# A failure of amenability

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# Soundness and Fine Structure

# The language $\mathcal{L}_{(\mathrm{gen})}$ and $\mathcal{L}_{(\mathrm{gen})}$-structures

## Definition

1. Let $\mathcal{L}_{(\mathrm{gen})}$ be the language of set theory together with unary predicates $\dot{\mathbb{P}}$ and $\dot{P}$.
2. Suppose that $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ and that $\mathbb{P}_\alpha \subseteq J_\alpha[\mathbb{P}]$.
   - Then $\mathcal{M}$ defines an $\mathcal{L}_{(\mathrm{gen})}$-structure where $\dot{\mathbb{P}}$ interpreted by $\mathbb{P}|\alpha$ and $\dot{P}$ interpreted by $\mathbb{P}_\alpha$.
3. Suppose $\mathcal{M}$ is a (transitive) $\mathcal{L}_{(\mathrm{gen})}$-structure.
   - Then $\mathbb{P}_\mathcal{M}$ is the interpretation of $\dot{\mathbb{P}}$ and $P_\mathcal{M}$ is the interpretation of $\dot{P}$.

- We shall only consider transitive $\mathcal{L}_{(\mathrm{gen})}$-structures.

## Definition (2)

1. ${\mathcal{L}_{(\mathrm{gen})}}^+$ is $\mathcal{L}_{(\mathrm{gen})}$ expanded by adding 3-ary predicates $\dot{T}_n$ for $1 \leq n < \omega$.
2. Suppose $\theta$ is a formula of ${\mathcal{L}_{(\mathrm{gen})}}^+$.
   - $\theta$ is $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$ if $\theta$ is a $\Sigma_1$-formula relative to $\mathcal{L}_{(\mathrm{gen})}$.
   - $\theta$ is $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_{n+1}$ if there is a $\Sigma_1$-formula $\varphi(x_0, \ldots, x_m, x_{m+1}, x_{m+2})$ of $\mathcal{L}_{(\mathrm{gen})}$ such that

$$\theta = \exists x_m \exists x_{m+1} \exists x_{m+2} \left( \dot{T}_n(x_m, x_{m+1}, x_{m+2}) \wedge \varphi \right).$$

## Definition

1. For each formula $\varphi(x_0, \ldots, x_n, x_{n+1})$ of $\mathcal{L}^+_{(\mathrm{gen})}$ (with free occurrences of $x_{n+1}$), $\tau_\varphi(x_0, \ldots, x_n)$ is the Skolem term given by $\varphi$.
2. For each $1 \leq n < \omega$, $({}_{\mathcal{L}_{(\mathrm{gen})}})\mathrm{Sk}_n$ is the smallest collection of terms closed under composition and containing all the terms $\tau_\varphi$ where $\psi$ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$.
3. A formula $\psi$ is a **generalized** $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-**formula**, where $1 \leq n < \omega$, if for some $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formula $\varphi[x_0, \ldots, x_m]$,

$$\psi = \varphi(x_0, \ldots, x_m : \sigma_0, \ldots, \sigma_m)$$

   where

   - for each $i \leq m$, $\sigma_i \in ({}_{\mathcal{L}_{(\mathrm{gen})}})\mathrm{Sk}_n$, and $\sigma_i$ is free for $x_i$ in $\varphi$,
   - $\varphi(x_0, \ldots, x_m : \sigma_0, \ldots, \sigma_m)$ is the formula obtained from $\psi$ by substituting $\sigma_i$ for each free occurrence of $x_i$.

Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\mathrm{gen})}$-structure. By induction on $1 \leq n < \omega$, we define:

1. The interpretation of $\dot{T}_n$ in $\mathcal{M}$, denoted $T_n^{\mathcal{M}}$,
2. The $n$-th projectum of $\mathcal{M}$, denoted $\rho_n^{\mathcal{M}}$.
3. The interpretations of $\tau_\varphi$, denoted $\tau_\varphi^{\mathcal{M}}$.

- To simplify notation a bit we adopt the following conventions.
  - $\varphi(x_0, \ldots, x_m)$ indicates the free variables of $\psi$ are included in $\{x_0, \ldots, x_m\}$ and that $x_m$ is a free variable of $\varphi$.
  - Suppose $\varphi(x_0, \ldots, x_m)$ is a formula, $m > 0$, and $s \in |\mathcal{M}|^{<\omega}$.
    - $\mathcal{M} \models \varphi[\bar{s}]$ means both $|s| = m + 1$ and that

$$\mathcal{M} \models \varphi[s_0, \ldots, s_m].$$

## Definition

Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\mathrm{gen})}$-structure. Suppose $1 \leq n < \omega$.

