigma}^1_{n+1}$ iff there is a $\boldsymbol{\Pi}^1_n$ set $B \subseteq \mathbb{R}^2$ such that $x \in A \leftrightarrow \exists y \ (x, y) \in B$.

A is projective iff A is $\boldsymbol{\Sigma}^1_n$ for some $n < \omega$.

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The influence of large cardinals on the reals

- (Martin-Steel) Suppose there are n Woodin cardinals below a measurable cardinal, then $\mathbf{\Pi}^1_{n+1}$ sets are determined.
- (Woodin) Suppose there are ω Woodin cardinals below a measurable cardinal, then $L(\mathbb{R}) \models$AD.

In this talk, we assume ZFC+$L(\mathbb{R}) \models$AD.

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Question

What about the effective version?

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DST way of thinking: Suppose $\emptyset \neq A = p[T]$.

- $(x, \vec{\alpha}) \in [T]$ is the leftmost branch of T iff whenever $(y, \vec{\beta}) \in [T]$, $(x(0), \alpha_0, x(1), \alpha_1, \ldots)$ is lexicographically smaller than $(y(0), \beta_0, y(1), \beta_1, \ldots)$.

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- If $(x, \vec{\alpha}) \in [T]$ is the leftmost branch of T, then x is the leftmost real associated to T.

Effective codes of ordinals (1)

Σ^1_2 sets are "effectively" ω_1-Suslin.

Codes of ordinals up to ω_1:

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Is there a canonical coding system for ordinals smaller than Θ^{UB}?
Next level: Σ^1_3 sets are "effectively" u_ω-Suslin.

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- L is Skolem generated by $\{c_\alpha : \alpha < \infty\}$,
- for every first-order formula φ, every $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$,

$$L \models \varphi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \leftrightarrow \varphi(c_{\beta_1}, \cdots, c_{\beta_n}).$$

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$(c_\alpha : \alpha < \infty)$ is called the class of Silver indiscernibles for L. In particular, every uncountable cardinal is a Silver indiscernible for L.

$$0^\# = \oplus_{n<\omega}\{\ulcorner\varphi\urcorner : \varphi \text{ has } n \text{ free variables, } L \models \varphi(\aleph_1, \ldots, \aleph_n)\}.$$

$0^\#$ is the unique wellfounded remarkable EM blueprint.

The EM blueprint formulation of sharps

Definition

An EM blueprint is a complete consistent theory in the language $\{\dot{\in}, \dot{c}_n : n < \omega\}$ that extends the following axioms:

- $ZFC + V = L$. In particular, there is a definable wellordering of the universe.
- $c_1 < c_2$
- For every formula φ, every increasing tuple $k_1 < \cdots < k_m$, $\varphi(\dot{c}_1, \ldots, \dot{c}_m) \rightarrow \varphi(\dot{c}_{k_1}, \ldots, \dot{c}_{k_m})$ is an axiom

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An EM blueprint Γ is remarkable iff Γ contains the remarkability axioms:

$$\tau(\dot{c}_1, \ldots, \dot{c}_{m+n}) < \dot{c}_{m+1} \to \tau(\dot{c}_1, \ldots, \dot{c}_{m+n}) = \tau(\dot{c}_1, \ldots, \dot{c}_m, \dot{c}_{m+n+1}, \ldots,$$

holds for every Skolem term τ.

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Suppose Γ is an EM blueprint. For every linear ordering $<^*$ on a set W, there is (up to isomorphism) a unique model $\mathcal{M}(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq \mathcal{M}(\Gamma, <^*)$ such that

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- $0^\#$ is the unique wellfounded remarkable EM blueprint
- $x^\#$ is the unique wellfounded remarkable EM blueprint over x.

Effective codes of ordinals (2)

Codes of ordinals up to u_ω:

γ is a uniform indiscernible iff γ is a Silver indiscernible for $L[x]$ for every $x \in \mathbb{R}$. The uniform indiscernibles are enumerated $(u_\alpha : 1 \leq \alpha < \infty)$.

$$u_1 = \omega_1. \ \ u_\omega = \sup_{n<\omega} u_n.$$

A sharp code is pair $(\ulcorner\tau\urcorner, x^\#)$ where $x \in \mathbb{R}$, τ is an n-ary Skolem term for some n, and $\tau^{L[x]}(u_1, \dots, u_n)$ is an ordinal.

