Mathias forcing for filters and combinatorial covering properties

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A subset $\mathcal{F}$ of $[\omega]^\omega$ is called a filter if $\mathcal{F}$ contains all cofinite sets, is closed under finite intersections of its elements, and under taking supersets.

$\mathbb{M}_\mathcal{F}$ consists of pairs $\langle s, F \rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F \rangle$ is stronger than $\langle t, U \rangle$ if $F \subset U$, $s$ is an end-extension of $t$, and $s \setminus t \subset U$.

$\mathbb{M}_\mathcal{F}$ is usually called Mathias forcing associated with $\mathcal{F}$.

$\mathbb{M}_\mathcal{F}$ is a natural forcing adding a pseudointersection of $\mathcal{F}$: if $G$ is a $\mathbb{M}_\mathcal{F}$-generic, then $X = \bigcup\{s : \exists F \in \mathcal{F}(\langle s, F \rangle \in G)\}$ is almost contained in any $F \in \mathcal{F}$.

Applications: killing mad families, making the ground model reals not splitting, etc.
Let $x, y \in \omega^\omega$. The notation $x \leq^* y$ means $x(n) \leq y(n)$ for all but finitely many $n$.

$b$ (resp. $d$) is the minimal size of an $\leq^*$-unbounded (resp. dominating) $A \subset \omega^\omega$.

A poset $\mathbb{P}$ is said to add a dominating real if in $V^\mathbb{P}$ there exists $x \in \omega^\omega$ such that $y \leq^* x$ for all ground model $y \in \omega^\omega$.

Example: Laver forcing, Hechler forcing. Miller and Cohen forcing do not add dominating reals.

Theorem (Canjar 1988)

$d = c$ implies the existence of an ultrafilter $\mathcal{F}$ such that $\mathbb{M}_\mathcal{F}$ does not add dominating reals.

Definition (Guzman-Hrusak-Martinez)

A filter $\mathcal{F}$ on $\omega$ is called Canjar if $\mathbb{M}_\mathcal{F}$ does not add dominating reals.

Let $B$ be an unbounded subset of $\omega^\omega$. A filter $\mathcal{F}$ on $\omega$ is called $B$-Canjar if $\mathbb{M}_\mathcal{F}$ adds no reals dominating all elements of $B$.

There is a combinatorial characterization of Canjar filters by Hrusak and Minami in terms of the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega}$ generated by $\{[F]^{<\omega} : F \in \mathcal{F}\}$. 
Theorem (Brendle 1998)

1) Every $\sigma$-compact filter is Canjar.
2) $(b = c)$. Let $A$ be a mad family. Then for any unbounded $B = \{b_\alpha : \alpha < b\} \subset \omega^\omega$ such that $b_\alpha \leq^* b_\beta$ for all $\alpha < \beta$, there exists a $B$-Canjar $F \supset F_A$. \hfill $\square$

If an ultrafilter $F$ is Canjar, then it is a $P$-filter and there is no monotone surjection $\varphi : \omega \to \omega$ such that $\varphi(F)$ is rapid. The converse is consistently not true by a recent result of Blass, Hrusak and Verner. Its proof relies on the following characterization

Theorem (Guzman-Hrusak-Martinez 2013; Blass-Hrusak-Verner 2011 for ultrafilters)

A filter $F$ is Canjar iff it is a coherent strong $P^+$-filter. \hfill $\square$

Recall that a filter $F$ is a coherent strong $P^+$-filter if for every sequence $\langle C_n : n \in \omega \rangle$ of compact subsets of $F^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in C_n$ for all $n$ and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$, then $\bigcup_{n \in \omega} (X_n \cap [k_n, k_{n+1})) \in F^+$.

Strong $P^+$-filters are defined by removing the coherence requirement.
A topological space $X$ has the Menger covering property (or simply is Menger), if for every sequence $\langle U_n : n \in \omega \rangle$ of open covers of $X$ there exists a sequence $\langle V_n : n \in \omega \rangle$ such that $V_n \in [U_n]^{<\omega}$ and $\{ \bigcup V_n : n \in \omega \}$ is a cover of $X$.

If, moreover, we can choose $V_n$ in such a way that for any $x \in X$ we have $x \in \bigcup V_n$ for all but finitely many $n \in \omega$, then $X$ is called Hurewicz.