1. Suppose that $\varphi(x_0, \ldots, x_{m+1})$ is a $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formula and that $\tau_\varphi(x_0, \ldots, x_{m+1})$ is the corresponding $({}_{\mathcal{L}_{(\mathrm{gen})}})\mathrm{Sk}_n$-term. Then for each
$$\langle a_i : i \leq m \rangle \in |\mathcal{M}|^{<\omega},$$
$\langle a_i : i \leq m \rangle \in \mathrm{dom}(\tau_\varphi^{\mathcal{M}})$ and $b = \tau_\varphi^{\mathcal{M}}(a_0, \ldots, a_m)$ if
    - $\mathcal{M} \models \varphi[a_0, \ldots, a_m, b]$,
    - for all $c <_{\mathcal{M}} b$, $\mathcal{M} \models (\neg\varphi)[a_0, \ldots, a_m, c]$.
2. For each $X \subseteq |\mathcal{M}|$, $\mathrm{Th}_n^{\mathcal{M}}(X)$ is the set:
$$\{(\varphi, s) \mid s \in X^{<\omega}, \psi \text{ is generalized } ({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n, \mathcal{M} \models \varphi[\bar{s}]\}.$$
3. $\rho_n^{\mathcal{M}}$ is the least ordinal $\rho \leq \alpha$, such that $\mathrm{Th}_n(\rho \cup \{q\}) \notin |\mathcal{M}|$ for some $q \in |\mathcal{M}|$, where $\alpha = |\mathcal{M}| \cap \mathrm{Ord}$.
4. $T_n^{\mathcal{M}}(\alpha, q, b)$ if and only if $\alpha < \rho_n^{\mathcal{M}}$, $q \in |\mathcal{M}|$, and $b = \mathrm{Th}_n^{\mathcal{M}}(\alpha \cup \{q\})$.

## Definition

Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\mathrm{gen})}$-structure. Suppose $X \subseteq |\mathcal{M}|$, $X \neq \emptyset$, $1 \leq n < \omega$, and $\rho_k^{\mathcal{M}} > 0$ for all $0 < k < n$.

1. $S_n^{\mathcal{M}}(X) = \{\tau^{\mathcal{M}}(s) \,|\, s \in \mathrm{dom}(\tau_\varphi^{\mathcal{M}}) \cap X^{<\omega}$ and $\tau \in ({}_{\mathcal{L}_{(\mathrm{gen})}})\mathrm{Sk}_n\}$.
2. $\mathcal{H}_n^{\mathcal{M}}(X)$ is the $\mathcal{L}_{(\mathrm{gen})}$-structure given by the transitive collapse of
$$\left(S_n^{\mathcal{M}}(X), \mathbb{P}_{\mathcal{M}} \cap S_n^{\mathcal{M}}(X), P_{\mathcal{M}} \cap S_n^{\mathcal{M}}(X)\right)$$

## Definition (6)

Suppose $\mathcal{M}$ is an amenable $\mathcal{L}_{(\mathrm{gen})}$-structure.

- Then $\mathcal{M}$ is $\omega$-sound if for each $k+1<\omega$, one of the following hold.
  1. $k>0$ and $\rho_k^{\mathcal{M}}=0$.
  2. There exists $a\in\mathcal{M}$ such that $\mathcal{M}=\mathcal{H}_{k+1}^{\mathcal{M}}(\rho_{k+1}^{\mathcal{M}}\cup\{a\})$.

## Claim

*If $\mathcal{M}$ is an $\mathcal{L}_{(\mathrm{gen})}$-structure such that*

$$\mathcal{M}\models\mathrm{ZFC}$$

*then $\mathcal{M}$ is $\omega$-sound.*

# Why sound structures are so useful

Suppose $\mathcal{M}$ is an amenable $\omega$-sound $\mathcal{L}_{(\mathrm{gen})}$-structure, $0 < k < \omega$, and

$$\rho_k^M > 0.$$

### (An equivalent structure)

*Let $\rho = \rho_k^{\mathcal{M}}$ and let $q \in \mathcal{M}$ be such that $\mathcal{M} = \mathcal{H}_k^{\mathcal{M}}(\rho, \{q\})$. Then the structure, $\mathcal{N} = (\mathcal{M}|\rho, T)$, is amenable where $T = \mathrm{Th}_k^{\mathcal{M}}(\rho \cup \{q\})$, naturally coded as a subset of $\mathcal{M}|\rho$.*

### (The key points)

- *$\mathcal{N}$ is $\omega$-sound and $\rho_1^{\mathcal{N}} = \rho_{k+1}^{\mathcal{M}}$.*
- *Suppose $A \subset \mathcal{M}|\rho$. Then the following are equivalent.*
  - *$A$ is $(\mathcal{L}_{(\mathrm{gen})})\Sigma_1$-definable in $\mathcal{N}$ from parameters.*
  - *$A$ is $(\mathcal{L}_{(\mathrm{gen})})\Sigma_{k+1}$-definable in $\mathcal{M}$ from parameters.*

# Amenable and sound predicates

**Definition**

Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $J[\mathbb{P}]$ is **amenable** if for all $\alpha \in \mathrm{dom}(\mathbb{P})$ the following hold.