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- The set of sharp codes is Π^1_2.
- The relation "v, w are sharp codes $\wedge\, |v| = |w|$" is Δ^1_3.

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- $u_1 = \omega_1, u_2, \ldots, u_n, \ldots$ list the first ω uncountable projective wellordered cardinals.
- The "projective successor" of u_ω is $\boldsymbol{\delta}^1_3$.
- IMT provides a deep insight into the structural theory related to the projective well-ordered cardinals.

The Martin-Solovay tree projecting to $\boldsymbol{\Pi}^1_2$

Theorem (Martin-Solovay)

If A is $\boldsymbol{\Pi}^1_2$, then $A = p[T]$ for some T on $\omega \times u_\omega$ such that T is $\boldsymbol{\Delta}^1_3$ in the sharp codes.

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Definition of the Martin-Solovay tree:

If $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is an order preserving map, then σ induces $j^\sigma : u_{m+1} \to u_{n+1}$ by

$$j^\sigma(\tau^{L[x]}(u_1, \ldots, u_m)) = \tau^{L[x]}(u_{\sigma(1)}, \ldots, u_{\sigma(m)}).$$

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- $(s, \vec{\alpha}) \in T_2$ iff $s \in T$ and for every m, n, for every order preserving $\sigma : \{1, \ldots, k_m\} \to \{1, \ldots, k_n\}$, if the formula

$$\tau_m(\dot{c}_{\sigma(1)}, \ldots, \dot{c}_{\sigma(k_m)}) = \tau_n(\dot{c}_1, \ldots, \dot{c}_{k_n})$$

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- (Martin-Solovay) $p[T_2] = \{x^\# : x \in \mathbb{R}\}$.
- It is easy to define $\widehat{T}_2$, a variant of T_2, such that $p[\widehat{T}_2]$ is a good universal Π^1_2 set.

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$\widehat{T}_2$ is a tree on $\omega \times u_\omega$. It is Δ^1_3 in the codes. $p[\widehat{T}_2]$ is a good universal Π^1_2 set.

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Theorem

Assume $\boldsymbol{\Delta}^1_2$-determinacy. Then $y^0_3 \equiv_m M_1^\#$ (many-one equivalence).

Effective codes of ordinals (3)

Codes of ordinals up to $\boldsymbol{\delta}^1_3$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$.

- $x \in \mathrm{WO}_{(3)}$ iff x codes a wellordering of u_ω via Kunen's coding of subsets of u_ω. (under AD, there is a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is Δ^1_3)

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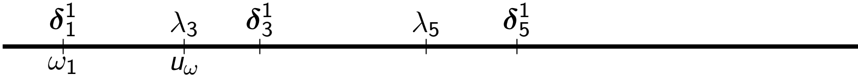

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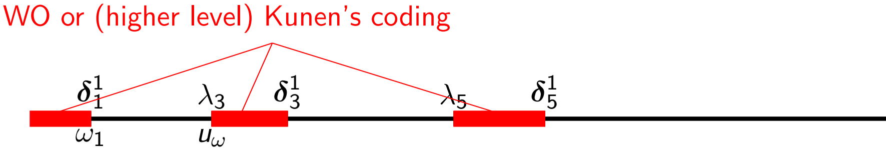

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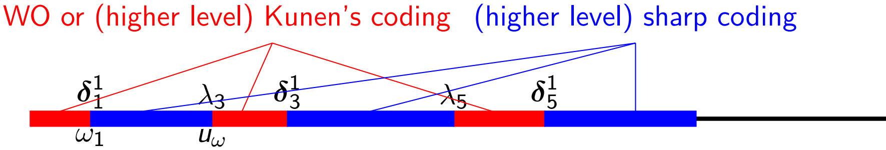

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- (Kechris) Assume ZF+DC+AD. Let $\lambda_{2n+1} = (\boldsymbol{\delta}^1_{2n+1})^-$. Then every $\boldsymbol{\Pi}^1_{2n}$ set is λ_{2n+1}-Suslin.

The model theoretic form of Π^1_3 sets

The model theoretic form of Π^1_1 subsets of $\mathbb{R}$

The following are equivalent:

- $A \subseteq \mathbb{R}$ is Π^1_1.
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(Kechris-Martin, Becker-Kechris) The following are equivalent:

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Higher Kleene's O: $(\ulcorner\varphi\urcorner, \alpha) \in \mathcal{O}^{T_2}$ iff φ is a Σ_1 formula with one free variable, $\alpha < u_\omega$ and $L_{\kappa_3}[T_2] \models \varphi(\alpha)$.