Example: every $\sigma$-compact space is Hurewicz. More generally: a union of fewer than $b$ (resp. $\mathfrak{d}$) compacts is Hurewicz (resp. Menger).

$\omega^\omega$ is not Menger as witnessed by $\langle U_n : n \in \omega \rangle$,
$U_n = \{ \{x : x(n) = k \} : k \in \omega \}$. 
Main results

Theorem (Chodounský-Repovš-Z. 2014)
$M_{\mathcal{F}}$ is Canjar iff $\mathcal{F}$ has the Menger covering property as a subspace of $\mathcal{P}(\omega)$.

Theorem (Chodounský-Repovš-Z. 2014)
Let $\mathcal{F}$ be a filter. Then $M_{\mathcal{F}}$ is almost $\omega^\omega$-bounding iff $\mathcal{F}$ is $B$-Canjar for all unbounded $B \subset \omega^\omega$ iff $\mathcal{F}$ is Hurewicz.

Recall that a poset $\mathbb{P}$ is almost $\omega^\omega$-bounding if for every $\mathbb{P}$-name $\dot{f}$ for a real and $q \in \mathbb{P}$, there exists $g \in \omega^\omega$ such that for every $A \in [\omega]^\omega$ there is $q_A \leq q$ such that $q_A \Vdash g \upharpoonright A \not\subseteq^* \dot{f} \upharpoonright A$. 

Some corollaries

Corollary

Let $\mathcal{F}$ be an analytic filter on $\omega$. Then $\mathbb{M}_\mathcal{F}$ does not add a dominating real iff $\mathcal{F}$ is $\sigma$-compact.

Answers a question of Hrusak and Minami. For Borel filters has been independently proved by Guzman, Hrusak, and Martinez.

Corollary (Hrušák-Martínez 2012)

There exists a mad family $\mathcal{A}$ on $\omega$ such that $\mathbb{M}_\mathcal{F}(\mathcal{A})$ adds a dominating real ($= \mathcal{F}(\mathcal{A})$ is not Canjar).

Answers a question of Brendle.

Corollary

($\mathfrak{d} = \mathfrak{c}$.) There exists a mad family $\mathcal{A}$ on $\omega$ such that $\mathbb{M}_\mathcal{F}(\mathcal{A})$ does not add a dominating real ($= \mathcal{F}$ is Canjar).

Under $\mathfrak{d} = \mathfrak{c} = \mathfrak{u}$ it was proved by Guzman, Hrusak, and Martinez.

Corollary

A filter $\mathcal{F}$ is Canjar iff it is a strong $P^+$-filter.
Theorem (Guzman-Hrusak-Martinez 2013)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong $P^+$-filter.

Recall that a filter $\mathcal{F}$ is a coherent strong $P^+$-filter if for every sequence $\langle C_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in C_n$ for all $n$

and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$,

then $\bigcup_{n \in \omega} (X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.

Strong $P^+$-filters are defined by removing the coherence requirement.
For \( n \in \omega \) and \( q \subset n \) we set \([n, q] := \{A \in \mathcal{P}(\omega) : A \cap n = q\}\). Sets \([n, q]\) form a standard base \( \mathcal{B} \) for the topology of \( \mathcal{P}(\omega) \). Set also \( \uparrow X = \{A \in \mathcal{P}(\omega) : A \supset X\} \) for every \( X \subset \omega \).

**Lemma**

*Suppose that \( \mathcal{X} \subset \mathcal{P}(\omega) \) is closed under taking supersets and \( \mathcal{O} \) is a cover of \( \mathcal{X} \) by sets open in \( \mathcal{P}(\omega) \). Then there exists a family \( Q \subset [\omega]^{<\omega} \) such that \( \mathcal{X} \subset \bigcup_{q \in Q} \uparrow q \) and for every \( q \in Q \) there exists \( \mathcal{O}' \in [\mathcal{O}]^{<\omega} \) covering \( \uparrow q \).*