1. $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \models \Sigma_1\text{-Replacement}$
2. $\mathbb{P}_\alpha \subseteq J_\alpha[\mathbb{P}]$,
3. For all $\beta < \alpha$, $\mathbb{P}_\alpha \cap J_\beta[\mathbb{P}] \in J_\alpha[\mathbb{P}]$.

**Definition**

Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is amenable. Then $J[\mathbb{P}]$ is **sound** if for each $\alpha \in \mathrm{Ord}$:

1. $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \emptyset)$ is $\omega$-sound,
2. if $\alpha \in \mathrm{dom}(\mathbb{P})$ then $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ is $\omega$-sound.

## Lemma (9)

*Suppose that* $\mathbb{P} \subset \mathrm{Ord} \times V$, $J[\mathbb{P}]$ *is amenable, and* $J[\mathbb{P}]$ *is sound. Then* $\mathrm{GCH}$ *holds in* $J[\mathbb{P}]$.

## Lemma (10)

*Assume* $\mathrm{GCH}$. *Suppose that* $\kappa$ *is* $\kappa^{+\omega}$*-supercompact and that* $\mathbb{P} \subset V_\kappa$. *Then there exist* $\delta < \kappa$ *and an elementary embedding*

$$\pi : (H(\gamma), \mathbb{P} \cap H(\gamma)) \to (H(\pi(\gamma)), \mathbb{P} \cap H(\pi(\gamma)))$$

*such that*

(1) $\pi \in V_\kappa$ *and* $\mathrm{CRT}(\pi) = \delta$,

(2) $\gamma = \delta^{+(\omega+1)}$ *and* $\pi(\gamma) = (\pi(\delta))^{+(\omega+1)}$.

# Proof.

Since $\mathrm{GCH}$ holds, $\kappa$ is $\kappa^{+(\omega+1)}$-supercompact. Let

$$j : V \to M$$

be an elementary embedding such that $\mathrm{CRT}(j) = \kappa$ and $M^\lambda \subset M$ where

$$\lambda = \kappa^{+(\omega+1)} = |V_{\kappa+\omega+1}| = |H(\kappa^{+(\omega+1)})|.$$

Let $N = j(M)$ and let

$$j(j) \circ j : V \to N$$

be the iteration embedding.

Then

- $j|H(\kappa^{+(\omega+1)}) \in j(j) \circ j(V_\kappa) = N_{j(j(\kappa))}$

witness that the conclusion of lemma holds in $N$ at $j(j) \circ j(\kappa)$ for $j(j) \circ j(\mathbb{P})$. $\square$

## Theorem (11)

*Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is amenable and sound. Then*

$$J[\mathbb{P}] \models \text{“There are no cardinals } \kappa \text{ which are } \kappa^{+\omega}\text{-supercompact”}.$$

Proof.

We work in $(J[\mathbb{P}], \mathbb{P})$. Assume toward a contradiction that there exists $\kappa$ such that $\kappa$ is $\kappa^{+\omega}$-supercompact. Therefore by Lemma 9 and Lemma 10, there exist $\delta < \kappa$ and an elementary embedding

$$\pi : (J_\gamma[\mathbb{P}], \mathbb{P}|\gamma) \to (J_{\pi(\gamma)}[\mathbb{P}], \mathbb{P}|\pi(\gamma))$$

such that

1. $\mathrm{CRT}(\pi) = \delta$ and $\gamma = \delta^{+(\omega+1)}$,
2. $\pi(\gamma) = (\pi(\delta))^{+(\omega+1)}$.

# Proof continued

Let $\eta = \text{sup}(\pi[\gamma])$. Thus

$$(J_\eta[\mathbb{P}], \mathbb{P}|\eta) \models \mathrm{ZFC} \backslash \mathrm{Powerset}.$$

**Claim (1)**

*There can be no closed set $C \subset \eta$ such that*

(i) $|C| < \pi(\delta^{+\omega})$ *and* $C$ *is cofinal in* $\eta$,

(ii) $C \cap \xi \in J_\eta[\mathbb{P}]$ *for all* $\xi < \eta$.

Assume toward a contradiction that $C$ exists. Let

- $D = \{\xi < \gamma \mid \pi(\xi) \in C\}$.

Thus $D$ is $\omega$-closed and by (i), $D$ is cofinal in $\gamma$. Let

- $\xi_0 \in D$ be such that $|D \cap \xi_0| \geq \delta^{+\omega}$.