The level-2 sharp

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By the previous slide, $0^{2\#} \equiv_m y_3^0$. It is the model theoretic definition of y_3^0.

Theorem

$0^{2\#} \equiv_m M_1^{\#}$. $x^{2\#} \equiv_m M_1^{\#}(x)$, *the many-one reductions independent of* x.

Π^1_3 sets, the structure of $L[T_3]$

Definition

T_3 is the tree on $\omega \times \boldsymbol{\delta}^1_3$ associated to a Π^1_3-scale on a good universal Π^1_3 set.

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2. (Steel) $L[T_3]$ is a mouse. Let $j : M_2^{\#} \to M_{2,\infty}^{\#}$ be the direct limit map of all countable iterates of $M_2^{\#}$, δ_∞ be the least Woodin of $M_{2,\infty}^{\#}$, then
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Intuition: $(L[T_3], \text{level-3 indiscernibles}) \sim M_2^{\#}$.

Level-3 EM blueprint

The role of level-3 EM blueprints

The (level-1) EM blueprint definition of sharps leads to

- Effective wellordered cardinals $(u_n : n \leq \omega)$, $u_n = \aleph_n^{L(\mathbb{R})}$,
- Effective (Δ^1_3) coding system of ordinals in u_ω,
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The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals $(u_{\xi;3} : \xi \leq \omega^{\omega^\omega})$, $u_{\xi;3} = \aleph^{L(\mathbb{R})}_{\omega+1+\xi}$,
- Effective (Δ^1_5) coding system of ordinals in $u_{\omega^{\omega^\omega};3}$,
- The level-4 Martin-Solovay tree T_4 projecting to Π^1_4,
- The model theoretic form of Π^1_5 subsets of $u_{\omega^{\omega^\omega};3} \times \mathbb{R}$,
- $0^{4\#} \equiv_m M_3^{\#}$.

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- $\mathcal{L}^R$ is the language $\{\dot{\in}, \dot{c}_s : s \in \mathrm{dom}(R)\}$.
- $\Gamma(R)$ is a complete consistent $\mathcal{L}^R$ theory containing the following axioms:
  - $ZFC + V = K +$"I am a 2-small mouse".
  - ($\varphi_\Sigma, \varphi_\Pi$ are fixed formulas) $\varphi_\Sigma, \varphi_\Pi$ define class trees T_Σ, T_Π resp. In any set generic extension, $\check{T}_\Sigma$, $\check{T}_\Pi$ project to the good universal Σ^1_3 set and its complement.
  - V is closed under the $M_1^\#$ operator, as certified by T_Π.
  - ...(The order relation of those c_s's)...

$\mathcal{M}(\Gamma, R)$ is (up to isomorphism) the unique model of $\Gamma(R)$ that is Skolem generated by $(\dot{c}_s^{\mathcal{M}(\Gamma,R)} : s \in \mathbb{R})$. $\mathcal{M}(\Gamma, R)$ is called the level-3 EM model associated to Γ and R.

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- Γ is coherent. In particular, if R is a subfunction of R', then $\Gamma(R) \subseteq \Gamma(R')$, so that $\mathcal{M}(\Gamma, R)$ elementarily embeds into $\mathcal{M}(\Gamma, R')$ in a unique way.

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- Remarkability. Generalization of the level-1 remarkability axiom.

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What does a level-3 EM blueprint Γ look like?

- Γ is coherent. In particular, if R is a subfunction of R', then $\Gamma(R) \subseteq \Gamma(R')$, so that $\mathcal{M}(\Gamma, R)$ elementarily embeds into $\mathcal{M}(\Gamma, R')$ in a unique way.
- Remarkability. Generalization of the level-1 remarkability axiom.
- Iterability. If a tower of level-3 trees $(R_n : n < \omega)$ is Π^1_3-wellfounded, then the direct limit of $(\mathcal{M}(\Gamma, R_n) : n < \omega)$ is a Π^1_3-iterable mouse.

Iterability is Π^1_4.

Theorem

Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then there is a unique iterable remarkable level-3 EM blueprint.

The level-3 sharp

Definition

Assume $\boldsymbol{\Pi}^1_3$-determinacy.

- $0^{3\#}$ is the unique level-3 EM blueprint.
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The proof uses the equivalence $x^{2\#} \equiv_m M_1^\#(x)$.