**Proof.** Wlog \( \mathcal{O} \subset \mathcal{B} \). Let us fix \( X \in \mathcal{X} \) and find 
\[ \{[n_i, q_i] : i \in m\} \subset \mathcal{O} \text{ such that } \uparrow X \subset \bigcup_{i \in m} [n_i, q_i]. \]
Breaking some of the sets \([n_i, q_i]\) into smaller pieces of the same form, we may assume if necessary that for some \( n \in \omega \) we have \( n_i = n \) for all \( i \in m \). Moreover, wlog no proper subcollection of 
\[ \mathcal{O}' = \{[n, q_i] : i < m\} \text{ covers } \uparrow X. \]
Therefore 
\[ \{q_i : i < m\} = \{t \subset n : X \cap n \subset t\}, \] and consequently 
\[ \bigcup_{i < m} [n, q_i] = \uparrow (X \cap n). \]
Thus \( X \in \uparrow X \subset \uparrow (X \cap n) \subset \bigcup \mathcal{O}' \). \( \square \)
Proof of “$\mathcal{F}$ is Hurewicz iff $\mathbb{M}_\mathcal{F}$ is almost $\omega^\omega$-bounding”.

Suppose that $\mathcal{F}$ is Hurewicz, but there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and an $\mathbb{M}_\mathcal{F}$-name $\dot{g}$ for a function dominating $X$ (as forced by $1_{\mathbb{M}_\mathcal{F}}$). For every $x \in X$ find $n^x \in \omega$ and a condition $\langle s^x, F^x \rangle$ forcing $x(n) < \dot{g}(n)$ for all $n \geq n^x$. Since $X$ cannot be covered by a countable family of bounded sets, wlog $s^x = s_*$ and $n^x = n_*$ for all $x \in X$.

For every $m \in \omega$ consider $S_m = \{s \in [\omega]^{<\omega} : \max s_* < \min s \land \exists F_s \in \mathcal{F} (\langle s_* \cup s, F_s \rangle \Vdash \dot{g}(m) = g_s(m))\}$.

For every $F \in \mathcal{F}$ there exists $s \in S_m$ such that $s \subset F$. In other words, $U_m := \{\uparrow s : s \in S_m\}$ is an open cover of $\mathcal{F}$. Since $\mathcal{F}$ is Hurewicz, for every $m$ there exists $V_m \in [U_m]^{<\omega}$ such that $\bigcup V_m : m \in \omega \}$ is a $\gamma$-cover of $\mathcal{F}$. Let $T_m \in [S_m]^{<\omega}$ be such that $V_m = \{\uparrow s : s \in T_m\}$ and $f(m) = \max\{g_s(m) : s \in T_m\}$. We will derive a contradiction by showing $x <^* f$ for each $x \in X$. 
Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in T_m$ such that $F^n_x \in \uparrow s_m$. Pick any $m \geq n_*, l$. Since $\langle s_*, F^n_x \rangle \models x(m) < \dot{g}(m), \langle s_* \cup s_m, F_{s_m} \rangle \models \dot{g}(m) \leq f(m)$, and these two conditions are compatible, it follows that $x(m) < f(m)$.

Now suppose that $\mathcal{F}$ is not Hurewicz as witnessed by a sequence $\langle U_n : n \in \omega \rangle$ of covers of $\mathcal{F}$ by sets open in $\mathcal{P}(\omega)$. Wlog $U_n = \{ \uparrow q_m(n) : m \in \omega \}$, where $q_m(n) \in [\omega]^{<\omega}$. For every $F \in \mathcal{F}$ consider the function $x_F \in \omega^\omega$, $x_F(n) = \min \{ m : F \in \uparrow q_m(n) \}$. $X = \{ x_F : F \in \mathcal{F} \}$ is unbounded.

Let $G$ be the generic pseudointersection of $\mathcal{F}$ added by $\mathbb{M}_\mathcal{F}$. For every $n$ there exists $g(n)$ such that $G \setminus n \in \uparrow q_{g(n)}(n)$. Fix $F \in \mathcal{F}$ and find $n$ such that $G \setminus n \subset F$. Then $G \setminus n \in \uparrow q_{g(n)}(n)$ yields $F \in \uparrow q_{g(n)}(n)$, which implies $x_F(n) \leq g(n)$. Thus $g \in \omega^\omega$ is dominating $X$, and therefore $\mathbb{M}_\mathcal{F}$ fails to preserve ground model unbounded sets. \qed
Questions

Question

Let $\mathcal{A} \subset [\omega]^\omega$ be a mad family. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{F}(\mathcal{A})$?

Question

(CH) Let $\mathcal{U}$ be a meager filter generated by a tower. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{U}$?
Thank you for your attention.