Then $C \cap \pi(\xi_0)$ covers $\pi[D \cap \xi_0]$ and by (ii),

$$\mathcal{P}(C \cap \pi(\xi_0)) \in J_\eta[\mathbb{P}].$$

# Proof continued

This implies that $\pi[\delta^{+\omega}] \in J_\eta[\mathbb{P}]$ and so

$$\pi | H(\delta^{+\omega}) \in J_\eta[\mathbb{P}].$$

But then $\pi[\gamma]$ is definable from parameters in $J_\eta[\mathbb{P}]$ and this is a contradiction since $\eta = \sup(\pi[\gamma])$ and

$$(J_\eta[\mathbb{P}], \mathbb{P}|\eta) \models \mathrm{ZFC} \setminus \mathrm{Powerset}.$$

This proves the Claim 1.
Let $\alpha > \eta$ be least such that for some $0 < k < \omega$,

$$\rho_k^{\mathcal{M}} < \eta$$

where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha)$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$ and

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

otherwise.

- We assume $\alpha \in \mathrm{dom}(\mathbb{P})$. The case that $\alpha \notin \mathrm{dom}(\mathbb{P})$ is easier.

# Proof continued

Fix $k$ to be least such that $\rho_k^{\mathcal{M}} < \eta$. Thus $\rho_k^{\mathcal{M}} = \pi(\delta)^{+\omega}$.

**Claim (2)**

$k > 1$.

Assume toward a contradiction that $k = 1$.

**(By soundness))**

- *There exists $q \in J_\alpha[\mathbb{P}]$ such that*

$$\mathcal{M} = \mathcal{H}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{q\})$$

Thus there is a partial function

$$f : \pi(\delta)^{+\omega} \to \eta$$

such that $f$ is a surjection and such that $f$ is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{M}$ from parameters.

# Proof continued

**Claim**

*We can reduce to the case that $f$ is $\Sigma_1$-definable from parameters in the structure*

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha).$$

- This is because $\mathcal{M}$ is amenable.

Fix a $\Sigma_1$-formula $\varphi(x_0, x_1, x_2)$ and $c_0 \in J_\alpha[\mathbb{P}]$ such that

$$f = \{(a, b) \in J_\alpha[\mathbb{P}] \mid (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha) \models \varphi[a, b, c_0]\}$$

**Claim**

*We can require that $\varphi(x_0, x_1, x_2)$ has the form*

$$(\exists \xi < \alpha)\psi\left[x_0, x_1, x_2, \mathbb{P}_\alpha \cap J_\xi[\mathbb{P}]\right]$$

*where $\psi$ is a $(\mathcal{L}_{(\text{gen})})\Sigma_1$-formula not mentioning $\dot{P}$*

- *which is the predicate for $\mathbb{P}_\alpha$.*

## Proof continued

Fix $\pi(\delta)^{+\omega} < \alpha_0 < \alpha$ such that $c_0 \in J_{\alpha_0}[\mathbb{P}]$.

*For each $\alpha_0 < \beta < \alpha$, let $f_\beta$ be the set of all $(a, b) \in J_\beta[\mathbb{P}]$ such that*

$$(J_\beta[\mathbb{P}], \mathbb{P}|\beta) \models \psi[a, b, c_0, \mathbb{P}_\alpha \cap J_\xi[\mathbb{P}]]$$

*for some $\xi < \beta$ such that $\mathbb{P}_\alpha \cap J_\xi[\mathbb{P}] \in J_\beta[\mathbb{P}]$.*

Thus

- $f_\beta \in J_\alpha[\mathbb{P}]$ and $f_\beta \subseteq f$,
- $f = \cup\{f_\beta \mid \alpha_0 < \beta < \alpha\}$.

**Case 1**: $\mathrm{cof}(\alpha) \neq \delta^{+(\omega+1)}$.

There must exist $\pi(\delta)^{+\omega} < \beta < \alpha$ such that $f_\beta$ has range cofinal in $\eta$. But

$$f_\beta \in J_\alpha[\mathbb{P}].$$

This contradicts the choice of $\alpha$.

# Proof continued

**Case 2**: $\mathrm{cof}(\alpha) = \delta^{+(\omega+1)}$.

Let $\theta < \pi(\delta)^{+\omega}$ be least such that $f|\theta$ has cofinal range in $\eta$. Thus

- $\mathrm{cof}(\theta) = \mathrm{cof}(\eta) = \delta^{+(\omega+1)}$.

Fix $X \subset \theta$ such that

- $|X| = \delta^{+(\omega+1)}$ and $f|X$ has cofinal range in $\eta$.