Level-3 EM blueprint

The role of level-3 EM blueprints

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals $(u_{\xi;3} : \xi \leq \omega^{\omega^\omega})$, $u_{\xi;3} = \aleph^{L(\mathbb{R})}_{\omega+1+\xi}$: level-3 uniform indiscernibles

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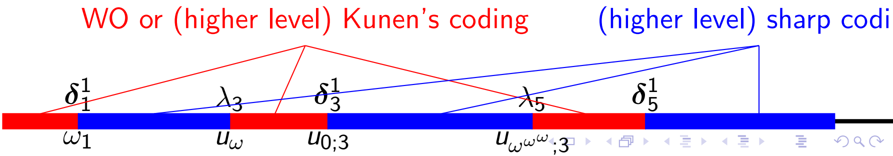
- The level-4 Martin-Solovay tree T_4 projecting to Π^1_4.
- The model theoretic form of Π^1_5 subsets of $u_{\omega^{\omega^\omega};3} \times \mathbb{R}$,
- $0^{4\#} \equiv_m M_3^\#$.



The model theoretic form of Π^1_5 sets

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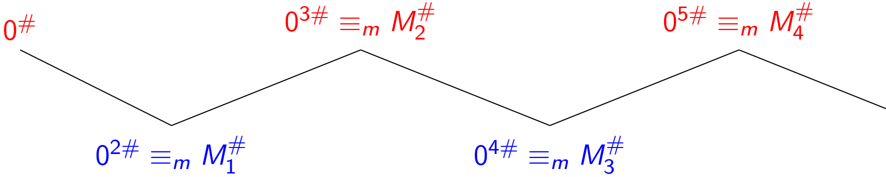
$0^{4\#}$ is obtained by squeezing $\mathcal{O}^{T_4}$ into a real. It is many-one equivalent to $M_3^{\#}$.

Continue!

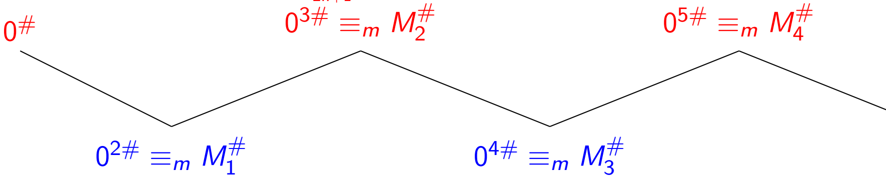


Question

1. *How to continue this analysis in $L(\mathbb{R})$ and beyond? I.e., an effective coding of ordinals in $\Theta^{L(\mathbb{R})}$, etc.*
2. *Is there a purely inner model theoretic approach? Test question: What is the fine structure of $L_{\kappa_3}[T_2]$?*

Continue!

Theory of the mouse $L_{\delta^1_{2n+1}}[T_{2n+1}]$ with level-$(2n+1)$ indiscernibles



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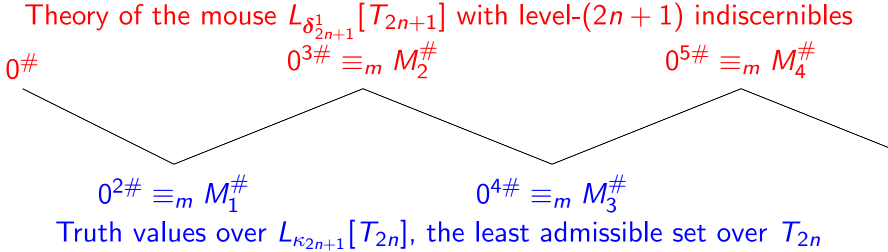

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Applications

Theorem

The following are equivalent.

1. $\boldsymbol{\Pi}^1_{2n+1}$ *and* Π^1_{2n+2} *sets are determined.*
2. *There is a countably iterable* $M^\#_{2n+1}$.

The $n = 0$ case is proved by Neeman and Woodin. The equivalence of $0^{(n+1)\#}$ and $M^\#_n$ allows the full generalization. (Open for $n > 0$) What if we replace $(2n+1, 2n+2)$ by $(2n, 2n+1)$?

The big hope

Every theorem in DST at the level of $\boldsymbol{\Sigma}^1_1$ under the assumption of sharps will generalize to projective sets under PD. E.g., the classification of thin projective equivalence relations on $\mathbb{R}$.

The first half of “the higher sharp” will be available on arXiv by the end of 2015.

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Thank you!