A new framework for Souslin-tree constructions

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This is joint work with Assaf Rinot, and still in progress.

Are these what Souslin trees look like?

Souslin Trees — History

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Is every ccc dense linear ordering necessarily separable?

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Theorem (Kurepa, 1935)

$$\exists \textit{ Souslin line} \iff \exists \textit{ Souslin tree.}$$

Definition

A tree T is Souslin if:

- it has height ω_1,
- every chain is countable, and
- every antichain is countable.

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Theorem

Souslin's problem is independent of ZFC.
Among other constructions:

$$\Diamond \implies \exists \text{ \textit{Souslin tree}} \quad \text{\textit{(Jensen, 1972)}}$$

$$\mathrm{MA}_{\aleph_1} \implies \nexists \text{ \textit{Souslin tree}} \quad \text{\textit{(Solovay \& Tennenbaum, 1971)}}$$

Souslin Trees — Higher Cardinals

What about κ-Souslin trees for higher cardinals?

Definition

For any regular cardinal κ, a tree T is κ-Souslin if:

- it has height κ,
- it has no chain of size κ,
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It seems to depend on the nature of κ (Jensen, 1972):

Successor of regular $\lambda^{<\lambda} = \lambda + \diamondsuit(E^{\lambda^+}_{\lambda}) \implies \exists\lambda^+$-Souslin tree.

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Inaccessible, not weakly compact $V = L \implies \exists\kappa$-Souslin tree.

A New Axiom: $\boxtimes^-(\kappa)$

Notation

For any set of ordinals D:

$$\mathrm{acc}(D) = \{\alpha \in D \mid \sup(D \cap \alpha) = \alpha > 0\}; \text{ and}$$
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- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

Building a Souslin Tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$

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- Each node $t \in T$ is a function $t : \alpha \to 2$ for some ordinal $\alpha < \kappa$;
- The tree order $<_T$ is simply extension of functions $\subset$;
- If $t : \alpha \to 2$ is in T, then $t \restriction \beta \in T$ for every $\beta < \alpha$.
- For all $t \in T$, $\mathrm{ht}(t) = \mathrm{dom}(t)$ and $t_\downarrow = \{t \restriction \beta \mid \beta < \mathrm{dom}(t)\}$.
- For all $\alpha < \kappa$, the level $T_\alpha = T \cap {}^{\alpha}2$.

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- For all $\alpha < \kappa$, the level $T_\alpha = T \cap {}^\alpha 2$.
  Motivation: ease of completing a branch at a limit level.
  If $\langle t_\alpha \mid \alpha < \beta\rangle$ (for some $\beta < \kappa$) is a $\subseteq$-increasing sequence of nodes in T, then the (unique) limit of this sequence, which may or may not be a member of T, is simply $\bigcup_{\alpha<\beta} t_\alpha$.

Refining an Old Axiom: From $\diamondsuit(\kappa)$ to $\diamondsuit^-(H_\kappa)$

Fix a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa\rangle$ such that for every $Z \subseteq \kappa$, the set $\{\beta < \kappa \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

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$\diamondsuit^-(H_\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa \rangle$ such that for every $p \in H_{\kappa^+}$ and $Z \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$ such that:

- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in \kappa$;
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- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in \kappa$;
- $\mathcal{M} \cap Z = Z_{\mathcal{M} \cap \kappa}$.

Proposition

$\diamondsuit(\kappa)$ is equivalent to $\diamondsuit^-(H_\kappa)$.

Preliminaries

Fix a $\diamondsuit^-(H_\kappa)$-sequence $\langle Z_\beta \mid \beta < \kappa \rangle$.
Fix a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ witnessing $\boxtimes^-(\kappa)$.
Fix a well-ordering $<_w$ on H_κ.

The Easy Part

Let $T_0 = \{\emptyset\}$.