We have $H(\pi(\delta)^{+\omega}) \subset J_\alpha[\mathbb{P}]$ and so clearly $X \in J_\alpha[\mathbb{P}]$. Therefore:

**Claim**

*There is an increasing cofinal continuous function*

$$g : \delta^{+(\omega+1)} \to \eta$$

*such that*

- *$g$ is $\Sigma_1$-definable from parameters in $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$*
- *$g|\xi \in J_\alpha[\mathbb{P}]$ for each $\xi < \delta^{+(\omega+1)}$.*

- Let $C$ be the range of $g$. Then $C$ satisfies (i)–(ii) of Claim 1, which is a contradiction.

This proves Claim 2.

# Proof continued

Let $n = k - 1 > 0$. Thus:

- $\rho_n^{\mathcal{M}} > \eta$.
- Either $\rho_n^{\mathcal{M}} = \alpha$ or $\rho_n^{\mathcal{M}}$ is a cardinal of $J_\alpha[\mathbb{P}]$.

Let:

- $\rho = \rho_n^{\mathcal{M}}$.
- Fix $q \in J_\alpha[\mathbb{P}]$ such that $\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho \cup \{q\})$.

**Claim**

*The structure,*

$$\mathcal{N} = (J_\rho[\mathbb{P}], T)$$

*is amenable where* $T = \mathrm{Th}_n^{\mathcal{M}}(\rho \cup \{q\})$, *naturally coded as a subset of* $\rho$.

The key points are

- $\rho_1^{\mathcal{N}} = \rho_{n+1}^{\mathcal{M}} = \rho_k^{\mathcal{M}} = \pi(\delta)^{+\omega}$.
- For some $p \in J_\rho[\mathbb{P}]$, $\mathcal{N} = \mathcal{H}_1^{\mathcal{N}}(\rho_1^{\mathcal{N}} \cup \{p\})$.

We can now just repeat the proof of Claim 2 that $k > 1$. $\square$

# Weakly amenable structures

**Definition**

Suppose that $\alpha \in \mathrm{dom}(\mathbb{P})$ and $\alpha$ is a limit ordinal. Then

$$(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

is $\omega$-**weakly amenable** if

$$\mathbb{P}_\alpha \subset \omega \times J_\alpha[\mathbb{P}]$$

and for each $n < \omega$, there exists a limit ordinal $\gamma_n \leq \alpha$ such that

1. $(\mathbb{P}_\alpha)_n \subset J_{\gamma_n}[\mathbb{P}]$,
2. $(\mathbb{P}_\alpha)_n \cap J_\xi[\mathbb{P}] \in J_\alpha[\mathbb{P}]$ for each $\xi < \gamma_n$.

**Definition**

Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $\mathbb{P}$ is $\omega$-**weakly amenable** if for all $\alpha \in \mathrm{dom}(\mathbb{P})$, $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ is $\omega$-weakly amenable.

# $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formulas for non-amenable structures

**Definition**

$\mathcal{L}^{-}_{(\mathrm{gen})}$ is $\mathcal{L}_{(\mathrm{gen})}$ reduced by eliminating $\dot{P}$.

**Definition (15)**

Suppose $\theta$ is a formula of ${\mathcal{L}_{(\mathrm{gen})}}^{+}$.

1. $\theta$ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$ if there is a $\Sigma_1$-formula $\varphi(x_0, x_1)$ of $\mathcal{L}^{-}_{(\mathrm{gen})}$ such that
$$\theta = \exists x_0 \left( \text{“}x_0 \subset \dot{P}\text{''} \wedge \text{“}x_0 \text{ is finite''} \wedge \varphi \right).$$
2. $\theta$ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_{n+1}$ if there is a $\Sigma_1$-formula $\varphi(x_0, \ldots, x_m, x_{m+1}, x_{m+2})$ of $\mathcal{L}^{-}_{(\mathrm{gen})}$ such that
$$\theta = \exists x_m \exists x_{m+1} \exists x_{m+2} \left( \dot{T}_n(x_m, x_{m+1}, x_{m+2}) \wedge \varphi \right).$$

# Soundness for non-amenable structures

*The notion of soundness is exactly as defined for the case of amenable* $\mathbb{P}$ *except that one uses the revised definition of* $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$*-formulas.*

- For the amenable $\mathcal{L}_{(\mathrm{gen})}$-structures, this amounts to simply replacing $\mathbb{P}$ with $\mathbb{P}^*$ where

$$\mathrm{dom}(\mathbb{P}) = \mathrm{dom}(\mathbb{P}^*)$$

and for each $\alpha \in \mathrm{dom}(\mathbb{P})$,

$$\mathbb{P}^*_\alpha = \{\mathbb{P}_\alpha \cap J_\xi[\mathbb{P}] \mid \xi < \alpha\}.$$

## Definition

Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $J[\mathbb{P}]$ is **sound** if for each $\alpha \in \mathrm{Ord}$, $\mathcal{M}$ is $\omega$-sound where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \emptyset)$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$, and

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

otherwise.

## Lemma (17)

*Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and $J[\mathbb{P}]$ is sound. Then $\mathrm{GCH}$ holds in $J[\mathbb{P}]$.*

# The approachability property

## Definition (Foreman-Magidor)

Suppose $\kappa$ is an infinite cardinal. Then $\mathcal{AP}_{\kappa^+}$ holds if there is a sequence

$$\langle C_\alpha : \alpha < \kappa^+ \rangle$$

such that for all limit $\alpha < \kappa^+$:

- $C_\alpha$ is a closed cofinal subset of $\alpha$ and $\mathrm{ordertype}(C_\alpha) \leq \kappa$.
- If $\mathrm{cof}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$.
- For all $\beta < \alpha$, $C_\alpha \cap \beta = C_\gamma$ for some $\gamma < \alpha$.

- This is called the **Approachability Property at $\kappa^+$** and is usually defined slightly differently.
  - First studied by Shelah.
- This definition highlights the principle as a very weak version of $\square$
  - and makes clear what a witness for $\mathcal{AP}_{\kappa^+}$ is.

# A lemma of Foreman–Magidor

- The following gives the equivalent formulation of $\mathcal{AP}_{\kappa^+}$.

**Lemma (19)**

*Suppose that $\kappa$ is an infinite cardinal. Then the following are equivalent.*

(1) $\mathcal{AP}_{\kappa^+}$ *holds.*

(2) *For all*

$$X \prec H(\kappa^{++})$$

*if $|X| = \kappa$ and $\kappa \subset X$ then there exists a closed cofinal subset $C \subset X \cap \kappa^+$ such that*

- $\mathrm{ordertype}(C) \leq \kappa$,
- *if* $\mathrm{cof}(X \cap \kappa^+) < \kappa$ *then* $|C| < \kappa$,
- $C \cap \xi \in X$ *for all* $\xi < X \cap \kappa^+$.

# Soundness and $\mathrm{GCH}$

**Lemma**

*Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then the following are equivalent.*

(1) $J[\mathbb{P}] \models \mathrm{GCH}$.

(2) *There exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:*
   - $J[\mathbb{P}^*]$ *is sound.*
   - $J[\mathbb{P}] = J[\mathbb{P}^*]$.
   - $\mathbb{P}^*$ *is $\Sigma_2$-definable in $(J[\mathbb{P}], \mathbb{P})$.*

# Soundness, amenability, and $\mathcal{AP}_{\kappa^+}$

## Lemma (21)

*Suppose that* $\mathbb{P} \subset \mathrm{Ord} \times V$ *and that* $J[\mathbb{P}] \models \mathrm{GCH}$. *Then the following are equivalent.*

(1) *For each uncountable cardinal* $\kappa$ *of* $J[\mathbb{P}]$,
- $\mathcal{AP}_{\kappa^+}$ *holds in* $J[\mathbb{P}]$,

(2) *There exists* $\mathbb{P}^* \subset \mathrm{Ord} \times V$ *such that:*
- $J[\mathbb{P}^*]$ *is amenable and sound.*
- $J[\mathbb{P}] = J[\mathbb{P}^*]$.
- $\mathbb{P}^*$ *is* $\Sigma_2$*-definable in* $(J[\mathbb{P}], \mathbb{P})$.

(3) *There exists* $\mathbb{P}^* \subset \mathrm{Ord} \times V$ *such that:*
- $J[\mathbb{P}^*]$ *is* $\omega$*-weakly amenable and sound.*
- $J[\mathbb{P}] = J[\mathbb{P}^*]$.
- $\mathbb{P}^*$ *is* $\Sigma_2$*-definable in* $(J[\mathbb{P}], \mathbb{P})$.

## Theorem (22)

*Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is $\omega$-weakly amenable and sound. Then*

$$J[\mathbb{P}] \models \text{"There are no cardinals } \kappa \text{ which are } \kappa^{+\omega}\text{-supercompact"}.$$

Proof.

By Lemma 21 there exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:

- $J[\mathbb{P}^*]$ is amenable and sound.
- $J[\mathbb{P}] = J[\mathbb{P}^*]$.
- $\mathbb{P}^*$ is $\Sigma_2$-definable in $(J[\mathbb{P}], \mathbb{P})$.

The theorem is an immediate corollary of this by Theorem 11. □

# Recall:

**Definition**

Suppose that $\mathbb{E}$ is an extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$. Then $\mathbb{E}$ is a **good partial extender sequence at** $\alpha$ if the following hold where $E$ is the partial extender $\mathbb{E}_\alpha$.