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For every $\alpha < \kappa$, define

$$T_{\alpha+1} = \{s^\frown\langle i\rangle \mid s \in T_\alpha, i < 2\}.$$

The Hard Part

What do we do at limit levels?
Fix a limit ordinal $\alpha < \kappa$, and assume $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$ has already been defined.
We need to decide which branches through $T \restriction \alpha$ will have their limits placed in the level T_α of the tree.
We need T_α to contain enough nodes so that the tree is normal.
That is, for every $x \in T \restriction \alpha$, we need to place some node $\mathbf{b}_x^\alpha$ in T_α above x.
The node $\mathbf{b}_x^\alpha$ will be the limit of some sequence b_x^α in $T \restriction \alpha$.
But we have to choose these sequences carefully, so that the resulting tree doesn't have large antichains.

Identifying Cofinal Branches

Recall that C_α is a club subset of α.
For every $x \in T \restriction C_\alpha$, we will use C_α to identify a cofinal branch b^α_x through $\langle T \restriction \alpha, \subseteq \rangle$, containing x, as follows:

- b^α_x will be an increasing, continuous sequence of nodes.
- $\mathrm{dom}(b^\alpha_x) = C_\alpha \setminus \mathrm{ht}(x)$.
- $b^\alpha_x(\mathrm{ht}(x)) = x$.
- We will need to identify $b^\alpha_x(\beta) \in T_\beta$ for all $\beta \in C_\alpha$ with $\beta > \mathrm{ht}(x)$.
  We will do this by recursion over β, considering the cases $\beta \in \mathrm{nacc}(C_\alpha)$ and $\beta \in \mathrm{acc}(C_\alpha)$ in turn.

Intersecting a Maximal Antichain at Levels in $\mathrm{nacc}(C_\alpha)$

Suppose $\beta \in \mathrm{nacc}(C_\alpha)$ with $\beta > \mathrm{ht}(x)$.
Denote $\beta^- = \max(C_\alpha \cap \beta)$.
This exists and is in $\mathrm{dom}(b_x^\alpha)$, so that $b_x^\alpha(\beta^-)$ has been defined.
We need to identify $b_x^\alpha(\beta) \in T_\beta$, extending $b_x^\alpha(\beta^-)$.
Consider two possibilities:

- If there is some $y \in Z_\beta$ and $z \in T_\beta$ such that $b_x^\alpha(\beta^-) \cup y \subseteq z$, then let $b_x^\alpha(\beta)$ be the $<_w$-least such z.
- Otherwise, let $b_x^\alpha(\beta)$ be the $<_w$-least element of T_β extending $b_x^\alpha(\beta^-)$. Such a node must exist, because we are ensuring that the tree is normal as we construct every level.

Notice that if Z_β is a maximal antichain through $T \restriction \beta$, then in particular there is some $y \in Z_\beta \cap (T \restriction \beta)$ compatible with $b_x^\alpha(\beta^-)$, so that $b_x^\alpha(\beta^-) \cup y \in T \restriction \beta$, and then by normality there is $z \in T_\beta$ extending this, so that the first option applies.

Will We Get Stuck at Levels in $\mathrm{acc}(C_\alpha)$?

Suppose $\beta \in \mathrm{acc}(C_\alpha)$ with $\beta > \mathrm{ht}(x)$.
We want b_x^α to be continuous, so the only possible definition is:

$$b_x^\alpha(\beta) = \bigcup_{\gamma \in \mathrm{dom}(b_x^\alpha) \cap \beta} b_x^\alpha(\gamma).$$

Clearly $b_x^\alpha(\beta) \in {}^\beta 2$, but how do we know that $b_x^\alpha(\beta) \in T_\beta$?
This question highlights the difference between the classical approach and our new framework.

Coherence to the Rescue!

Since $\beta \in \mathrm{acc}(C_\alpha)$, our choice of the sequence satisfying $\boxtimes^-(\kappa)$ gives $C_\beta = C_\alpha \cap \beta$.

For every $\gamma \in \mathrm{dom}(b^\alpha_x) \cap \beta$, the value of $b^\beta_x(\gamma)$ was determined in exactly the same way as $b^\alpha_x(\gamma)$:

- starting with $b^\beta_x(\mathrm{ht}(x)) = x = b^\alpha_x(\mathrm{ht}(x))$;
- for $\gamma \in \mathrm{nacc}(C_\alpha)$: depending only on $b^\alpha_x(\gamma^-)$, Z_γ, and T_γ;
- for $\gamma \in \mathrm{acc}(C_\alpha)$: taking limits.

It follows that

$$b^\alpha_x(\beta) = \bigcup_{\gamma \in \mathrm{dom}(b^\alpha_x) \cap \beta} b^\alpha_x(\gamma) = \bigcup_{\gamma \in \mathrm{dom}(b^\beta_x)} b^\beta_x(\gamma) = \mathbf{b}^\beta_x.$$

Since $\beta < \alpha$, the level T_β has already been constructed, and the construction guarantees that we have included the limit $\mathbf{b}^\beta_x$ of the sequence b^β_x into T_β. But we have just shown that this is exactly $b^\alpha_x(\beta)$, so that $b^\alpha_x(\beta) \in T_\beta$, as required.

Completing the Construction of T_α

The sequence b_x^α just identified determines a cofinal branch through $T \restriction \alpha$ containing x.
As promised, we take its limit

$$\mathbf{b}_x^\alpha = \bigcup_{\beta \in \mathrm{dom}(b_x^\alpha)} b_x^\alpha(\beta),$$

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Finally, we collect all nodes constructed in this way, and let

$$T_\alpha = \{\mathbf{b}_x^\alpha \mid x \in T \restriction C_\alpha\}.$$

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Finally, we collect all nodes constructed in this way, and let

$$T_\alpha = \{\mathbf{b}_x^\alpha \mid x \in T \restriction C_\alpha\}.$$

Having constructed all levels of the tree, we then let

$$T = \bigcup_{\alpha < \kappa} T_\alpha.$$

Here We Use $\diamondsuit^{-}(H_\kappa)$

Claim

Suppose $A \subseteq T$ is a maximal antichain. Then the set

$$B = \{\beta < \kappa \mid A \cap (T \restriction \beta) = Z_\beta \text{ is a maximal antichain in } T \restriction \beta\}.$$

is a stationary subset of κ.

Proof.

Let $D \subseteq \kappa$ be an arbitrary club. We must show that $D \cap B \neq \emptyset$. Put $p = \{A, T, D\}$. Using the fact that the sequence $\langle Z_\beta \mid \beta < \kappa\rangle$ satisfies $\diamondsuit^{-}(H_\kappa)$, pick $\mathcal{M} \prec H_{\kappa^+}$ with $p \in \mathcal{M}$ such that $\beta = \mathcal{M} \cap \kappa$ is in κ and $Z_\beta = A \cap \mathcal{M}$. Since $D \in \mathcal{M}$ and D is club in κ, we have $\beta \in D$. We claim that $\beta \in B$. For all $\alpha < \beta$, by $\alpha, T \in \mathcal{M}$, we have $T_\alpha \in \mathcal{M}$, and by $\mathcal{M} \models |T_\alpha| < \kappa$, we have $T_\alpha \subseteq \mathcal{M}$. So $T \restriction \beta \subseteq \mathcal{M}$. As $\mathrm{dom}(z) \in \mathcal{M}$ for all $z \in T \cap \mathcal{M}$, we conclude that $T \cap \mathcal{M} = T \restriction \beta$. So, $Z_\beta = A \cap (T \restriction \beta)$. As $H_{\kappa^+} \models A$ is a maximal antichain in T and $T \cap \mathcal{M} = T \restriction \beta$, we get that $A \cap (T \restriction \beta)$ is maximal in $T \restriction \beta$. □

Verifying that T is Souslin

Claim

The tree $\langle T, \subset \rangle$ is a κ-Souslin tree.

Proof.

Let $A \subseteq T$ be a maximal antichain. From the previous claim,

$$B = \{\beta < \kappa \mid A \cap (T \restriction \beta) = Z_\beta \text{ is a maximal antichain in } T \restriction \beta\}$$

is a stationary subset of κ.