1. $J_\alpha[\mathbb{E}]$ is strongly acceptable and $J_\alpha[\mathbb{E}] \models \mathrm{ZFC} \setminus \mathrm{Powerset}$.
2. $E$ is a $J_\alpha[\mathbb{E}]$-extender.
3. (Indexing) $E$ is the Jensen completion of $E|\nu_E$ and $\alpha = \mathrm{LTH}(E_\alpha)$.
4. (Coherence) Let

$$j_{E_\alpha} : J_\alpha[\mathbb{E}] \to \mathrm{Ult}(J_\alpha[\mathbb{E}], E)$$

   be the ultrapower embedding and let $\xi = j_{E_\alpha}(\kappa)$. Then

$$j_{E_\alpha}(\mathbb{E}|\xi)|(\alpha + 1) = \mathbb{E}|\alpha).$$

# The weak initial segment condition

**Definition**

Suppose that $\mathbb{E}$ is a good partial extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$.
Let

$$j_E : J_\alpha[\mathbb{E}] \to \mathrm{Ult}(J_\alpha[\mathbb{E}], \mathbb{E}_\alpha)$$

be the ultrapower embedding.

- Then $\mathbb{E}$ satisfies the **weak initial segment condition** at $\alpha$ if

$$\mathbb{E}_\alpha | j_E(\eta) \in J_\alpha[\mathbb{E}]$$

for all $\eta < \iota_{\mathbb{E}_\alpha}$.

# The weak initial segment condition and amenability

**Lemma**

*Suppose that $\mathbb{E}$ is a good partial extender sequence which satisfies the weak initial segment condition at all $\alpha \in \mathrm{dom}(\mathbb{E})$.*

- *Then for all $\alpha \in \mathrm{dom}(\mathbb{E})$, $\mathbb{E}_\alpha$ can be coded by a set $E$ such that for some $\beta \leq \alpha$,*
  - $\beta \leq \nu_{\mathbb{E}_\alpha}$ *and* $(J_\beta[\mathbb{E}], \mathbb{E}|\beta) \models \Sigma_1\text{-Replacement}$.
  - $E \subset J_\beta[\mathbb{E}]$ *and* $(J_\beta[\mathbb{E}], E)$ *is amenable.*
  - $(J_\beta[\mathbb{E}], E)$ *and* $(J_\alpha[\mathbb{E}], \mathbb{E}_\alpha)$ *are logically equivalent.*

- $(J_\beta[\mathbb{E}], E)$ is a **reshaping** of $(J_\alpha[\mathbb{E}], \mathbb{E}_\alpha)$.

## Theorem (Weak $(\omega_1 + 1)$ Iteration Hypothesis)

*Assume there is an extendible cardinal. Then there exists a good partial extender sequence $\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \mathrm{dom}(\mathbb{E}) \rangle$ such that the following hold.*

(1) *$J[\mathbb{E}]$ is weakly backgrounded and $\mathbb{E}$ is weakly $\Sigma_2$-definable.*

(2) *$J[\mathbb{E}]$ satisfies comparison.*

(3) *For each $\xi$ there exists $\alpha \in \mathrm{dom}(\mathbb{E})$ such that*
- $\alpha > \xi$,
- *$\mathbb{E}_\alpha$ is a $J[\mathbb{E}]$-extender which witnesses that $\kappa$ is $\omega$-extendible in $J[\mathbb{E}]$ where $\kappa = \mathrm{CRT}(\mathbb{E}_\alpha)$.*

(4) *$\mathbb{E}$ satisfies the weak initial segment condition at all $\alpha \in \mathrm{dom}(\mathbb{E})$.*

(5) *The levels of $J[\mathbb{E}]$ are $\omega$-sound modulo reshaping.*

# Weak extender models and comparison

- One can naturally generalize the formulation of comparison to arbitrary inner models $N$.

**Definition**

Suppose that $N$ is a weak extender model for $\delta$ is supercompact and that $N \models$ "$V = \mathrm{HOD}$".
Then $N$ satisfies **comparison** if for all $X, Y \prec_{\Sigma_2} N$ the following hold where $N_X$ is the transitive collapse of $X$ and $N_Y$ is the transitive collapse of $Y$.
Suppose that $N_X$ and $N_Y$ are finitely generated models of $\mathrm{ZFC}$ and

$$N_X \cap \mathbb{R} = N_Y \cap \mathbb{R}.$$

- Then there exist a transitive set $N^*$ and elementary embeddings

$$\pi_X : N_X \to N^*$$

and

$$\pi_Y : N_Y \to N^*$$

such that $\pi_X$ is close to $N_X$ and $\pi_Y$ is close to $N_Y$.