Thus we apply $\boxtimes^-(\kappa)$ to obtain a limit ordinal $\alpha < \kappa$ satisfying

$$\sup(\text{nacc}(C_\alpha) \cap B) = \alpha.$$

Consider any $v \in T_\alpha$. By construction, $v = \mathbf{b}_x^\alpha = \bigcup_{\beta \in \text{dom}(b_x^\alpha)} b_x^\alpha(\beta)$ for some $x \in T \restriction C_\alpha$. Fix $\beta \in \text{nacc}(C_\alpha) \cap B$ with $\text{ht}(x) < \beta < \alpha$. So $Z_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$. Thus we chose $b_x^\alpha(\beta)$ to extend some $y \in Z_\beta$. Altogether, $y \subseteq b_x^\alpha(\beta) \subseteq \mathbf{b}_x^\alpha = v$, as required. $\square$

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Theorem

$\diamondsuit(\kappa) + \boxtimes(\kappa)$ holds, assuming any of the following:

- *$\kappa = \aleph_1$ and $\diamondsuit(\aleph_1)$ holds;*
- *$\kappa = \lambda^+$ for λ singular, and $\square_\lambda + \mathrm{CH}_\lambda$ holds;*
- *$\kappa = \lambda^+$ for λ regular uncountable, and $\boxed{\diamondsuit}_\lambda$ holds;*
- *$\kappa = \lambda^+$, λ is not subcompact, and V is a Jensen-type extender model of the form $L[E]$;*
- *κ is a regular uncountable cardinal that is not weakly compact, and $V = L$;*
- *$\kappa = \lambda^+$ for λ regular uncountable and $V = W^{\mathrm{Add}(\lambda,1)}$, where*

$$W \models \mathrm{ZFC} + \square_\lambda + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda.$$

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$$W \models \mathsf{ZFC} + \square_\lambda + \mathsf{CH}_\lambda + \lambda^{<\lambda} = \lambda.$$

Thus, we get a κ-Souslin tree uniformly in all these scenarios!

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At each limit level $\alpha < \kappa$, we put (at most) one node into T_α for every $x \in T \restriction \alpha$.

It follows that $|T_\alpha| \leq \max\{|\alpha|, \aleph_0\}$ for every $\alpha < \kappa$.

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Thus we say the tree is slim.
What if we consider an opposite property?

Complete Souslin Trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

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But if there are elements of T_α that are not of the form $\mathbf{b}_x^\alpha$, then how do we kill the antichains using C_α?

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We need to tweak an axiom to avoid such levels.

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To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?
For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.
But if there are elements of T_α that are not of the form $\mathbf{b}_x^\alpha$, then how do we kill the antichains using C_α?
We need to tweak an axiom to avoid such levels.
Recall that the previous construction used $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$.
Which axiom should we strengthen: $\diamondsuit(\kappa)$ or $\boxtimes^-(\kappa)$?

Complete Souslin Trees

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Which axiom should we strengthen: $\diamondsuit(\kappa)$ or $\boxtimes^-(\kappa)$?
Classical constructions of χ-complete Souslin trees would replace $\diamondsuit(\kappa)$ with $\diamondsuit(E^\kappa_{\geq\chi})$. But we'll try something different. . . .

A Stronger Parameter: $\boxtimes^{-}(S)$

Definition

Fix a regular uncountable cardinal κ.
The principle $\boxtimes^{-}(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

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Theorem

For any regular uncountable cardinal κ and any infinite $\chi < \kappa$ satisfying $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$, $\diamondsuit(\kappa) + \boxtimes^-(E^\kappa_{\geq\chi})$ implies the existence of a χ-complete κ-Souslin tree.

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$\exists$ models satisfying $\diamondsuit(\kappa)$ and $\boxtimes^-(E^\kappa_{\geq\chi})$ in which $\diamondsuit(E^\kappa_{\geq\chi})$ fails.

Strengthening $\boxtimes^-(S)$ to $\boxtimes(S)$

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- for every sequence $\langle B_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that for all $i < \alpha$

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- for every sequence $\langle B_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that for all $i < \alpha$

$$\sup\{\beta < \alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha.$$

Hold On to Your Chairs . . .

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Why all the parameters?
By tweaking the various parameters, we can control the properties that the Souslin tree satisfies.

Example

For example, using $P(\aleph_7, 2, \sqsubseteq, \aleph_7, \{E^{\aleph_7}_{\aleph_6}\}, 2, \omega, ([\aleph_7]^{<\aleph_7})^2)$, we obtain:

Theorem

If $\boxed{\diamondsuit}_{\aleph_6} + \mathrm{GCH}$ holds, then there exists an $\aleph_7$-Souslin tree $(T, <_T)$ and a sequence of uniform ultrafilters $\langle \mathcal{U}_n \mid n < 7 \rangle$ such that:

- *If $n \in \{0, 1, 4, 5\}$, then $T^{\aleph_n}/\mathcal{U}_n$ is an $\aleph_7$-Aronszajn tree;*
- *If $n \in \{2, 3, 6\}$, then $T^{\aleph_n}/\mathcal{U}_n$ is not an $\aleph_7$-Aronszajn tree.*

Another Example

Baumgartner, Malitz & Reinhardt (1970) proved that every $\aleph_1$-Aronszajn tree can be made special in some cofinality-preserving extension. The next example is of a λ^+-Souslin tree that cannot be specialized without reducing it to the BMR scenario.

Theorem

Suppose $\square_\lambda + \mathrm{CH}_\lambda$ *holds for a given singular cardinal* λ *of countable cofinality.*
Then there exists a λ^+*-Souslin tree* $(T, <_T)$ *satisfying the following. If* W *is a* ZFC *extension of the universe in which* $(T, <_T)$ *is a special* $|\lambda|^+$*-tree, then* $W \models |\lambda| = \aleph_0$.

More Examples

Theorem

Assuming $P(\kappa, 2, \sqsubseteq, \kappa, \{\kappa\}, 2, 1, ([\kappa]^{<\kappa})^2)$*, there exists a coherent* κ*-Souslin tree.*

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Theorem

Assuming $P(\kappa, \kappa, {}_{\chi}{\sqsubseteq}, \kappa, \{E^{\kappa}_{\geq\chi}\}, 2, 1, ([\kappa]^{<\kappa})^2)$ *and* $\lambda^{<\chi} < \kappa$ *for all* $\lambda < \kappa$*, there exists a* χ*-complete,* χ*-free* κ*-Souslin tree.*

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Theorem

Assuming $\mathsf{GCH} + P(\lambda^+, \lambda^+, {}_{\chi}{\sqsubseteq}^*, 1, \{E^{\lambda^+}_{\lambda}\}, \lambda^+, 1, =^*)$*, there exists a* λ*-complete specializable* λ^+*-Souslin tree.*

Still More

By further tweaking the parameters and varying the construction slightly, we can construct a Souslin tree from weaker axioms than those mentioned earlier.

Theorem

A form of the proxy principle $P(\dots)$ holds enabling the construction of a λ^+-Souslin tree for uncountable λ, assuming any of the following:

- $\lambda^{<\lambda} = \lambda + \diamondsuit(E^{\lambda^+}_{\lambda})$;
- $V = W^{\mathrm{Add}(\lambda,1)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$;
- $V = W^{\mathrm{Prikry}(\lambda)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda$ *is measurable;*
- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary set of* $E^{\lambda^+}_{<\lambda}$.

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- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary set of* $E^{\lambda^+}_{<\lambda}$.
- $2^{<\lambda} = \lambda + \square^*_\lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary subset of* $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

Even More

Theorem

Assuming the consistency of a supercompact cardinal, there is a model of ZFC that satisfies:

1. *Martin's Maximum holds, and hence:*
   1.1 $\square^*_\lambda$ *fails for every singular cardinal* λ *of countable cofinality;*
   1.2 $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal* λ*;*
   1.3 *There does not exists any* $\aleph_1$*-Souslin or* $\aleph_2$*-Souslin tree.*
2. *There are no inaccessible cardinals;*
3. $P(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal* λ*;*
4. $P(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal* λ*.*

From (2), (3) and (4), it follows that there exists a free κ*-Souslin tree for all* $\kappa > \aleph_2$*.*

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