## A test question for Ultimate-$L$

*Can a weak extender model for supercompactness satisfy comparison?*

## Summary

*Going beyond the level of $\omega$-extendible cardinals requires:*

- *Allowing the weak initial segment condition to fail.*
- *Allowing levels which are not even $\omega$-weakly amenable.*
- *Altering how comparison is proved if one reaches supercompactness.*

## Claim

- *Therefore we cannot just use good partial extender sequences.*
- *A natural alternative is to augment extender models with their iteration strategies.*
    - *These are* strategic-extender *models.*

# Epilogue

# The picture which seems to be emerging

- At the lowest levels, reaching past measurable cardinals, the fine structure models can be simply defined and there is no distinction between the extender and strategic-extender hierarchies.
- Ascending to levels below that of one Woodin cardinal there is still no distinction (at reasonable closure levels) between the extender and strategic-extender hierarchies.
  - However the fine structure models no longer are known to have a simple definition.

- Passing one Woodin cardinal and up to the finite levels of supercompactness, the extender and strategic-extender hierarchies strongly diverge but both exist and are weakly amenable hierarchies. But:
  - Existence within the extender hierarchy **requires** an iteration hypothesis
  - Existence is still **open** for the strategic-extender hierarchy from any iteration hypothesis.
- Reaching beyond the finite levels of supercompactness requires a complete failure of amenability.
- Finally at some point past the finite levels of supercompact, the extender hierarchy fails and one is left with just the strategic-extender hierarchy.

# Defining the axiom $V = L$ without defining $L$

## Claim

*In the context of a a strong cardinal which is a limit of Woodin cardinals, there are naturally defined approximations to* $\mathrm{Ultimate}\text{-}L$ *and the collection is rich enough to make a definiton of the axiom,* $V = \mathrm{Ultimate}\text{-}L$*, possible*

- *without yet knowing the detailed level-by-level definition of* $\mathrm{Ultimate}\text{-}L$.

## Lemma

*The following are equivalent.*

(1) $V = L$.

(2) *For each* $\Sigma_2$*-sentence* $\varphi$*, if* $V \models \varphi$ *then there exists a countable ordinal* $\alpha$ *such that* $N \models \varphi$ *where*

- $N = \cap\{M \mid M \models \mathrm{ZFC}\backslash\mathrm{Powerset} \text{ and } \mathrm{Ord}^M = \alpha\}$.

# The universally Baire sets

**Definition (Feng-Magidor-Woodin)**

A set $A \subseteq \mathbb{R}$ is **universally Baire** if for all topological spaces $\Omega$ and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of $A$ by $\pi$ has the property of Baire in the space $\Omega$.

**Theorem**

*Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then*

1. *Every set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.*
2. $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

# $\mathrm{HOD}^{L(A,\mathbb{R})}$ and large cardinal axioms

**Definition**

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals $\alpha$ such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\Theta^{L(A,\mathbb{R})}$ is a measure of the complexity of $A$.

**Theorem**

*Suppose that there is a proper class of Woodin cardinals and that $A$ is universally Baire.*

*Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes $\mathrm{HOD}$ as defined **within** $L(A, \mathbb{R})$.
  - $L(A, \mathbb{R}) \models \mathrm{ZF}$ but $L(A, \mathbb{R}) \not\models \mathrm{ZFC}$.
    - $\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{ZFC}$.

# The axiom $V = \text{Ultimate-}L$

Assume there are infinitely many Woodin cardinals with a measurable cardinal above. Then for many universally Baire sets $A \subset \mathbb{R}$,

$$\mathrm{HOD}^{L(A,\mathbb{R})}$$

has been verified to be a strategic-extender model.

- The natural conjecture is that must be true for **all** the universally Baire sets.

### The axiom for $V = \text{Ultimate-}L$

- *There is a strong cardinal which is a limit of Woodin cardinals.*
- *For each $\Sigma_3$-sentence $\varphi$, if $\varphi$ holds in $V$ then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

*where $\Theta = \Theta^{L(A,\mathbb{R})}$.*

# Consequences of $V = \mathrm{Ultimate}\text{-}L$

**Theorem ($V = \mathrm{Ultimate}\text{-}L$)**

*The Continuum Hypothesis holds.*

**Theorem ($V = \mathrm{Ultimate}\text{-}L$)**

*$V$ is **not** a generic extension of any transitive class $N \subset V$.*

**Theorem ($V = \mathrm{Ultimate}\text{-}L$)**

$V = \mathrm{HOD}$.

**Question**

*Does $V = \mathrm{Ultimate}\text{-}L$ imply comparison?*

# The Ultimate-$L$ Conjecture

**Question**

*Is there a generalization of Scott's theorem to $V = \mathrm{Ultimate}\text{-}L$?*

**$\mathrm{Ultimate}\text{-}L$ Conjecture**

(ZFC) *Suppose that $\delta$ is an extendible cardinal. Then there is a transitive class $N$ such that:*

1. *$N$ is a weak extender model of $\delta$ is supercompact.*
2. *$N \subseteq \mathrm{HOD}$.*
3. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*

- The conjecture implies there is no generalization of Scott's theorem to the case of $V = \mathrm{Ultimate}\text{-}L$.